ON PRINCIPAL FREQUENCIES AND INRADIUS IN CONVEX SETS

LORENZO BRASCO

To Michelino Brasco, master craftsman and father, on the occasion of his 70th birthday

Abstract. We generalize to the case of the $p-$Laplacian an old result by Hersch and Protter. Namely, we show that it is possible to estimate from below the first eigenvalue of the Dirichlet $p-$Laplacian of a convex set in terms of its inradius. We also prove a lower bound in terms of isoperimetric ratios and we briefly discuss the more general case of Poincaré-Sobolev embedding constants. Eventually, we highlight an open problem.

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1. Introduction

1.1. Overview. For every open set $\Omega \subset \mathbb{R}^N$, we consider its *principal frequency* or *first eigenvalue of the Laplacian with Dirichlet conditions*, defined by

$$\lambda(\Omega) = \inf_{u \in C_0^\infty(\Omega) \setminus \{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2 \, dx}{\displaystyle\int_\Omega |u|^2 \, dx}.$$

We recall that, whenever the completion $\mathcal{D}_0^{1,2}(\Omega)$ of $C_0^\infty(\Omega)$ with respect to the norm $\|\nabla u\|_{L^2(\Omega)}$ is compactly embedded into[1] $L^2(\Omega)$, the number $\lambda(\Omega)$ coincides with the smallest $\lambda \in \mathbb{R}$ such that the boundary value problem

$$-\Delta u = \lambda\, u, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega,$$

does admit a nontrivial solution $u \in \mathcal{D}_0^{1,2}(\Omega)$.

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Dipartimento di Matematica, Università di Bologna

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[1]For example, this happens if Ω is bounded or has finite $N-$dimensional Lebesgue measure.

For general sets, the explicit determination of $\lambda(\Omega)$ can be a challenging task. It is thus important to look for sharp estimates on $\lambda(\Omega)$ in terms of simpler quantities, typically of geometric flavour. The most celebrated instance of such an estimate is the so-called *Faber-Krahn inequality*. This asserts that $\lambda(\Omega)$ can be estimated from below by a negative power of the $N-$dimensional measure of Ω. Precisely, we have

$$\lambda(\Omega) \geq \Big(|B|^{\frac{2}{N}}\,\lambda(B)\Big)\,\frac{1}{|\Omega|^{\frac{2}{N}}}, \tag{1}$$

where B is any $N-$dimensional ball. Equality (1) is sharp in the sense that the dimensional constant $|B|^{\frac{2}{N}}\,\lambda(B)$ is attained whenever Ω is itself a ball (actually, this is the only possibility, up to sets of zero capacity).

In despite of its elegance, sharpness and simplicity, the lower bound dictated by (1) loses its interest for open sets such that

$$|\Omega| = +\infty \qquad \text{and} \qquad \lambda(\Omega) > 0.$$

This happens for example for the infinite slab $\Omega = \mathbb{R}^{N-1} \times (0, 1)$.

For such cases, it could be natural to ask whether a lower bound on $\lambda(\Omega)$ can be given in terms of the *inradius* R_Ω, i.e. the radius of the largest open ball contained in Ω. In other words, we can ask whether we can have an inequality like

$$\frac{C}{R_\Omega^2} \leq \lambda(\Omega). \tag{2}$$

The power -2 on R_Ω is imposed by scale invariance, once it is observed that $\lambda(\Omega)$ has the physical dimensions "*length to the power* -2". However, an estimate like (2) can not be true for general open sets, in dimension $N \geq 2$. Indeed, it is sufficient to consider the set

$$\Omega = \mathbb{R}^N \setminus \mathbb{Z}^N.$$

It is easy to see that $R_\Omega < +\infty$, while $\lambda(\Omega) = \lambda(\mathbb{R}^N) = 0$, since points have zero capacity in $\mathbb{R}^N$, if $N \geq 2$.

On the other hand, if we impose further geometric restrictions on the open set Ω, then it is possible to prove (2). An old result due to Hersch (see [9]) shows that for an open *convex* set $\Omega \subset \mathbb{R}^2$, it holds

$$\left(\frac{\pi}{2}\right)^2 \frac{1}{R_\Omega^2} \leq \lambda(\Omega). \tag{3}$$

The inequality is sharp and it is strict among bounded convex sets. The proof by Hersch is based on a method that he called "*évaluation par défaut*". Later on, Protter generalized this result to higher dimensions by using the same technique, see [17, page 68].

We also point out that the Hersch-Protter estimate has been recently generalized in [4, Theorem 5.1] to the *anisotropic case*, i.e. to the case of

$$\lambda_H(\Omega) = \inf_{u \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega H(\nabla u)^2\,dx}{\displaystyle\int_\Omega |u|^2\,dx},$$

where $H : \mathbb{R}^N \to [0, +\infty)$ is any norm. In this case, the definition of inradius has to be suitably adapted, in order to take into account the anisotropy H.

Remark 1.1 (More general sets I). *We have already observed that* (2) *can not be true in general. However, the planar case $N = 2$ is peculiar and well-studied: in this case, if Ω is simply connected, then it is possible to prove* (2), *but the main open issue in this case is the determination of the sharp constant C. The first result in this direction is due to Hayman* [8]. *We refer to* [1] *for a review of this kind of results.*

Actually, Osserman in [14] *showed that* (2) *still holds for planar sets with finite connectivity, the constant C depending on the connectivity k and degenerating as k goes to ∞ (this is in perfect accordance with the above example of $\mathbb{R}^2 \setminus \mathbb{Z}^2$). The result by Osserman has then been improved by Croke in* [5].

For the higher dimensional case $N \geq 3$, some results for classes of open sets more general than convex ones have been given by Hayman [8, Theorem 2] *and Taylor* [18, Theorem 3].

