CRITICAL EXPONENTS AND WHERE TO FIND THEM

ESPONENTI CRITICI E DOVE TROVARLI

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Abstract. In this expository paper we present a list of different semilinear wave-type problems with time-variable coefficients. The aim of this work is to understand the influence of such coefficients on the critical exponents for polynomial nonlinearities. Statements of global existence and blow-up will follow according to exponents which are below or above these critical ones.

Sunto. Vengono qui elencati alcuni problemi di evoluzione semilineari di tipo onde a coefficienti variabili. Si vuole capire l'influenza dei coefficienti variabili sull'esponente critico della nonlinearità di tipo polinomiale. Dopo aver congetturato tali esponenti, si danno risultati di esistenza globale e blow-up nei casi sovracritici e sottocritici.

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1. Introduction

Let $L(x, t, D_x, \partial_t)$ be a linear evolution operator; we consider the semilinear equation

$$Lu = f(u) \quad |f(u)|, \simeq |u|^p ;$$

whose analysis is influenced by the behavior of the function f when u is large or small. We look for:

- $R_L \subset \mathbb{R}$ the range of p for which we are able to prove *local* existence in time;
- $R_G \subset R_L$ the range of p for which we are able to prove *global* existence in time;
- $R_{\nexists}$ the range of p where even the weak solutions do not exist globally;
- $R_B \supset R_{\nexists}$ the range of p for which the local solutions *blow up* in a certain norm.

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An exponent p_c is critical if we are able to prove that R_L, R_G, $R_{\nexists}$, R_B are intervals contained in $[1, +\infty)$ and

$$\bar{R}_G \cap \bar{R}_{\nexists} = \{p_c\} \quad or \quad \bar{R}_G \cap \bar{R}_B = \{p_c\}.$$

Let us list the critical exponents for well–known semilinear equations; we roughly find the following situation:

Equation	space	critical exponents	known results
$-\Delta u = \|u\|^{p-2}u$	$H^1(\Omega)$ $u_{\|\partial\Omega} = 0$	$2^* = \frac{2n}{n-2}$	$p < 2^*$ $\exists$ $p > 2^*$ $\nexists$
$\partial_t - \Delta u = u^p$	weak sol. $u \geq 0$	$p_F(n) = 1 + \frac{2}{n}$	$p < p_F$ $\nexists$ $p > p_F$ global $\exists$
$iu_t + \Delta u = \pm\|u\|^{p-1}u$	$L^2(\mathbb{R}^N)$	$p_{L^2}(n) = 1 + \frac{4}{n}$	$p < p_F$ no scattering $p_F < p < p_{L^2}$ global $\exists$
$iu_t + \Delta u = -\|u\|^{p-1}u$	$H^1(\mathbb{R}^N)$	$p_{So}(n) = \frac{n+2}{n-2}$ $= 2^* - 1$	$p_{L^2} < p < p_{So}$ global $\exists$
$u_{tt} - \Delta u = \|u\|^{p-1}u$	weak sol.	$p_K = 1 + \frac{2}{n-1}$	$p < p_K$ $\nexists$
$u_{tt} - \Delta u = \pm\|u\|^{p-1}u$	small data	$p_{St} =$ $\frac{p_k + \sqrt{p_K^2 + 4p_K - 4}}{2}$	$p < p_{St}$ $\nexists$ blow up $p > p_{St}$ global $\exists$
$u_{tt} - \Delta u = -\|u\|^{p-1}u$	$H^1(\mathbb{R}^N)$	$p_{cc}(n) = \frac{n+3}{n-1}$ $p_{So}(n)$	$p_{cc}(n) < p < p_{So}(n)$ global $\exists$
$u_{tt} - \Delta u + u = \pm\|u\|^{p-1}u$	small data	p_F, p_{So}	$p < p_F$ blow-up $p_F < p < p_{So}$ global $\exists$

We point out that in this table we are omitting boundary regularity assumptions, possible sign assumptions for the solutions, the dimension where the result of the last column holds, and so on. We refer the reader to [17] for details. We also neglect to specify when a critical exponent belongs to R_B or $R_{\nexists}$ and when it belongs to R_G.

We observe that some exponents are simply a shift of others:

$$p_K(n) = p_F(n-1)\,; \quad p_{L^2}(n) = p_{So}(n+2)\,; \quad p_{cc}(n) = p_{So}(n+1)\,.$$

Hence the meaningful exponents are only p_F, p_{St} and p_{So}.

