ON MAXIMAL REGULARITY FOR THE CAUCHY-DIRICHLET MIXED PARABOLIC PROBLEM WITH FRACTIONAL TIME DERIVATIVE

REGOLARITÀ MASSIMALE PER IL PROBLEMA MISTO PARABOLICO DI CAUCHY-DIRICHLET CON DERIVATA TEMPORALE FRAZIONARIA

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Abstract. In this seminar we illustrate some results of maximal regularity for the Cauchy-Dirichlet mixed problem, with a fractional time derivative of Caputo type in spaces of continuous and Hölder continuous functions.

Sunto. In questo seminario presentiamo alcuni risultati di regolarità massimale per il problema misto di Cauchy-Dirichlet, con una derivata temporale frazionaria di Caputo, in spazi di funzioni continue e hölderiane.

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Let us consider the following classical Cauchy-Dirichlet mixed parabolic problem:

$$
\begin{cases}
D_t u(t,x) = A(x, D_x)u(t,x) + f(t,x), \quad t \in [0,T], x \in \Omega, \\
\\
u(t,x') = g(t,x'), \quad (t,x') \in [0,T] \times \partial\Omega, \\
\\
u(0,x) = u_0(x), \quad x \in \overline{\Omega}.
\end{cases} \tag{1}
$$

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Here $A(x, D_x)$ is a linear elliptic partial differential operator of second order, in the bounded domain Ω of $\mathbb{R}^n$. Suppose, for example, that

$$A(x, D_x) = \sum_{|\rho| \le 2} a_\rho(x) D_x^\rho,$$

with $a_\rho : \overline{\Omega} \to \mathbb{R}$,

$$\sum_{|\rho|=2} a_\rho(x)\xi^\rho \ge \nu|\xi|^2, \tag{2}$$

for some ν positive, independent of x in $\overline{\Omega}$ and ξ in $\mathbb{R}^n$.

Then several theorems of maximal regularity for (1) are known in mathematical literature. A theorem of maximal regularity is a statement establishing the existence of a linear and topological isomorphism between a certain class of data (in this case a certain class of triples (f, g, u_0)) and a certain class of solutions u. Apart its intrinsic interest, maximal regularity is very helpful to treat certain nonlinear problems as perturbations of linear ones.

We are going to recall one of the most classical results of this type for (1). We begin with some well known definitions.

If $\beta \in \mathbb{N}_0$ and Ω is an open, bounded subset of $\mathbb{R}^n$, we shall indicate with $C^\beta(\overline{\Omega})$ the class of complex valued functions which are continuous in $\overline{\Omega}$, together with their derivatives (extensible by continuity to $\overline{\Omega}$) of order not exceeding β. If $\beta \in \mathbb{R}^+ \setminus \mathbb{N}$, $C^\beta(\overline{\Omega})$ will indicate the class of functions in $C^{[\beta]}(\overline{\Omega})$ whose derivatives of order $[\beta]$ are Hölder continuous of order $\beta - [\beta]$ in $\overline{\Omega}$. These definitions admit natural extensions to functions with values in a Banach space X. In this case, we shall use the notation $C^\beta(\overline{\Omega}; X)$ (in particular $C^\beta([a,b]; X)$, in case $\Omega = (a,b) \subseteq \mathbb{R}$). By local charts, if $\partial\Omega$ is sufficiently regular, we can consider the spaces $C^\beta(\partial\Omega)$. All these classes will be assumed to be equipped of natural norms. We shall use the notation

$$C_0^\beta(\overline{\Omega}) := \{f \in C^\beta(\overline{\Omega}) : \gamma f = f_{|\partial\Omega} = 0\}.$$

If $\alpha, \beta \in [0, \infty)$, $T \in \mathbb{R}^+$ and Ω is an open bounded subset of $\mathbb{R}^n$, we set

$$C^{\alpha,\beta}([0,T] \times \overline{\Omega}) := C^\alpha([0,T]; C(\overline{\Omega})) \cap B([0,T]; C^\beta(\overline{\Omega})).$$

$B([0,T];C^{\beta}(\overline{\Omega}))$ indicates the class of bounded functions with values in $C^{\beta}(\overline{\Omega})$. An analogous meaning will have $C^{\alpha,\beta}([0,T]\times\partial\Omega)$. If X is a Banach space, $Lip([0,T];X)$ will indicate the class of Lipschitz continuous functions from $[0,T]$ to X, equipped with a natural norm.