1.2. **The results of this paper.** We now fix an exponent $1 < p < +\infty$, then for an open set $\Omega \subset \mathbb{R}^N$, we introduce the quantity

$$\lambda_p(\Omega) = \inf_{u \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^p\,dx}{\displaystyle\int_\Omega |u|^p\,dx}.$$

As in the quadratic case $p = 2$, whenever the completion $\mathcal{D}_0^{1,p}(\Omega)$ of $C_0^\infty(\Omega)$ with respect to the norm $\|\nabla u\|_{L^p(\Omega)}$ is compactly embedded into $L^p(\Omega)$, the number $\lambda_p(\Omega)$ coincides with the smallest $\lambda \in \mathbb{R}$ such that the boundary value problem

$$-\Delta_p u = \lambda\,|u|^{p-2}\,u, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega,$$

does admit a nontrivial solution $u \in \mathcal{D}_0^{1,p}(\Omega)$. Here Δ_p is the quasilinear operator

$$\Delta_p u = \mathrm{div}\,(|\nabla u|^{p-2}\,\nabla u),$$

known as $p-$*Laplacian*. For this reason, $\lambda_p(\Omega)$ is called *first eigenvalue of the* $p-$*Laplacian with Dirichlet conditions on* Ω. In this case as well, we have the sharp lower bound

$$\lambda_p(\Omega) \ge \Big(|B|^{\frac{p}{N}}\,\lambda_p(B)\Big)\,\frac{1}{|\Omega|^{\frac{p}{N}}},$$

which generalizes (1) to $p \neq 2$. The main goal of this paper is to generalize the Hersch-Protter estimate (3) to the case of λ_p. At this aim, we introduce the one-dimensional Poincaré constant

$$\pi_p = \inf_{\varphi \in C^1([0,1])\setminus\{0\}} \left\{ \frac{\|\varphi'\|_{L^p([0,1])}}{\|\varphi\|_{L^p([0,1])}} \,:\, \varphi(0) = \varphi(1) = 0 \right\}.$$

We will prove the following

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be an open convex set. Then we have*

$$\lambda_p(\Omega) \ge \left(\frac{\pi_p}{2}\right)^p \frac{1}{R_\Omega^p}. \tag{4}$$

The estimate is sharp, equality being attained for example:

- *by an infinite slab, i.e. a set of the form*

$$\Big\{x \in \mathbb{R}^N \,:\, a < \langle x, \omega\rangle < b\Big\},$$

for some $a < b$ and $\omega \in \mathbb{S}^{N-1}$;

- *asymptotically by the family of "collapsing pyramids"*

$$C_\alpha = \mathrm{convex\,hull}\Big((-1,1)^{N-1} \cup \{(0,\dots,0,\alpha)\}\Big),$$

in the sense that

$$\lim_{\alpha\to 0^+} R^p_{C_\alpha}\,\lambda_p(C_\alpha) = \left(\frac{\pi_p}{2}\right)^p;$$

- *more generally, asymptotically by the family of infinite slabs with section given by a $k-$dimensional collapsing pyramid, i.e.*

$$\mathbb{R}^{N-k} \times C_\alpha, \qquad \text{for } N \geq 3 \text{ and } 2 \leq k \leq N-1.$$

Remark 1.2. *After the completion of this paper, we have been informed by Vladimir Bobkov that the same result is contained* [11, Theorem 2.1]*. In turn, more recently this result has been generalized in* [6, Theorem 5.1]*, to cover the anisotropic case. However, in both references the proof of the lower bound is different, as they do not use the original idea by Hersch. In* [11]*, the so-called method of interior parallels is used (see* [11, Lemmas 4.5 & 4.6]*), while* [6] *exploits a method based on maximum principles and the so-called $P-$functions. We also point out that our result contains a finer analysis of the equality cases, since in* [6, 11] *the sequence of collapsing pyramids is not identified.*

Remark 1.3 (More general sets II)**.** *For $p \neq 2$, the case of more general sets has been investigated by Poliquin in* [15]*. In* [15, Theorem 1.4.1] *it is proved that for $p > N$ and $\Omega \subset \mathbb{R}^N$ open bounded set, one has*

$$\lambda_p(\Omega) \geq \frac{C}{R^p_\Omega},$$

for a constant $C = C(N,p) > 0$. Then in [15, Theorem 1.4.2] *the same estimate is proved, for $p > N-1$ and Ω having a connected boundary. In both cases, the constant C is not explicit. In* [16, Proposition 3.5] *the same author proved such a lower bound for convex sets with an explicit constant, which is however not sharp.*

As already observed by Makai in the case $p = N = 2$ (see [13]), the estimate of Theorem 1.1 in turn implies another interesting lower bound on $\lambda_p(\Omega)$, this time in terms of the quantity

$$\frac{P(\Omega)}{|\Omega|},$$

where $P(\Omega)$ is the perimeter of Ω. The resulting estimate, which seems to be new for $N \geq 3$ and $p \neq 2$, is contained in Corollary 5.1 below.

Remark 1.4 (Upper bound). *Up to now, we never mentioned the possibility of having an upper bound of the type*

$$\lambda_p(\Omega) \leq \frac{C}{R_\Omega^p}.$$

The reason is simple: such an estimate is indeed true and very simple to obtain in a sharp form, without any assumption on the set Ω. Indeed, by definition of λ_p it is easy to see that this is a monotone decreasing quantity, with respect to set inclusion. Thus, if $\Omega \subset \mathbb{R}^N$ is an open set with $R_\Omega < +\infty$, there exists a ball $B_{R_\Omega}(\xi) \subset \Omega$ and we have

$$\lambda_p(\Omega) \leq \lambda_p(B_{R_\Omega}(\xi)).$$

If we now use the scaling properties of λ_p, the previous can be rewritten as

$$\lambda_p(\Omega) \leq \frac{\lambda_p(B_1(0))}{R_\Omega^p}.$$

Observe that this estimate is sharp, equality being (uniquely) attained by balls.

1.3. **Plan of the paper.** In Section 2 we introduce the notation used throughout the whole paper and the technical facts needed to handle the proof of Theorem 1.1. Section 3 contains a rougher version of our main result, based on Hardy's inequality for convex sets. This is a sort of *divertissement*, that we think to be interesting in its own. The proof of Theorem 1.1 is then contained in Section 4. We combine this result with a geometric estimate, to obtain a further lower bound on λ_p of geometric nature: this is Section 5, which also contains a lower bound on the *Cheeger constant*. Finally, in the last Section 6 we consider the same type of lower bound in terms of the inradius, with λ_p replaced by a general Poincaré-Sobolev sharp constant. The paper ends with an open problem.

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2. Preliminaries

2.1. **Notation.** For an open set $\Omega \subset \mathbb{R}^N$, we indicate by $|\Omega|$ its $N-$dimensional Lebesgue measure. For an open bounded set $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary, we define the *distance function*

$$d_\Omega(x) = \inf_{y\in\partial\Omega} |x-y|, \qquad x \in \Omega.$$

Then we recall that the inradius R_Ω of Ω coincides with

$$R_\Omega = \sup_{x\in\Omega} d_\Omega(x).$$

We will set $\nu_\Omega(x)$ to be the outer normal versor at $\partial\Omega$, whenever this is well-defined.