A unified vision of these exponents is given in [16] as follows:

$$\frac{n}{2}(p-1)=\begin{cases} 1 & \Longrightarrow \quad p=p_F \\ \frac{p+1}{2}+\frac{1}{p} & \Longrightarrow \quad p=p_{St} \\ p+1 & \Longrightarrow \quad p=p_{So} \end{cases}$$

We will see that these critical exponents reveal a ‘barycentre” among the scaling properties of the linear operator, the growth of the nonlinear polynomial term and the kind of solution we are looking for. In turn, the scaling properties of the linear operator depend on the space dimension, the order of the operator and the decomposition of the variable in suitable subspaces due to the physical model that the equation describes (for example space-time variables).

Our aim is to consider time-variable coefficients in the linear operator. We want to understand the influence of the growth of this coefficients on the critical exponents. The main difficulty is the loss of the scaling invariant properties due to the time-coefficients. In particular it is very difficult to obtain $L^p - L^q$ estimates. A successful idea is to treat these cases by finding a transformation of variables that put the operator in the constant coefficients form. Clearly this is not simple to be done, hence we will present only some particular cases and we will observe again a shift of the basic critical exponents connected to the variable coefficients.

The bibliography on critical exponents is huge, so we do not pretend to summarize it here. We propose only a path through the works of the author as a simple example of this topic. The seminal paper on the critical exponents is [11]; other important hints for a unified vision can be found in the books [16], [17].

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2. Fujita-type critical exponents

In [7] a non-existence result has been obtained for weak solutions of semilinear equations when the operator is quasi-homogeneous in two sets of variables.

Definition 2.1. *Let $L(x, y, D_x, D_y)$ be a differential operator of order $m \in \mathbb{N}$ defined on $D(L)$, a suitable set of functions on $\mathbb{R}^n_x \times \mathbb{R}^d_y$. Let $N = n + d$. We say that L is a* quasi-homogeneous operator of type (δ_1, δ_2), *with $\delta_1, \delta_2 > 0$, if for any $\lambda > 0$, $(x, y), (\xi, \eta) \in \mathbb{R}^N$, one has*

$$L(\lambda^{-\delta_1}x, \lambda^{-\delta_2}y, \lambda^{\delta_1}\xi, \lambda^{\delta_2}\eta) = \lambda^m L(x, y, \xi, \eta).$$

We call quasi-homogeneous dimension *of L the quantity*

$$Q_L = \delta_1 n + \delta_2 d.$$

It can be proved that for a quasi-homogeneous operator L of type (δ_1, δ_2) and order m, one has

$$LS^I_{\lambda^{\delta_1}} S^{II}_{\lambda^{\delta_2}} f = \lambda^m S^I_{\lambda^{\delta_1}} S^{II}_{\lambda^{\delta_2}} Lf \quad \text{ for } f \in D(L).$$

Here $S^I_\lambda f(x, y) := f(\lambda x, y)$ and $S^{II}_\lambda f(x, y) := f(x, \lambda y)$.

In this context, the main result of [7] can be stated in the following form.

Theorem 2.1. *Suppose that L is a linear differential operator of order $m \geq 1$ of the form*

$$L(x, y, D_x, D_y) = \sum_{|(\alpha,\beta)| \leq m} l_{\alpha,\beta}(x, y) D^\alpha_x D^\beta_y.$$

We assume that

(i) *L is quasi-homogeneous of type (δ_1, δ_2), with $\delta_1, \delta_2 > 0$;*
(ii) *for any $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^d$, it holds $D^\alpha_x D^\beta_y l_{\alpha,\beta}(x, y) = 0$.*

Let $p > 1$ and $p' = \frac{p}{p-1}$. Let $l_{\alpha,\beta}(x, y) \in L^{p'}_{loc}(\mathbb{R}^n)$ for any $\alpha_1 \leq \alpha$ and $\beta_1 \leq \beta$, $|(\alpha, \beta)| \leq m$. If

$$(Q - m)p \leq Q\,, \tag{1}$$

then

$$Lu = |u|^p$$

has no nontrivial weak solutions $u \in L^p_{loc}(\mathbb{R}^N)$.