Then the following maximal regularity theorem is well known (see [8]):

Theorem 1. *Consider problem (1), with the following conditions:*

(α_1) Ω is an open, bounded subset in $\mathbb{R}^n$ lying on one side of its boundary $\partial\Omega$, which is a $n-1-$submanifold of $\mathbb{R}^n$ of class $C^{2+\theta}$, with $\theta\in(0,2)\setminus\{1\}$.

(α_2) $A(x,D_x)$ is strongly elliptic, in the sense of (2), with coefficients a_ρ of class $C^{\theta}(\overline{\Omega})$.

Then the following conditions on f,g,u_0 are necessary and sufficient in order that there exists a unique solution u in the class $C^{1+\frac{\theta}{2},2+\theta}([0,T]\times\overline{\Omega})$:

(I) $f\in C^{\frac{\theta}{2},\theta}([0,T]\times\overline{\Omega})$;

(II) $u_0\in C^{2+\theta}(\overline{\Omega})$;

(III) $g\in C^{1+\frac{\theta}{2},2+\theta}([0,T]\times\partial\Omega)$:

(IV) $u_0(x')=g(0,x')\ \forall x'\in\partial\Omega$;

(V) $A(x',D_x)u_0(x')+f(0,x')=D_tg(0,x')\ \forall x'\in\partial\Omega$.

Another maximal regularity theorem we are considering is the following (for a proof, see [3]) :

Theorem 2. *Consider problem (1), with the conditions (α_1)-(α_2). Let $\theta\in(0,2)\setminus\{1\}$. Then the following conditions on f,g,u_0 are necessary and sufficient in order that there exists a unique solution u such that $u\in C^{1,2}([0,T]\times\overline{\Omega})$ with $D_tu\in B([0,T];C^{\theta}(\overline{\Omega}))$, $u\in B([0,T];C^{2+\theta}(\overline{\Omega}))$:*

(I) $f\in C([0,T];C(\overline{\Omega}))\cap B([0,T];C^{\theta}(\overline{\Omega}))$;

(II) $u_0\in C^{2+\theta}(\overline{\Omega})$;

(III) $g\in C^{1,2}([0,T]\times\partial\Omega)$, $D_tg\in B([0,T];C^{\theta}(\partial\Omega))$, $g\in B([0,T];C^{2+\theta}(\partial\Omega))$;

(IV) $u_0(x')=g(0,x')\ \forall x'\in\partial\Omega$;

(V) $f_{|[0,T]\times\partial\Omega}-D_tg\in C^{\frac{\theta}{2}}([0,T];C(\partial\Omega))$;

(VI) $A(x',D_x)u_0(x')+f(0,x')=D_tg(0,x')\ \forall x'\in\partial\Omega$.

Remark 1. Among the assumptions of Theorem 2, the least obvious is (V). The necessity of it can be seen in the following way: if $D_t u$ is bounded with values in $C^\theta(\overline{\Omega})$, then u is Lipschitz continuous with values in $C^\theta(\overline{\Omega})$. If $u \in C^{2+\theta}(\overline{\Omega})$,

$$\|u\|_{C^2(\overline{\Omega})} \leq C\|u\|_{C^\theta(\overline{\Omega})}^{\theta/2}\|u\|_{C^{2+\theta}(\overline{\Omega})}^{1-\theta/2}$$

for some C positive independent of u. We deduce that if $D_t u \in B([0,T]; C^\theta(\overline{\Omega}))$ and $u \in B([0,T]; C^{2+\theta}(\overline{\Omega}))$, then $u \in C^{\theta/2}([0,T]; C^2(\overline{\Omega}))$, so that $A(\cdot, D_x)u \in C^{\theta/2}([0,T]; C(\overline{\Omega}))$ and

$$f_{|[0,T]\times\partial\Omega} - D_t g = A(\cdot, D_x)u_{|[0,T]\times\partial\Omega} \in C^{\theta/2}([0,T]; C(\partial\Omega)).$$