Definition 2.1. *We say that $\Omega \subset \mathbb{R}^N$ is an open polyhedral convex set if there exists a finite number of open half-spaces $\mathcal{H}_1, \dots, \mathcal{H}_k \subset \mathbb{R}^N$ such that*

$$\Omega = \bigcap_{i=1}^{k} \mathcal{H}_i \neq \emptyset.$$

If Ω is an open polyhedral convex set, we say that $F \subset \partial\Omega$ is a face of Ω if the following facts hold:

- *$F \neq \emptyset$;*
- *$F \subset \partial\mathcal{H}_i$, for some $i = 1, \dots, k$;*
- *for any $E \subset \partial\Omega \cap \partial\mathcal{H}_i$ such that $F \subset E$, we have $E = F$.*

If $\Omega \subset \mathbb{R}^N$ is an open convex set with $R_\Omega < +\infty$, we know that there exists $\xi \in \Omega$ such that $B_{R_\Omega}(\xi) \subset \Omega$. Accordingly, we define the *contact set*

$$\mathcal{C}_{\Omega,\xi} = \partial\Omega \cap \partial B_{R_\Omega}(\xi).$$

Finally, we recall the definition

$$\pi_p = \inf_{\varphi\in C^1([0,1])\setminus\{0\}} \left\{ \frac{\|\varphi'\|_{L^p((0,1))}}{\|\varphi\|_{L^p((0,1))}} \,:\, \varphi(0) = \varphi(1) = 0 \right\}.$$

It is not difficult to see that.

2.2. **A geometric lemma.** The following geometric result is one of the building blocks of the proof of Theorem 1.1. It is a higher-dimensional analogue of a simple two-dimensional fact used by Hersch in [9]. This is the same as [4, Lemmas 5.2 & 5.3], to which we refer for the proof.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded convex set. Let $\xi \in \Omega$ be such that $B_{R_\Omega}(\xi) \subset \Omega$. Then there exists $m \geq 2$ and $\{P^1, \dots, P^m\} \subset \mathcal{C}_{\Omega,\xi}$ distinct points such that the open polyhedral convex domain*

$$T = \bigcap_{i=1}^{m} \{x \in \mathbb{R}^N \,:\, \langle x - P^i, \nu_\Omega(P^i)\rangle < 0\},$$

has the following properties:

- $\Omega \subset T$*;*
- $R_T = R_\Omega$*;*
- *every face of T touches $\partial B_{R_\Omega}(\xi)$.*

Remark 2.1. *The previous result is similar to an analogous geometric lemma contained in Protter's paper, see* [17, page 68]*. Such a result in* [17] *is credited to a private communication by David Gale, without giving a proof. It should be noticed that the statement in* [17] *is slightly more precise, since it is said that m can be chosen to be smaller than or equal to $N + 1$. However, in the statement contained* [17] *the crucial feature that all the faces of T touches the internal ball $B_{R_\Omega}(\xi)$ seems to have been accidentally omitted. For this reason we prefer to refer to the result proved in* [4]*.*

2.3. **Eigenvalues of special sets.**

Lemma 2.2 (Product sets)**.** *Let $1 < p < +\infty$ and $k \in \{1, \dots, N-1\}$. We take the open set $\Omega = \mathbb{R}^{N-k} \times \omega$, with $\omega \subset \mathbb{R}^k$ open bounded set. Then we have*

$$\lambda_p(\Omega) = \lambda_p(\omega).$$

Proof. The proof is standard, we include it for completeness.

We use the notation $(x, y) \in \mathbb{R}^{N-k} \times \mathbb{R}^k$, for a point in $\mathbb{R}^N$. We first prove that

$$\lambda_p(\Omega) \leq \lambda_p(\omega). \tag{5}$$

For every $\varepsilon > 0$, we take $u_\varepsilon \in C^\infty_0(\omega)$ to be an almost optimal function for the problem on ω, i.e.

$$\int_\omega |\nabla_y u_\varepsilon|^p\, dy < \lambda_p(\omega) + \varepsilon \qquad \text{and} \qquad \int_\omega |u_\varepsilon|^p\, dy = 1.$$

We take $\eta \in C^\infty_0(\mathbb{R})$ such that

$$0 \le \eta \le 1, \qquad \eta \equiv 1 \text{ on } \left[-\frac{1}{2}, \frac{1}{2}\right], \qquad \eta \equiv 0 \text{ on } \mathbb{R} \setminus [-1, 1],$$

then for every $R > 0$, we choose

$$\varphi(x, y) = \eta_R(|x|)\, u_\varepsilon(y), \qquad \text{where } \eta_R(t) = R^{\frac{k-N}{p}}\, \eta\left(\frac{t}{R}\right).$$

By using Fubini's Theorem, we obtain

$$\lambda_p(\Omega) \le \frac{\displaystyle\int_{B_R(0)} \int_\omega \left(|\nabla_x \eta_R\,(|x|)|^2\ |u_\varepsilon(x_N)|^2 + |\nabla_y u_\varepsilon(y)|^2\, \eta_R(|x|)^2\right)^{\frac{p}{2}}\ dx\, dy}{\displaystyle\int_{B_R(0)} \eta_R(|x|)^p\, dx},$$

where $B_R(0) = \{x \in \mathbb{R}^{N-k} : |x| < R\}$. We now use the definition of η_R and the change of variables $x = R\, x'$, so to get

$$\lambda_p(\Omega) \le \frac{\displaystyle\int_{B_1(0)} \int_\omega \left[R^{\frac{2}{p}\,(k-N)-2}\ |\eta'(|x'|)|^2\ |u_\varepsilon(y)|^2 + R^{\frac{2}{p}\,(k-N)}\, |\nabla_y u_\varepsilon(y)|^2\, |\eta(|x'|)|^2\right]^{\frac{p}{2}} R^{N-k}\, dx'\, dy}{\displaystyle\int_{B_1(0)} \eta(|x'|)^p\, dx'}$$

$$= \frac{\displaystyle\int_{B_1(0)} \int_0^1 \left[\frac{1}{R^2}\ |\eta'(|x'|)|^2\ |u_\varepsilon(y)|^2 + |\nabla_y u_\varepsilon(y)|^2\, |\eta(|x'|)|^2\right]^{\frac{p}{2}}\ dx'\, dy}{\displaystyle\int_{B_1(0)} \eta(|x'|)^p\, dx'}.$$

By taking the limit as R goes to $+\infty$ and using the Dominated Convergence Theorem, from the previous estimate we get

$$\lambda_p(\Omega) \le \frac{\displaystyle\int_{B_1(0)} \int_\omega |\nabla_y u_\varepsilon(y)|^p\, |\eta(|x'|)|^p\, dx'\, dy}{\displaystyle\int_{B_1(0)} \eta(|x'|)^p\, dx'} = \int_0^1 |\nabla_y u_\varepsilon|^p\, dy < \lambda_p(\omega) + \varepsilon.$$

The arbitrariness of $\varepsilon > 0$ implies (5).