In this theorem *weak solution* means a distribution $u \in L^p_{loc}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} |u|^p \varphi \, dxdy = \int_{\mathbb{R}^N} uL^* \varphi \, dxdy, \quad \forall \, \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N, \mathbb{R}_+) \, ,$$

where L^* denotes the adjoint of L, which satisfies

$$\int (Lf) g \, dxdy = \int f L^* g \, dxdy,$$

for any $f \in D(L)$, $g \in D(L^*)$. Clearly, the domains $D(L)$ and $D(L^*)$ depend on the regularity of the coefficients $l_{\alpha,\beta}$. Starting from (1), we put

$$p_F = \begin{cases} 1 + \frac{m}{Q-m} \, , & Q > m \\ \infty \, , & Q \leq m. \end{cases}$$

Let us consider the case $d = 1$. Interpreting $y := t$ as a time-variable, this result then applies to evolution equations. For example, for the heat equation we find the Fujita critical exponent

$$L = \partial_t - \Delta \, ; \; m = 2 \, ; \; (\delta_1, \delta_2) = (1, 2) \, ; \; Q = n + 2 \, ; \; p_F = 1 + \frac{2}{n} \, .$$

The same exponent appears for the nonlinear Schrödinger equation.
If we consider the wave equation, we get

$$L = \partial_{tt} - \Delta \, ; \; m = 2 \, ; \; (\delta_1, \delta_2) = (1, 1) \, ; \; Q = n + 1 \, ; \; p_F = 1 + \frac{2}{n-1} \, .$$

This corresponds to the exponent p_K in Section 1. We choose this notation since a nonexistence result for $L^1_{loc}(\mathbb{R}^n)$ solutions of

$$u_{tt} - \triangle u = |u|^p \, , \qquad u(0, x) = u_0(x) \quad u_t(0, x) = u_1(x) \, ,$$

under the assumptions $u_0, u_1 \in L^1$ compactly supported functions and

$$\int_{\mathbb{R}^n} u_1(x) \, dx > 0$$

had been established by Kato in 1980. More precisely, for compacted supported data, with average of the the initial velocity strictly positive, the average of the solution $\int_{\mathbb{R}^n} u(t, x) \, dx$ satisfies an ODE and blows up in finite time.

If we have

$$\sum_{0 \neq |(\alpha,\beta)| \leq m} l_{\alpha,\beta}(x, y) D_x^\alpha D_y^\beta u = f(u).$$

with $|f(u)| \neq |u|^p$, the scaling argument is not sharp in predicting critical exponents, but some partial results can be established. On the contrary, our theorem does not say nothing for the Klein-Gordon equation

$$u_{tt} - \triangle u + u = |u|^p \,.$$

The main problem is that $l_{0,0} \neq 0$. In other words, $f(u) = -u + |u|^p$ seems not treatable by scaling. This happens also in presence of memory-type nonlinear terms as considered in [1] (whose title underlines this phenomenon).

It is not possible to apply directly Theorem 2.1 when the operator is not quasi-homogeneous. In particular we want to extend Theorem 2.1 to operators which are quasi-homogeneous only in the principal part. For example, let us consider the classical wave damping equation

$$u_{tt} - \Delta u + u_t = |u|^p \,. \tag{2}$$

There is a competition between the heat operator $u_t - \Delta$ and the wave operator $u_{tt} - \Delta$. The idea is to transform (2) into an equation involving a quasi-homogeneous operator. In [5] we apply this idea to a more general equation $Lu = |u|^p$. We assume that there exists $g(x)$ such that L^*M_g has a non-zero order term; here M_g is the multiplication operator by g. In addiction, with respect to the result of [7], we consider the support of the test functions in a parallelepiped whose edges are established in relation with g and the coefficients a_α. In particular this technique enables us to establish non-existence results for $L = \partial_{tt} - a(t)\Delta + b(t)\partial_t$.

3. Sobolev-type critical exponents

Applying Theorem 2.1 to the Tricomi equation

$$u_{tt} - |t|^\lambda \Delta u = |u|^p \tag{3}$$

we find the critical exponent

$$p_F = 1 + \frac{2}{\frac{2+\lambda}{2}n - 1} = 1 + \frac{4}{(2+\lambda)n - 2} \,.$$

Indeed the operator has order $m = 2$ and it is quasi-homogeneous of type $(\frac{2+\lambda}{2}, 1)$, so that $Q = \frac{1+\lambda}{2}n + 1$. This first result for variable coefficients equations follows directly by Theorem 2.1: if $u \in L^p_{loc}$ is the solution of (3) then $u \equiv 0$.