Now we introduce the notion of fractional derivative in the sense of Caputo. Let X be a complex Banach space. We introduce the following operator B_X:

$$\text{(3)} \qquad \begin{cases} D(B_X) := \{u \in C^1([0,T]; X) : u(0) = 0\}, \\ \\ B_X u(t) = u'(t). \end{cases}$$

Then it is easy to see that $\rho(B_X) = \mathbb{C}$. Moreover, $\forall \lambda \in \mathbb{C}$, $\forall f \in C([0,T]; X)$,

$$\text{(4)} \qquad (\lambda - B_X)^{-1} f(t) = -\int_0^t e^{\lambda(t-s)} f(s) ds.$$

B_X is positive of type $\frac{\pi}{2}$ in the sense of the following

Definition 1. *Let X be a complex Banach space and let $B : D(B) \subseteq X \to X$ be a linear (unbounded) operator. We shall say that it is positive of type $\omega \in (0, \pi)$ if*

$$\{\lambda \in \mathbb{C} \setminus \{0\} : |Arg(\lambda)| > \omega\} \cup \{0\} \subseteq \rho(B)$$

and for every $\omega' \in (\omega, \pi)$ there exists $M(\omega') > 0$ such that $\|(1 + |\mu|)(\mu - B)^{-1}\|_{\mathcal{L}(X)} \leq M(\omega')$ in case $|Arg(\mu)| \geq \omega'$.

$\rho(B)$ stands for the resolvent set of B.

Let B be a positive operator in X. Suppose that it is of type ω, for some $\omega \in (0, \pi)$. We fix θ in (ω, π) and R in $\mathbb{R}^+$ such that $\{\mu \in \mathbb{C} : |\mu| \leq R\} \subseteq \rho(B)$. Then, if $\alpha > 0$ (so that $-\alpha < 0$), we set

$$\text{(5)} \qquad B^{-\alpha} := -\frac{1}{2\pi i}\int_{\gamma(\theta,R)} \lambda^{-\alpha}(\lambda - B)^{-1} d\lambda.$$

with $\gamma(\theta, R)$ piecewise C^1 path, describing

$$\{\lambda \in \mathbb{C} : |\lambda| \geq R, |Arg(\lambda)| = \theta\} \cup \{\lambda \in \mathbb{C} : |\lambda| = R, |Arg(\lambda)| \leq \theta\}.$$

This definition is clearly inspired by the elementary case of B real positive number. The complex integral is convergent in the Banach space $\mathcal{L}(X)$. The following facts can be checked (see, for example, [10]):

(a) (5) is consistent with the usual definition of $B^{-\alpha}$ in case $\alpha \in \mathbb{N}$;

(b) if $\alpha, \beta \in \mathbb{R}^+$, $B^{-\alpha}B^{-\beta} = B^{-(\alpha+\beta)}$;

(c) $\forall \alpha$ in $\mathbb{R}^+$ $B^{-\alpha}$ is injective;

(d) in case B is positive and self-adjoint in the Hilbert space X, (2) is equivalent with the definition of fractional power obtained employing the spectral resolution.

So we can define, for $\alpha \in \mathbb{R}^+$,

$$B^\alpha := (B^{-\alpha})^{-1}. \tag{6}$$

Of course, the domain $D(B^\alpha)$ of B^α is the range of $B^{-\alpha}$. We observe, that, if B is unbounded and $\beta \in \mathbb{R}$, B^β is bounded only if $\beta \leq 0$. Moreover, employing (a)-(c), it is easy to show that, if $0 < \alpha < \beta$, $D(B^\beta) \subseteq D(B^\alpha)$,

$$D(B^\beta) = \{x \in D(B^\alpha) : B^\alpha x \in D(B^{\beta-\alpha})\}$$

and, if $x \in D(B^\beta)$,

$$B^\beta x = B^{\beta-\alpha}B^\alpha x.$$

For example, in the case of B_X defined in (3) we have, for any positive α,

$$B_X^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

For a proof, see [4].