We now prove the reverse inequality

$$\lambda_p(\Omega) \geq \lambda_p(\omega). \tag{6}$$

For every $\varepsilon > 0$, we take $\varphi_\varepsilon \in C_0^\infty(\Omega) \setminus \{0\}$ such that

$$\frac{\displaystyle\int_\Omega |\nabla \varphi_\varepsilon|^p \, dx \, dy}{\displaystyle\int_\Omega |\varphi_\varepsilon|^p \, dx \, dy} < \lambda_p(\Omega) + \varepsilon.$$

Observe that

$$\begin{aligned}\int_\Omega |\nabla \varphi_\varepsilon|^p \, dx \, dy &\geq \int_{\mathbb{R}^{N-k}} \left(\int_\omega |\nabla_y \varphi_\varepsilon|^p \, dy \right) dx \\ &\geq \lambda_p(\omega) \int_{\mathbb{R}^{N-k}} \left(\int_\omega |\varphi_\varepsilon|^p \, dy \right) dx = \lambda_p(\omega) \int_\Omega |\varphi_\varepsilon|^p \, dx \, dy,\end{aligned}$$

where we used that $y \mapsto \varphi_\varepsilon(x, y)$ is admissible for the one-dimensional problem, for every x. We thus obtained

$$\lambda_p(\omega) \leq \lambda_p(\Omega) + \varepsilon.$$

The arbitrariness of $\varepsilon > 0$ implies (6). □

The following technical result is the core of the proof of Theorem 1.1. It enables to estimate from below an eigenvalue with mixed boundary conditions, when the set is a "pyramid-like" one. We have to pay attention to possibly unbounded sets. In what follows $W^{1,p}(\Omega)$ is the usual Sobolev space of $L^p(\Omega)$ functions, having their distributional gradient in $L^p(\Omega)$, as well.

Lemma 2.3. *Let $\Sigma \subset \mathbb{R}^{N-1}$ be an open polyhedral convex set. Let $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$ be a point whose projection on $\mathbb{R}^{N-1}$ belongs to Σ and such that $\xi_N > 0$. We consider the $N-$dimensional polyhedral convex set*

$$T = \operatorname{convex\,hull}\big(\Sigma \cup \{\xi\}\big),$$

and define

$$\mu(T) = \inf_{u \in C^1(\overline{T}) \cap W^{1,p}(T) \setminus \{0\}} \left\{ \frac{\displaystyle\int_T |\nabla u|^p \, dx}{\displaystyle\int_T |u|^p \, dx} \, : \, u = 0 \text{ on } \Sigma \right\}.$$

Then we have

$$\mu(T) \geq \left(\frac{\pi_p}{2}\right)^p \frac{1}{(\xi_N)^p}.$$

Proof. By recalling the definition of π_p, we have that for $a > 0$ and for every $\varphi \in C^1([0, a])$ such that $\varphi(0) = 0$ it holds

$$\int_0^a |\varphi'(t)|^p \geq \left(\frac{\pi_p}{2}\right)^p \frac{1}{a^p} \int_0^a |\varphi(t)|^p \, dt, \tag{7}$$

see [2, Lemma A.1]. We now take a function $u \in C^1(\overline{T}) \cap W^{1,p}(T)$ which is admissible for the problem defining $\mu(T)$. By hypothesis, there exists an affine function $\Psi : \Sigma \to [0, \xi_N]$ such that

$$T = \left\{(x', x_N) \,:\, \mathbb{R}^{N-1} \times \mathbb{R} \,:\, x' \in \Sigma,\, 0 < x_N < \Psi(x')\right\}.$$

Thus by Fubini's Theorem and (7) we have

$$\begin{aligned}
\int_T |\nabla u|^p \, dx \geq \int_T |u_{x_N}|^p \, dx &= \int_\Sigma \left(\int_0^{\Psi(x')} |u_{x_N}|^p \, dx_N\right) dx' \\
&\geq \int_\Sigma \left(\left(\frac{\pi_p}{2}\right)^p \frac{1}{\Psi(x')^p} \int_0^{\Psi(x')} |u|^p \, dx_N\right) dx' \\
&\geq \left(\frac{\pi_p}{2}\right)^p \frac{1}{\xi_N^p} \int_\Sigma \left(\int_0^{\Psi(x')} |u|^p \, dx_N\right) dx' = \left(\frac{\pi_p}{2}\right)^p \frac{1}{\xi_N^p} \int_T |u|^p \, dx.
\end{aligned}$$

By taking the infimum over admissible functions u, we get the desired conclusion. □

3. A *divertissement* on Hardy's inequality

Before proving the sharp estimate *à la* Hersch-Protter (4), we present a rougher estimate. This is a consequence of *Hardy's inequality for convex sets*. Even if the resulting estimate is not sharp, we believe that the proof has its own interest and we reproduce it for the reader's convenience.

Proposition 3.1. *Let $1 < p < +\infty$ and let $\Omega \subset \mathbb{R}^N$ be an open bounded convex set. Then we have*

$$\left(\frac{p-1}{p}\right)^p \frac{1}{R_\Omega^p} \leq \lambda_p(\Omega).$$

Proof. We recall that the following Hardy's inequality holds for a convex set

$$\left(\frac{p-1}{p}\right)^p \int_\Omega \left|\frac{u}{d_\Omega}\right|^p dx \le \int_\Omega |\nabla u|^p\, dx, \qquad \text{for every } u \in C_0^\infty(\Omega). \tag{8}$$

By using this inequality, it is easy to obtain the claimed estimate. Indeed, by recalling that

$$R_\Omega = \|d_\Omega\|_{L^\infty(\Omega)},$$

from (8) we get

$$\left(\frac{p-1}{p}\right)^p \frac{1}{R_\Omega^p} \int_\Omega |u|^p\, dx \le \int_\Omega |\nabla u|^p\, dx.$$

By taking the infimum over admissible test functions, we finally obtain the lower bound on $\lambda_p(\Omega)$.