Coming back to Section 1, we ask which exponents for this equation plays the role of p_{So} and p_{St}. In particular, we expect a dependence on λ. By using a heuristic argument, based on the homogeneity properties of our equation, we may substitute $p_{So}(Q - \delta_2 d)$ for $p_{So}(n)$ obtaining

$$p_c(\lambda, n) := \frac{n(\lambda+2)/2+2}{n(\lambda+2)/2-2} = \frac{n(\lambda+2)+4}{n(\lambda+2)-4}\,. \tag{4}$$

We may expect a smooth large data solution of

$$u_{tt} - |t - t_0|^\lambda \Delta u = -u|u|^p\,, \quad t_0 > 0,$$

for some $p \le p_c(\lambda, n)$. In particular one expects classical solutions for

$$\begin{aligned} &n = 1, && \text{when either } \lambda \le 2 \text{ or } p \le \frac{\lambda+6}{\lambda-2}; \\ &n = 2, && \text{when either } \lambda = 0 \text{ or } p \le 1 + \frac{4}{\lambda}; \\ &n = 3, && \text{for } p \le \frac{3\lambda+10}{3\lambda+4}; \\ &n = 4, && \text{for } p \le \frac{\lambda+3}{\lambda+1}. \end{aligned}$$

The known results give a positive feeling for this conjecture:

- in [8] the case $n = 3$ is considered. The author proves classical global solution when $\frac{3}{2} < p < p_c(\lambda, 3)$;
- in [13] the case $n = 3$ critical is added under radial assumption;
- in [14] the case $n = 4$ and $p < p_c(\lambda, 3)$ is reached with radial assumption.

In this direction one can see also [10], [12]. However, some problems appear in low dimensions, indeed in [9] the case $n = 1$ is sharp only for $0 \le \lambda \le 1$, while for larger λ the authors consider only $p < \frac{\lambda+3}{\lambda-1} < p_c(\lambda, 1)$. For $n = 2$ the result in [9] holds for

$$p < 1 + \frac{1}{\sqrt{2\lambda^2 + 8\lambda + 4} - 2}\,.$$

This value is strictly less then $p_c(\lambda,2) = 1 + \frac{4}{\lambda}$. To prove global existence for $n = 2$ and p close to $p_c(\lambda,2)$ is still an open problem.

The quoted results are mainly obtained by using the Liouville transformation that we introduce in the next section.

4. Strauss-type critical exponents

Given $a(t) : \mathbb{R}_+ \to \mathbb{R}_+$, we consider the Cauchy problem for the wave equation with **variable speed**

$$\begin{cases} u_{tt} - a(t)\Delta u = f(t,u), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n \\ \partial_t u(0,x) = u_1(x), & x \in \mathbb{R}^n\,. \end{cases} \tag{5}$$

Let $t_0 > 0$. We associate with $a(t)$ the function ϕ which satisfies

$$\begin{cases} \phi'(S) = a(\phi(S))^{-1/2} & S \in [0, T_0), \\ \phi(0) = 0, \end{cases}$$

with

$$T_0 = \int_0^{t_0} a(s)^{1/2}\, d\, s$$

Following [12], one sees that if u solves (5) in $[0, t_0)$, then the function

$$v(x,T) = u(x, \phi(T))$$

is a solution of the **variable damping equation**

$$(v_{TT} - \triangle v - \phi''(\phi')^{-1} v_T)(x,T) = (\phi'(T))^2 f(\phi(T), u(x, \phi(T)))$$

on $T \in [0, T_0)$. One can also remove the damping term by taking

$$w(T,x) = (\phi'(T))^{-1/2} u(\phi(T), x) = a(\phi(T))^{1/4} u(\phi(T), x)$$

defined in $[0, T_0)$. This solves the **variable mass equation**

$$(w_{TT} - \Delta w)(x,T) - \left(\frac{3}{4}\frac{(\phi''(T))^2}{(\phi'(T))^2} - \frac{1}{2}\frac{\phi'''(T)}{\phi'(T)}\right) w(x,T) = -(\phi'(T))^{3/2} f(\phi(T), u(x, \phi(T)))\,.$$