Now we are able to define the fractional time derivative in the sense of Caputo $\mathbb{D}_X^\alpha u$ in case $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$. Let $m < \alpha < m+1$, with $m \in \mathbb{N}_0$. If $u \in C^{m+1}([0,T]; X)$, we set

$$\mathbb{D}_X^\alpha u(t) := B_X^{\alpha-(m+1)}(D^{m+1}u)(t) = \tfrac{1}{\Gamma(m+1-\alpha)} \int_0^t (t-s)^{m-\alpha} D^{m+1}u(s) ds.$$

We observe that

$$D^{m+1}u = D^{m+1}[u - \sum_{k=0}^{m} \frac{t^k}{k!} D^k u(0)]$$

and that, if $u \in C^{m+1}([0,T];X)$, $u - \sum_{k=0}^{m} \frac{t^k}{k!} u^{(k)}(0) \in D(B_X^{m+1})$. So

$$\mathbb{D}_X^\alpha u = B_X^{\alpha-(m+1)} B_X^{m+1}[u - \sum_{k=0}^{m} \frac{t^k}{k!} u^{(k)}(0)] = B_X^\alpha [u - \sum_{k=0}^{m} \frac{t^k}{k!} u^{(k)}(0)].$$

as $u - \sum_{k=0}^{m} \frac{t^k}{k!} u^{(k)}(0) \in D(B_X^{m+1})$. This suggests the following

Definition 2. *Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$, $m < \alpha < m+1$, with $m \in \mathbb{N}_0$. We shall say that $u \in D(\mathbb{D}_X^\alpha)$ if $u \in C^m([0,T];X)$ and $u - \sum_{k=0}^{m} \frac{t^k}{k!} u^{(k)}(0) \in D(B^\alpha)$. In this case, we set*

$$\mathbb{D}_X^\alpha u := B_X^\alpha (u - \sum_{k=0}^{m} \frac{t^k}{k!} u^{(k)}(0)).$$

In case $\alpha \in \mathbb{N}$, we set

$$D(\mathbb{D}_X^\alpha) = C^\alpha([0,T];X),$$

$$\mathbb{D}_X^\alpha u := B_X^\alpha (u - \sum_{k=0}^{m} \frac{t^k}{k!} u^{(k)}(0)) = D_t^\alpha u.$$

Now we consider the following generalization of (1):

$$(7) \qquad \begin{cases} \mathbb{D}_{C(\overline{\Omega})}^\alpha u(t,x) = A(x, D_x) u(t,x) + f(t,x), \quad t \in [0,T], x \in \Omega, \\ \\ u(t,x') = g(t,x'), \quad (t,x') \in [0,T] \times \partial\Omega, \\ \\ D_t^k u(0,x) = u_k(x), \quad k \in \mathbb{N}_0, k < \alpha, \quad x \in \overline{\Omega}. \end{cases}$$

We introduce the following definition of strict solution of (7):

Definition 3. *Let $f \in C([0,T] \times \overline{\Omega}) = C([0,T]; C(\overline{\Omega}))$, $u_k \in C(\partial\Omega)$ for each $k \in \mathbb{N}_0$, $k < \alpha$. A strict solution u of (7) is an element of $D(\mathbb{D}_{C(\overline{\Omega})}^\alpha) \cap C([0,T]; C^2(\overline{\Omega}))$, such that all the conditions in (7) are satisfied pointwise.*

We begin by stating the following extension of Theorem 2:

Theorem 3. *Suppose that the following assumptions are fulfilled:*

(A1) Ω is an open, bounded subset in $\mathbb{R}^n$ lying on one side of its boundary $\partial\Omega$, which is a $n-1-$submanifold of $\mathbb{R}^n$ of class $C^{2+\theta}$, with $\theta \in (0,2) \setminus \{1\}$.