For completeness, we now recall how to prove (8). Let us consider the distance function

$$d_\Omega(x) = \min_{y\in\partial\Omega} |x-y|, \qquad x \in \Omega.$$

This is a $1-$Lipschitz function, which is concave on Ω, due to the convexity of Ω. This implies that d_Ω is weakly superharmonic on Ω, i.e.

$$\int_\Omega \langle \nabla d_\Omega, \nabla\varphi\rangle\, dx \ge 0,$$

for every nonnegative $\varphi \in C_0^\infty(\Omega)$. By observing that

$$|\nabla d_\Omega| = 1, \qquad \text{almost everywhere in } \Omega, \tag{9}$$

from the previous inequality we also get

$$\int_\Omega \langle |\nabla d_\Omega|^{p-2}\, \nabla d_\Omega, \nabla\varphi\rangle\, dx \ge 0, \tag{10}$$

for every nonnegative $\varphi \in C_0^\infty(\Omega)$, i.e. d_Ω is weakly $p-$superharmonic as well. By a standard density argument, we easily see that we can enlarge the class of test functions up to nonnegative $\varphi \in W_0^{1,p}(\Omega)$, i.e. the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$.

We now insert in (10) the test function

$$\varphi = \frac{|u|^p}{(d_\Omega+\varepsilon)^{p-1}},$$

where $u \in C_0^\infty(\Omega)$ and $\varepsilon > 0$. We thus obtain

$$0 \le -(p-1)\int_\Omega \left|\frac{\nabla d_\Omega}{d_\Omega+\varepsilon}\right|^p |u|^p\,dx + p\int_\Omega \left\langle \frac{|\nabla d_\Omega|^{p-2}\,\nabla d_\Omega}{(d_\Omega+\varepsilon)^{p-1}}, \nabla u\right\rangle |u|^{p-2}\,u\,dx.$$

that is

$$\int_\Omega \left|\frac{\nabla d_\Omega}{d_\Omega+\varepsilon}\right|^p |u|^p\,dx \le \frac{p}{p-1}\int_\Omega \left|\left\langle \frac{|\nabla d_\Omega|^{p-2}\,\nabla d_\Omega}{(d_\Omega+\varepsilon)^{p-1}}, \nabla u\right\rangle\right| |u|^{p-1}\,dx.$$

We can now use Young's inequality in the following form

$$|\langle a,b\rangle| \le \delta\,\frac{p-1}{p}\,|a|^{\frac{p}{p-1}} + \frac{\delta^{1-p}}{p}\,|b|^p, \qquad \text{for } a,b\in\mathbb{R}^N,\ \delta>0.$$

This yields

$$\int_\Omega \left|\frac{\nabla d_\Omega}{d_\Omega+\varepsilon}\right|^p |u|^p\,dx \le \delta\int_\Omega \left|\frac{\nabla d_\Omega}{d_\Omega+\varepsilon}\right|^p |u|^p\,dx + \frac{\delta^{1-p}}{p-1}\int_\Omega |\nabla u|^p\,dx,$$

which can be recast into

$$(p-1)\,\delta^{p-1}\,(1-\delta)\int_\Omega \left|\frac{\nabla d_\Omega}{d_\Omega+\varepsilon}\right|^p |u|^p\,dx \le \int_\Omega |\nabla u|^p\,dx.$$

Finally, we observe that the quantity $\delta^{p-1}\,(1-\delta)$ is maximal for

$$\delta = \frac{p-1}{p},$$

thus by taking the limit as ε goes to 0 and recalling (9), by Fatou's Lemma we end up with (8), as desired. □

Remark 3.1. *We observe that the boundedness of Ω can be dropped, both in* (8) *and in the lower bound on $\lambda_p(\Omega)$. We also point out that, even if the constant*

$$\left(\frac{p-1}{p}\right)^p,$$

is not sharp, it only depends on p, just like the sharp one.

4. Proof of Theorem 1.1

We start with a particular case of Theorem 1.1, when the convex set is *polyhedral.* Its proof heavily relies on Lemma 2.3.

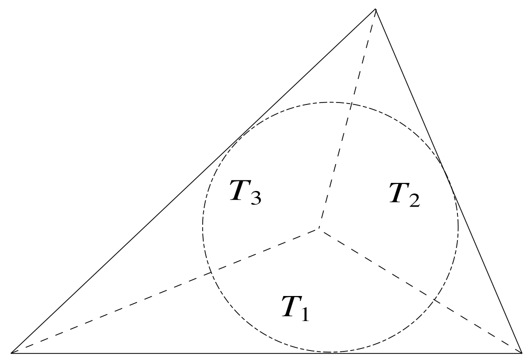

FIGURE 1. The construction for the proof of Proposition 4.1, when $N = 2$ and T has $j = 3$ faces.

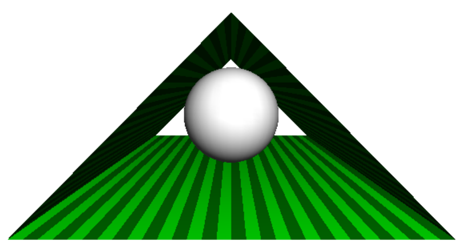

FIGURE 2. The construction for the proof of Proposition 4.1, when $N = 3$ and T is an unbounded set with $j = 3$ faces. In this case, the subsets T_1, T_2, T_3 (not drawn in the picture) are unbounded, as well.