Sometimes this transformation gives the possibility to obtain quasi-homogeneous operators. For example let us consider the scale invariant damping wave equation. It is then equivalent to treating

$$\begin{aligned}
&u_{tt} - \Delta u + \frac{\mu}{1+t} u_t = |u|^p\,;\\
&v_{TT} - (1+T)^{\frac{2\mu}{1-\mu}} \Delta v = (1+T)^{\frac{2\mu}{1-\mu}} |v|^p\,;\\
&w_{TT} - \Delta w + \left(\frac{\mu}{2} - \frac{\mu^2}{4}\right) \frac{1}{(1+T)^2} w = (1+T)^{\frac{\mu}{2}(1-p)} |w|^p\,.
\end{aligned}$$

If we know blow-up or global existence for one of them, then we get the same result for the others. In particular the equation in $\tilde{v}(x,T) := v(x,T-1)$ is quasi-homogeneous of dimension $Q = \frac{n}{1-\mu} + 1$ and we may apply Theorem 2.1.

There is another property of the Liouville transformation: in some lucky cases the operator becomes an operator with constant coefficients. In particular in the last example this happens for $\mu = 2$. In [6] this trick revealed the wave-nature of the equation and for high space-dimension we gave the first example of Strauss critical exponent which appears in a damping equation. The main result of [6] can be stated as following.

Theorem 4.1. *Given $(u_0, u_1) \in \mathcal{C}^2_c(\mathbb{R}^n) \times \mathcal{C}^1_c(\mathbb{R}^n)$, we consider*

$$\begin{cases}
u_{tt} - \triangle u + \frac{2}{1+t} u_t = |u|^p, & t > 0, \quad x \in \mathbb{R}^n,\\
u(0,x) = u_0(x), & x \in \mathbb{R}^n,\\
u_t(0,x) = u_1(x), & x \in \mathbb{R}^n\,.
\end{cases} \tag{6}$$

- *If $u_1, u_0 \geq 0$, and $(u_0, u_1) \not\equiv (0,0)$. Assume that $u \in \mathcal{C}^2([0,T) \times \mathbb{R}^n)$ is a solution to the Cauchy problem (6). If $1 < p \leq \max\{p_{St}(n+2); p_F(n)\}$, then $T < \infty$.*
- *If $n = 2$ and $p > 2 = p_{St}(4) = p_F(4)$, then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, if $u_0 = \varepsilon \bar{v}_0$ and $u_1 = \varepsilon \bar{u}_1$, then the Cauchy problem (6) admits a unique global small data solution $u \in \mathcal{C}([0,\infty), H^2) \cap \mathcal{C}^1([0,\infty), H^1) \cap \mathcal{C}^2([0,\infty), L^2)$.*
- *Let $n = 3$ and $p > p_{St}(5)$. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, if $u_0 = \varepsilon \bar{u}_0$ and $u_1 = \varepsilon \bar{u}_1$, radial initial data then the Cauchy problem (6) admits a unique global small data radial solution $u \in \mathcal{C}([0,\infty) \times \mathbb{R}^3) \cap \mathcal{C}^2([0,\infty) \times (\mathbb{R}^3 \setminus \{0\}))$.*

This result has been generalized in many directions, but the main idea is that a transformation puts the equation in a new form which avoids the damping term and a competition between Fujita and Strauss exponent appears. For example, in [15] the equation

$$v_{tt} - \Delta v + \frac{\mu_1}{1+t} v_t + \frac{\mu_2}{(1+t)^2} v = |v|^p$$

becomes

$$u_{tt} - (1+t)^{2\ell} \Delta u = (\ell+1)(1+t)^k |u|^p$$

by means of the the transformation

$$\tau = (1+t)^{\ell+1} - 1\,, \quad y = (1+\ell)x\,, u = (1+\tau)^{\frac{\mu_1-1}{2}+\frac{\sqrt{\delta}}{2}} v\,.$$

where $\delta = (\mu_1 - 1)^2 - 4\mu_2 \in (0,1]$.

5. Generalizations

5.1. **Pseudo-differential equations.** Having in mind Definition 2.1 of a quasi-homogeneous operator, we can apply previous heuristic arguments to find critical exponents also to pseudo-differential operators. Few works are devoted to this kind of equations since a double difficulty appears. First, the non-existence results are strongly based on test functions methods, and the test functions have compact support, while pseudo-differential operators are not local. Secondly, for proving existence results one needs some descriptions of fundamental solutions or at least their properties, but this information is very difficult to derive for pseudo-differential operators.