(A2) $\alpha \in (0,2)$, $A(x, D_x) = \sum_{|\rho|\le 2} a_\rho(x) D_x^\rho$, *with* $a_\rho \in C^\theta(\overline{\Omega})$, a_ρ *complex valued;* $A(x, D_x)$ *is assumed to be elliptic, in the sense that* $\sum_{|\rho|=2} a_\rho(x)\xi^\rho \neq 0$ $\forall \xi \in \mathbb{R}^n \setminus \{0\}$*; we suppose, moreover, that*

$$|Arg(\sum_{|\rho|=2} a_\rho(x)\xi^\alpha)| < (1 - \frac{\alpha}{2})\pi, \quad \forall x \in \overline{\Omega}, \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

(A3) $\alpha\theta < 2$.

Then the following conditions are necessary and sufficient, in order that (7) has a unique strict solution u *such that* $\mathbb{D}^\alpha_{C(\overline{\Omega})} u$ *belongs to* $B([0,T]; C^\theta(\overline{\Omega}))$ *and* u *belongs to* $B([0,T]; C^{2+\theta}(\overline{\Omega}))$*:*

(I) $f \in C([0,T]; C(\overline{\Omega})) \cap B([0,T]; C^\theta(\overline{\Omega}))$.

(II) $u_0 \in C^{2+\theta}(\overline{\Omega})$ *and, in case* $\alpha \in (1,2)$, $u_1 \in C^{\theta + 2(1-\frac{1}{\alpha})}(\overline{\Omega})$.

(III) $g \in C([0,T]; C^2(\partial\Omega)) \cap B([0,T]; C^{2+\theta}(\partial\Omega))$, $\mathbb{D}^\alpha_{C(\partial\Omega)} g$ *exists and belongs to* $C([0,T]; C(\partial\Omega)) \cap B([0,T]; C^\theta(\partial\Omega))$*;*

(IV) $u_{0|\partial\Omega} = g(0)$ *and, in case* $\alpha \in (1,2)$, $u_{1|\partial\Omega} = D_t g(0)$

(V) $f_{|[0,T]\times\partial\Omega} - \mathbb{D}^\alpha_{C(\partial\Omega)} g \in C^{\frac{\alpha\theta}{2}}([0,T]; C(\partial\Omega))$.

(VI) $(A(\cdot, D_x)u_0 + f(0))_{|\partial\Omega} = \mathbb{D}^\alpha_{C(\partial\Omega)} g(0)$.

The following theorem is an extension of Theorem 1:

Theorem 4. *Suppose that* $(\alpha_1) - (\alpha_2)$ *in the statement of Theorem 3 are fulfilled. Let* $\alpha, \theta \in (0,2)$, $\theta \neq 1$, $\alpha\theta < 2$. *Then the following conditions are necessary and sufficient, in order that (1) has a unique solution* u *in* $C([0,T]; C^2(\overline{\Omega})) \cap B([0,T]; C^{2+\theta}(\overline{\Omega}))$, *such that* $\mathbb{D}^\alpha_{C(\overline{\Omega})} u$ *exists and* $\mathbb{D}^\alpha_{C(\overline{\Omega})} u$ *and* $A(\cdot, D_x)u$ *belong to* $C^{\frac{\alpha\theta}{2}, \theta}([0,T] \times \overline{\Omega})$*:*

(I) $f \in C^{\frac{\alpha\theta}{2}, \theta}([0,T] \times \overline{\Omega})$.

(II) $u_0 \in C^{2+\theta}(\overline{\Omega})$ *and, in case* $\alpha \in (1,2)$, $u_1 \in C^{\theta + 2(1-\frac{1}{\alpha})}(\overline{\Omega})$.

(III) $g \in C([0,T]; C^2(\partial\Omega)) \cap B([0,T]; C^{2+\theta}(\partial\Omega))$, $\mathbb{D}^\alpha_{C(\partial\Omega)} g$ *exists and belongs to* $C^{\frac{\alpha\theta}{2}, \theta}([0, T] \times \partial\Omega)$*;*

(IV) $u_{0|\partial\Omega} = g(0)$ *and, in case* $\alpha \in (1,2)$, $u_{1|\partial\Omega} = D_t g(0)$.