Proposition 4.1. *Let $1 < p < +\infty$ and let $T \subset \mathbb{R}^N$ be an open polyhedral convex set. We suppose that $R_T < +\infty$ and we assume further that there exists a ball $B \subset T$ with radius R_T and such that each face of T touches B. Then we have*

$$\lambda_p(T) \geq \left(\frac{\pi_p}{2}\right)^p \frac{1}{R_T^p}.$$

Proof. Let us indicate by $F_1, \dots, F_j \subset \partial T$ the faces of T. We take the center ξ of B and then define

$$T_i = \operatorname{convex\,hull}\big(F_i \cup \{\xi\}\big), \qquad i = 1, \dots, j,$$

see Figures 1 and 2. We now consider T_i for a fixed $i = 1, \dots, j$ and estimate from below

$$\mu_i = \inf_{u\in C^1(\overline{T_i})\cap W^{1,p}(T_i)\setminus\{0\}} \left\{ \frac{\displaystyle\int_{T_i} |\nabla u|^p\,dx}{\displaystyle\int_{T_i} |u|^p\,dx} \,:\, u = 0 \text{ on } F_i \right\}.$$

Up to a rigid motion, we can assume that T_i satisfies the assumptions of Lemma 2.3. Observe that in this case, we have

$$\xi_N = R_T,$$

by construction. Thus Lemma 2.3 entails

$$\mu_i \geq \left(\frac{\pi_p}{2}\right)^p \frac{1}{(\xi_N)^p} = \left(\frac{\pi_p}{2}\right)^p \frac{1}{R_T^p}. \tag{11}$$

On the other hand, for every $\varepsilon > 0$, we take $\varphi_\varepsilon \in C_0^\infty(T) \setminus \{0\}$ such that

$$\lambda_p(T) \leq \frac{\displaystyle\int_T |\nabla\varphi_\varepsilon|^p\,dx}{\displaystyle\int_T |\varphi_\varepsilon|^p\,dx} \leq \lambda_p(T) + \varepsilon.$$

We observe that the restriction of φ_ε to each T_i is admissible for the problem defining μ_i. Then, we obtain

$$\lambda_p(T) + \varepsilon \geq \frac{\displaystyle\int_T |\nabla\varphi_\varepsilon|^p\,dx}{\displaystyle\int_T |\varphi_\varepsilon|^p\,dx} = \frac{\displaystyle\sum_{i=1}^j \int_{T_i} |\nabla\varphi_\varepsilon|^p\,dx}{\displaystyle\sum_{i=1}^j \int_{T_i} |\varphi_\varepsilon|^p\,dx} \geq \frac{\displaystyle\sum_{i=1}^j \mu_i \int_{T_i} |\varphi_\varepsilon|^p\,dx}{\displaystyle\sum_{i=1}^j \int_{T} |\varphi_\varepsilon|^p\,dx} \geq \min_{i=1,\dots,j} \mu_i.$$

By recalling the lower bound (11), we get the the desired conclusion, thanks to the arbitrariness of $\varepsilon > 0$. □

We eventually come to the proof of Theorem 1.1.

Proof of Theorem 1.1. We first prove the inequality and then analyze the equality cases.

Part 1: proof of the inequality. Let us first assume that Ω is bounded. By appealing to Lemma 2.1, we know that there exists $T \subset \mathbb{R}^N$ an open polyhedral convex set such that

$$\Omega \subset T \qquad \text{and} \qquad R_\Omega = R_T.$$

Moreover, each face of T touches a maximal ball $B_{R_\Omega}(\xi)$. By applying Proposition 4.1 to the set T, we get

$$\lambda_p(\Omega) \geq \lambda_p(T) \geq \left(\frac{\pi_p}{2}\right)^p \frac{1}{R_T^p} = \left(\frac{\pi_p}{2}\right)^p \frac{1}{R_\Omega^p}.$$

This concludes the proof, in the case Ω is bounded.

If Ω in unbounded, we can suppose that $R_\Omega < +\infty$, otherwise there is nothing to prove. Then we can consider the bounded set $\Omega_R = \Omega \cap B_R(0)$ for R large enough. By applying

$$\lambda_p(\Omega_R) \geq \left(\frac{\pi_p}{2}\right)^p \frac{1}{R_{\Omega_R}^p},$$

and taking on both sides the limit as R goes to $+\infty$, we get the conclusion.

Part 2: sharpness of the inequality. It is easy to see that equality is attained on a slab. Indeed, by Lemma 2.2 we have

$$\lambda_p(\mathbb{R}^{N-1} \times (0,1)) = \lambda_p((0,1)) = \Big(\pi_p\Big)^p \qquad \text{and} \qquad R_{\mathbb{R}^{N-1}\times(0,1)} = \frac{1}{2}.$$

As for the "collapsing pyramids"

$$C_\alpha = \text{convex hull}\Big((-1,1)^{N-1} \cup \{(0,\dots,0,\alpha)\}\Big),$$

we are going to use a purely variational argument, thus we do not need the explicit determination of λ_p for these sets. We first observe that

$$C_\alpha \subset \mathbb{R}^{N-1} \times (0,\alpha),$$

thus we have

$$\lambda_p(C_\alpha) \geq \lambda_p(\mathbb{R}^{N-1} \times (0,\alpha)) = \left(\frac{\pi_p}{\alpha}\right)^p.$$

In order to prove the reverse estimate, we observe that for $0 < \alpha < 1$

$$Q_\alpha := \Big(-(1-\sqrt{\alpha}), 1-\sqrt{\alpha}\Big)^{N-1} \times \Big(0, \alpha\,(1-\sqrt{\alpha})\Big) \subset C_\alpha,$$

thus by monotonicity and scaling

$$\lambda_p(C_\alpha) \leq \lambda_p(Q_\alpha) = \Big(\alpha\,(1-\sqrt{\alpha})\Big)^{-p} \lambda_p\left(\left(-\frac{1}{\alpha}, \frac{1}{\alpha}\right)^{N-1} \times (0,1)\right).$$

By observing that

$$\lim_{\alpha\to 0^+} \lambda_p\left(\left(-\frac{1}{\alpha}, \frac{1}{\alpha}\right)^{N-1} \times (0,1)\right) = \lambda_p(\mathbb{R}^{N-1} \times (0,1)) = \Big(\pi_p\Big)^p,$$

we thus get that

$$\lambda_p(Q_\alpha) \sim \left(\frac{\pi_p}{\alpha}\right)^p, \qquad \text{for } \alpha \to 0^+.$$

In conclusion, we obtained that

$$\lim_{\alpha\to 0^+} \alpha^p\, \lambda_p(C_\alpha) = \Big(\pi_p\Big)^p.$$

We are left with observing that

$$R_{C_\alpha} = \frac{\alpha}{1+\sqrt{1+\alpha^2}} \sim \frac{\alpha}{2}, \qquad \text{for } \alpha \to 0^+.$$

This concludes the proof of the optimality of the sequence $\{C_\alpha\}_\alpha$.