5.2. **Quasilinear equations.** Previous arguments applies also to quasilinear equations, when the equation is linear in higher order derivatives but the nonlinear terms can involve lower order derivatives. For example, in [3] and [4] the following equation is considered:

$$u_{tt} + \Delta^{2\theta} u + 2\mu(-\Delta)^\theta u_t = f(u, u_t)\,.$$

The operator $L = \partial_{tt} + \Delta^{2\theta} + 2\mu(-\Delta)^\theta \partial_t$ is quasi-omogeneaus of order $m = 4\theta$ with quasi-homogeneous dimension $Q = n + 2\theta$.
If $f(u) = |u|^p$, then we conjecture that the critical Fujita exponent is given by

$$p = 1 + 4\theta/(n - 2\theta)\,.$$

Corresponding results are proved in [3]. If we take $f(u_t) = |u_t|^p$, then we put $v = \partial_t u$ and formally consider $\tilde{L}v = L(\partial_t)^{-1}v = |v|^p$ which has symbol $(\tau^2 - |\xi|^{2\theta} + 2\mu|\xi|\tau)\tau^{-1}$. Hence we are dealing with another quasi-homogeneous operator having dimension $\tilde{Q} = 2\theta + n$ but now order $m = 2\theta$. The critical exponent becomes

$$p = 1 + 2\theta/n\,.$$

The related existence/non-existence theorems are given in [3] for small data, in [4] for large data under a sign assumption on the nonlinear term.

Some critical exponents for fully non-linear equations are considered in [2].

5.3. **Changing the space of initial data.** In this paper we do not discuss another issue that deeply influences the critical exponents: the space of initial data. Here p_F, p_{So} and p_{Str} concern classical solutions or their approximations. If we want to discuss the well posedness of a nonlinear wave-type equation in a certain Sobolev-space, thenthe critical exponents may change. In particular p_{So} is related to H^1 solutions. Indeed if we put $r = 2$ and $s = 1$ we have $H^1 = H^{s,r}$ and

$$p_{So} = \frac{n+2}{n-2} \iff \frac{1}{p_{So}+1} = \frac{1}{2} - \frac{1}{n} = \frac{1}{r} - \frac{s}{n}.$$

This means that we are considering the critical exponent of $L^{p+1} \subset H^{r,s}$ given by

$$p_{So,s,r} = r - 1 + \frac{r^2 s}{N - rs}\,.$$

The choice $s = 1$ for the wave equation corresponds to the energy space. To put it simply, this choice is connected with the order of the wave operator: in the energy space $s = m/2$, and $r = 2$, that is

$$p_{So} = 1 + \frac{2m}{N - m}\,.$$

Extending this argument, the critical Sobolev exponent for the H^s solution of $L(\partial_t, D_x)u = -|u|^{p-1}u$ with quasi-homogeneous L is given by

$$p_{So} = 1 + \frac{4s}{Q - \delta_2 - 2s}. \tag{7}$$

In particular we find again (4). In a future paper, we plan to analyze

$$u_{tt} + \Delta^{2\theta}u + 2\mu(-\Delta)^{\theta}u_t = -|u|^{p-1}u,$$

and prove the global existence of energy solutions with large data for

$$p < 1 + \frac{8\theta}{N - 4\theta}\,;$$

that is in (7) we put $Q = 2\theta + n$, $\delta_2 = 2\theta$ and $2s = m = 4\theta$.

5.4. **Concluding remarks.** The previous arguments could predict a critical exponents: the idea is finding a transformation that brings a generic PDE with variable coefficients into a quasi-homogeneous equation. If we are able to find out this transformation, then we are putting the equation in a sort of "canonical form". A first example in this direction was given in [11]: in order to treat the variable coefficient equation

$$u_t - \Delta u - \frac{x}{2} \cdot \nabla u - \frac{1}{p-1} u = |u|^p\,,$$

after the transformation

$$v = e^{\frac{s}{p-1}} u(e^s - 1, e^{s/2} x)\,,$$

one can consider

$$v_s = \Delta v + |v|^p\,.$$

Clearly, it is very difficult to find a similar "canonical form". Liouville transformation or the multiplication by g as in [7] are two examples in this direction. On the other hand, this is only the first step. In the variable coefficients context it is difficult to identify even the range of local existence. More difficult is determining the range of global existence and the blow up dynamic. In particular the existence results require decay estimates which in turn depend on the fundamental solution of the operator. The broken symmetry and homogeneity due to the variable coefficients make these fundamental solutions more complicated to express and estimate.

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