(V) $(A(\cdot, D_x)u_0 + f(0))_{|\partial\Omega} = \mathbb{D}^\alpha_{C(\partial\Omega)} g(0)$.

Complete proofs of Theorems 3 and 4 are given in [6]. For simplicity, we are going to sketch a proof of Theorem 4 in the particular case $g \equiv 0$. We shall employ Theorem 2, together with the following abstract result of maximal regularity (see [1], [2], [5]):

Theorem 5. *Let $\alpha \in (0,2)$. Let X be a complex Banach space, A a (usually) unbounded linear operator in X, such that, for some $\lambda_0 \in \mathbb{R}$, $\lambda_0 - A$ is positive of type ω less than $\pi(1-\frac{\alpha}{2})$. Consider the abstract problem*

$$\begin{cases} \mathbb{D}_X^\alpha u(t) = Au(t) + f(t), \quad t \in [0,T], \\ \\ D_t^k u(0) = u_k, \quad k \in \mathbb{N}_0, k < \alpha. \end{cases} \tag{8}$$

Let $\beta \in (0, \min\{1,\alpha\})$. Then the following conditions are necessary and sufficient in order that (8) has a unique strict solution u, with $\mathbb{D}_X^\alpha u$ and Au belonging to $C^\beta([0,T];X)$:

(a) $f \in C^\beta([0,T];X)$;
(b) $u_0 \in D(A)$;
(c) $Au_0 + f(0) \in (X, D(A))_{\beta/\alpha,\infty}$;
(d) if $\alpha > 1$, $u_1 \in (X, D(A))_{1-\frac{1-\beta}{\alpha},\infty}$.

Here $(X, D(A))_{\theta,\infty}$ stands for the real interpolation space with these parameters. We shall employ Theorem 5 in the following situation: we suppose that the conditions on Ω and $A(x, D_x)$ in the statement of Theorem 1 are fulfilled and set

$$X = C(\overline{\Omega}), \tag{9}$$

$$\begin{cases} D(A) = \{u \in \cap_{1 \le p < \infty}(W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)) : A(\cdot, D_x)u \in C(\overline{\Omega})\}, \\ \\ Au(x) = A(x, D_x)u. \end{cases} \tag{10}$$

A satisfies the assumptions of Theorem 5 (see [3]). Moreover, if $\theta \in (0,1) \setminus \{\frac{1}{2}\}$, we have

$$(C(\overline{\Omega}), D(A))_{\theta,\infty} = \{f \in C^{2\theta}(\overline{\Omega}) : f_{|\partial\Omega} = 0\}. \tag{11}$$

See for this [3]. So from Theorem 5 we obtain:

Corollary 1. *Consider problem (7). Let* $\alpha \in (0,2)$ *and suppose that* $g \equiv 0$. *Let* $\beta \in (0, \min\{1,\alpha\}) \setminus \{\frac{\alpha}{2}, 1-\frac{\alpha}{2}\}$. *Then the following conditions are necessary and sufficient, in order that there exits a unique solution* u *such that* $\mathbb{D}^{\alpha}_{C(\overline{\Omega})}u$ *and* $A(x,D_x)u$ *belonging to* $C^{\beta}([0,T];X) : C(\overline{\Omega}))$*:*

(I) $f \in C^{\beta}([0,T];C(\overline{\Omega}))$*;*

(b) $u_0 \in D(A)$*;*

(c) $A(x,D_x)u_0 + f(0) \in C^{\frac{2\beta}{\alpha}}(\overline{\Omega})$, $(A(\cdot,D_x)u_0 + f(0))_{|\partial\Omega} = 0$*;*

(d) if $\alpha > 1$, $u_1 \in C^{2(1-\frac{1-\beta}{\alpha})}(\overline{\Omega})$, $u_{1|\partial\Omega} = 0$.