Finally, we observe that for the sets

$$\mathbb{R}^{N-k} \times C_\alpha, \qquad \text{for } N \geq 3 \text{ and } 2 \leq k \leq N-1,$$

it is sufficient to use the computations above and the fact that by Lemma 2.2

$$\lambda_p(\mathbb{R}^{N-k} \times C_\alpha) = \lambda_p(C_\alpha),$$

together with $R_{\mathbb{R}^{N-k}\times C_\alpha} = R_{C_\alpha}$. □

Remark 4.1. *By comparing the sharp estimate* (3) *with the estimate of Proposition 3.1, we get*

$$\frac{\pi_p}{2} > \frac{p-1}{p}.$$

By recalling (**??**), *we have that both sides converge to* 1*, as* p *goes to* $+\infty$*. This shows that even if the estimate of Proposition 3.1 is not sharp for every finite* p*, it is "asymptotically" optimal for* $p \to +\infty$*.*

5. A further lower bound

It what follows, we will use the notation $P(\Omega)$ to denote the distributional perimeter of a set $\Omega \subset \mathbb{R}^N$. On convex sets, this coincides with the $(N-1)-$dimensional Hausdorff measure of the boundary.

We recall that for bounded convex sets, it is possible to bound $\lambda_p(\Omega)$ *from above* in terms of the isoperimetric–type ratio

$$\frac{P(\Omega)}{|\Omega|}.$$

Namely, we have

$$\lambda_p(\Omega) < \left(\frac{\pi_p}{2}\right)^p \left(\frac{P(\Omega)}{|\Omega|}\right)^p,$$

see [2, Main Theorem] and [7, Theorem 4.1]. The inequality is strict and the estimate is sharp.

As a straightforward consequence of Theorem 1.1, we get that the previous estimate can be reverted. Thus

$$\lambda_p(\Omega) \qquad \text{and} \qquad \left(\frac{P(\Omega)}{|\Omega|}\right)^p,$$

are equivalent quantities on open bounded convex sets. For $N = p = 2$, this result is due to Makai, see [13]. For all the other cases, to the best of our knowledge it is new.

Corollary 5.1. *Let $1 < p < +\infty$ and let $\Omega \subset \mathbb{R}^N$ be an open bounded convex set. Then we have*

$$\lambda_p(\Omega) \geq \left(\frac{\pi_p}{2\,N}\right)^p \left(\frac{P(\Omega)}{|\Omega|}\right)^p. \tag{12}$$

The inequality is sharp, equality being attained asymptotically by the sequence of "collapsing pyramids" of Theorem 1.1.

Proof. In order to prove (12), it is sufficient to recall that for an open bounded convex set, we have the sharp estimate (see for example [2, Lemma B.1])

$$\frac{R_\Omega}{N} \leq \frac{|\Omega|}{P(\Omega)}. \tag{13}$$

By inserting this in (4), we get the claimed estimate.

We now come to the sharpness issue. Observe that (12) has been obtained by joining the two inequalities (4) and (13). We already know that the family of "collapsing pyramids" is asymptotically optimal for the first one, thus we only need to verify that the same family is asymptotically optimal for (13), as well. Let us set as before

$$C_\alpha = \text{convex hull}\left(\Big(-1,1\Big)^{N-1} \cup \{(0,,\dots,0,\alpha)\}\right).$$

We recall that

$$R_{C_\alpha} \sim \frac{\alpha}{2},$$

while

$$|C_\alpha| = 2^{N-1} \int_0^\alpha \left(1 - \frac{z}{\alpha}\right)^{N-1} dz = \frac{\alpha\, 2^{N-1}}{N},$$

and

$$P(C_\alpha) \sim 2 \left| \Big(-1, 1 \Big)^{N-1} \right| = 2^N.$$

Thus we get

$$\frac{|C_\alpha|}{P(C_\alpha)} \sim \frac{\alpha}{2\,N} \sim \frac{R_{C_\alpha}}{N}, \qquad \text{for } \alpha \to 0,$$

as desired. □

We recall the definition of *Cheeger constant* of an open bounded set $\Omega \subset \mathbb{R}^N$, i.e.

$$h_1(\Omega) = \inf_{E \subset \Omega} \left\{ \frac{P(E)}{|E|} \,:\, |E| > 0 \right\}.$$

Observe that if $P(\Omega) < +\infty$, then Ω itself is admissible in the previous variational problem. Thus we have the trivial estimate

$$\frac{P(\Omega)}{|\Omega|} \geq h_1(\Omega).$$

For convex sets, this estimate can be reverted. Indeed, by recalling that (see [12, Corollary 6])

$$\lim_{p \searrow 1} \lambda_p(\Omega) = h_1(\Omega) \qquad \text{and} \qquad \lim_{p \searrow 1} \pi_p = \pi_1 = 2,$$

if we take the limit as p goes to 1 in (12), we get the following

Corollary 5.2. *Let $1 < p < +\infty$ and let $\Omega \subset \mathbb{R}^N$ be an open bounded convex set. Then we have*

$$h_1(\Omega) \geq \frac{1}{N} \frac{P(\Omega)}{|\Omega|}.$$

Remark 5.1 (The case $p = +\infty$)**.** *The limit as p goes to $+\infty$ of* (12) *is less interesting. Indeed, by taking the $p-$th root on both sides and recalling that (see* [10, Lemma 1.5]*)*

$$\lim_{p \to +\infty} \Big(\lambda_p(\Omega) \Big)^{\frac{1}{p}} = \frac{1}{R_\Omega},$$

from (12) *we get again* (13)*.*

6. More general principal frequencies

By appealing to its variational characterization, the first eigenvalue $\lambda_p(\Omega)$ is nothing but the sharp constant for the Poincaré inequality

$$C_\Omega \int_\Omega |u|^p\,dx \le \int_\Omega |\nabla u|^p\,dx, \qquad \text{for every } u \in C_0^\infty(\Omega).$$

From a theoretical point of view, it is thus quite natural to consider more generally the "principal frequencies"

$$\lambda_{p,q}(\Omega) = \inf_{u\in C_0^\infty(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^p\,dx}{\displaystyle\left(\int_\Omega |u|^q\,dx\right)^{\frac{p}{q}}}, \qquad \text{for } q \neq p.$$

Of course, such a quantity is interesting only if q is such that

$$\begin{cases} 1 \le q < p^*, & \text{if } p \le N, \\ 1 \le q \le +\infty, & \text{if } p > N, \end{cases} \qquad \text{where } p^* = \frac{N\,p}{N-p}.$$