We deduce the following particular case of Theorem 4:

Proposition 1. *Suppose that the assumptions on* Ω *and* $A(x,D_x)$ *in the statement of Theorem 3 are fulfilled. Let* $\alpha,\theta \in (0,2)$, $\theta \neq 1$, $\alpha\theta < 2$. *Then the following conditions are necessary and sufficient, in order that (1), with* $g \equiv 0$, *has a unique solution* u *in* $C([0,T];C^2(\overline{\Omega})) \cap B([0,T];C^{2+\theta}(\overline{\Omega}))$, *such that* $\mathbb{D}^{\alpha}_{C(\overline{\Omega})}u$ *exists and* $\mathbb{D}^{\alpha}_{C(\overline{\Omega})}u$ *and* $A(\cdot,D_x)u$ *belong to* $C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega})$*:*

(I) $f \in C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega})$.

(II) $u_0 \in C^{2+\theta}(\overline{\Omega})$ *and, in case* $\alpha \in (1,2)$, $u_1 \in C^{\theta+2(1-\frac{1}{\alpha})}(\overline{\Omega})$.

(III) $u_{0|\partial\Omega} = 0$ *and, in case* $\alpha \in (1,2)$, $u_{1|\partial\Omega} = 0$.

(IV) $(A(\cdot,D_x)u_0 + f(0))_{|\partial\Omega} = 0$.

Proof. It follows immediately from Theorem 3 and Corollary 1, taking $\beta = \frac{\alpha\theta}{2}$. □

Remark 2. In case $\alpha = 1$, the assumptions of Theorem 4 imply that u belongs to $C^{1+\frac{\theta}{2}}([0,T];C(\overline{\Omega}))$, so that u belongs to $C^{1+\frac{\theta}{2},2+\theta}([0,T]\times\overline{\Omega})$. This suggest that in the general case u should belong to $C^{\alpha+\frac{\alpha\theta}{2},2+\theta}([0,T]\times\overline{\Omega})$. This does not happen, as, in case $\alpha \neq 1$, u does not necessarily belong to any space $C^{\alpha+\epsilon}([0,T];C(\overline{\Omega}))$ for any ϵ positive. Consider the following example: let $\alpha \in (0,2)\setminus\{1\}$. Fix f_0 in $C_0^{2+\theta}(\overline{\Omega})\setminus\{0\}$, $\theta \in (0,2)\setminus\{1\}$, and define

$$\begin{cases} u : [0,T] \times \overline{\Omega} \to \mathbb{C}, \\ \\ u(t,x) = \frac{t^\alpha}{\Gamma(\alpha+1)} f_0(x). \end{cases}$$

Then u solves (1), if we take $f(t,x) = f_0(x) - \frac{t^\alpha}{\Gamma(\alpha+1)}[A(\cdot, D_x)f_0](x)$, $g \equiv 0$, $D_t^k u(0,\cdot) = 0$ if $k \in \mathbb{N}_0$, $k < \alpha$. It is easily seen that in this case the assumptions (I)-(V) of Theorem 4 are satisfied. However, u does not belong to any space $C^{\alpha+\epsilon}([0,T]; C(\overline{\Omega}))$, for any ϵ positive.

Nevertheless, let $v \in D(B_X^\alpha)$ be such that $B_X^\alpha v \in C^\beta([0,T]; X)$, with $\alpha + \beta, \beta \in \mathbb{R}^+ \setminus \mathbb{N}$. Then v can be represented in the form

$$v(t) = \sum_{k \in \mathbb{N}_0, k < [\beta]} t^{k+\alpha} v_k + w(t),$$

with $v_k \in X$ for each k, $w \in C^{\alpha+\beta}([0,T]; X)$, $w^{(j)}(0) = 0$, for each j in $\mathbb{N}_0$, $j < \alpha + \beta$ (see [5], Proposition 12 or [7], again Proposition 12). We deduce that in the situation of Theorem 4, at least in case $\alpha(1 + \frac{\theta}{2}) \notin \mathbb{N}_0$, the solution u can be written in the form

$$u(t) = U(t) + t^\alpha v_0,$$

with $v_0 \in C(\overline{\Omega})$, $U \in C^{\alpha + \frac{\alpha\theta}{2}}([0,T]; C(\overline{\Omega}))$.

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