For $p < N$ and $q = p^*$, the quantity $\lambda_{p,q}(\Omega)$ does not depend on Ω and is a universal constant, coinciding with the sharp constant in the Sobolev inequality

$$C\left(\int_{\mathbb{R}^N} |u|^{p^*}\,dx\right)^{\frac{p}{p^*}} \le \int_{\mathbb{R}^N} |\nabla u|^p\,dx, \qquad \text{for every } u \in C_0^\infty(\mathbb{R}^N).$$

In this section, we briefly investigate the possibility to have a lower bound of the type

$$\frac{C}{R_\Omega^\beta} \le \lambda_{p,q}(\Omega),$$

among convex sets, in this case as well. Observe that by scale invariance, the only possibility for the exponent β is

$$\beta = -N + p + N\,\frac{p}{q}.$$

In the case $q < p$, such an estimate is not possible, as shown in the following

Proposition 6.1 (Sub-homogeneous case)**.** *Let $1 < p < +\infty$ and $1 \le q < p$. Then*

$$\inf\left\{R_\Omega^{N\,\frac{p}{q}-N+p}\,\lambda_{p,q}(\Omega)\,:\,\Omega\subset\mathbb{R}^N \text{open bounded convex set}\right\} = 0.$$

Proof. By scale invariance, we can impose the further restriction that $R_\Omega = 1$. We recall that for $q < p$ we have

$$\lambda_{p,q}(\Omega) > 0 \qquad \Longleftrightarrow \qquad \text{the embedding } \mathcal{D}^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega) \text{ is compact,}$$

see [3, Theorem 1.2]. We now observe that for the open convex set $\Omega = \mathbb{R}^{N-1} \times (-1, 1)$ the embedding above can not be compact, due to the translation invariance of the set Ω in the first $N-1$ coordinate directions. Thus we get

$$\lambda_{p,q}(\mathbb{R}^{N-1} \times (-1, 1)) = 0.$$

By taking the sequence

$$\Omega_L = \left(-\frac{L}{2}, \frac{L}{2}\right)^{N-1} \times (-1, 1), \qquad L > 0,$$

and using that

$$\lim_{L\to+\infty} \lambda_{p,q}(\Omega_L) = \lambda_{p,q}(\mathbb{R}^{N-1} \times (-1, 1)),$$

we get the desired conclusion. □

Before analyzing the case $q > p$, we notice that for the case $q < p$, it is possible to have a lower bound on $\lambda_{p,q}$ in terms of an integral norm of the distance from the boundary. In a sense, this is the natural counterpart of the Hersch-Protter estimate.

Proposition 6.2. *Let $1 < p < +\infty$ and $1 \le q < p$. Then for every $\Omega \subset \mathbb{R}^N$ open bounded convex set, we have*

$$\lambda_{p,q}(\Omega) \ge \left(\frac{p-1}{p}\right)^p \frac{1}{\left(\displaystyle\int_\Omega d_\Omega^{\frac{p\,q}{p-q}}\, dx\right)^{\frac{p-q}{q}}}. \tag{14}$$

Proof. We observe that by Hölder inequality, for every $u \in C^\infty_0(\Omega)$ we have

$$\int_\Omega |u|^q\, dx \le \left(\int_\Omega \left|\frac{u}{d_\Omega}\right|^p dx\right)^{\frac{q}{p}} \left(\int_\Omega d_\Omega^{\frac{p\,q}{p-q}}\, dx\right)^{\frac{p-q}{p}}.$$

By taking the power p/q on both sides and using Hardy's inequality (8), we get

$$\left(\int_\Omega |u|^q\, dx\right)^{\frac{p}{q}} \le \left(\frac{p}{p-1}\right)^p \int_\Omega |\nabla u|^p\, dx \left(\int_\Omega d_\Omega^{\frac{p\,q}{p-q}}\, dx\right)^{\frac{p-q}{q}}.$$

By taking the infimum over $u \in C^\infty_0(\Omega)$, we get the desired estimate. □

On the contrary, for $q > p$ it is possible to have a lower bound on $\lambda_{p,q}$ in terms of the inradius.

Proposition 6.3 (Super-homogeneous case)**.** *Let $1 < p < \infty$ and $q > p$ such that*

$$\begin{cases} q < p^*, & \text{if } p \le N, \\ q \le +\infty, & \text{if } p > N. \end{cases}$$

Then there exists a constant $C = C(N, p, q) > 0$ such that for every $\Omega \subset \mathbb{R}^N$ open convex set, we have

$$\lambda_{p,q}(\Omega) \ge \frac{C}{R_\Omega^{N\,\frac{p}{q} - N + p}}. \tag{15}$$

Proof. By using the classical Gagliardo-Nirenberg inequalities, we have for every $u \in C_0^\infty(\Omega)$

$$\left(\int_\Omega |u|^q \, dx\right)^{\frac{p}{q}} \le C \left(\int_\Omega |u|^p \, dx\right)^{\vartheta} \left(\int_\Omega |\nabla u|^p \, dx\right)^{1-\vartheta}, \tag{16}$$

where $C = C(N, p, q) > 0$ and

$$\vartheta = \frac{N}{q} - \frac{N}{p} + 1.$$

For every $\varepsilon > 0$, we take $\varphi \in C_0^\infty(\Omega)$ such that

$$\lambda_{p,q}(\Omega) + \varepsilon > \frac{\displaystyle\int_\Omega |\nabla \varphi|^p \, dx}{\left(\displaystyle\int_\Omega |\varphi|^q \, dx\right)^{\frac{p}{q}}}.$$

By using (16) to estimate the denominator, we end up with

$$\lambda_{p,q}(\Omega) + \varepsilon > \left(\frac{\displaystyle\int_\Omega |\nabla \varphi|^p \, dx}{\displaystyle\int_\Omega |\varphi|^p \, dx}\right)^{\vartheta} \ge \Big(\lambda_p(\Omega)\Big)^{\vartheta}.$$

If we now use Theorem 1.1 and recall the definition of ϑ, we get the desired conclusion. □

The previous proofs very likely do not produce the sharp constants in (14) and (15). Moreover, in the case $q > p$ the Hersch's argument used for the case $p = q$ does not seem to work. Thus, we leave an open problem, which is quite interesting in our opinion.

Open problems 1. *Find the sharp constants in estimates* (14) *and* (15)*, among open bounded convex sets.*

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Dipartimento di Matematica e Informatica
Università degli Studi di Ferrara
Via Machiavelli 35, 44121 Ferrara, Italy
E-mail address: lorenzo.brasco@unife.it