IDENTIFICATION FOR DEGENERATE DIFFERENTIAL PROBLEMS OF HYPERBOLIC TYPE
IDENTIFICAZIONE PER PROBLEMI DIFFERENZIALI DEGENERI DI TIPO IPERBOLICO

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ABSTRACT. A degenerate identification problem in Hilbert space is considered. An application to second order evolution equations of hyperbolic type is also provided. The abstract results are applied to concrete differential problems.

SUNTO. Un problema degenere di identificazione in uno spazio di Hilbert viene descritto. Una applicazione ad equazioni di evoluzione del secondo ordine é anche fornita. Tutti i risultati astratti sono applicati a problemi differenziali concreti.
KEYWORDS. Degenerate differential equations in Hilbert spaces. Inverse problems. Evolution equations of hyperbolic type.

At the beginning of this seminar I would like to remember our dear Professor Bruno Pini, who chaired these seminars for many years and devoted his life to the purpose of mathematical analysis.

1. Introduction

Various papers and monographs have treated inverse problems in Hilbert spaces related to degenerate differential equations.

Taking previous key results concerning existence and uniqueness of solutions to degenerate differential equations described in the monograph [5] by Favini and Yagi, some papers investigated the inverse problem in a Hilbert space Y, which is that of identifying the solution $(y, f) \in C([0,\tau]; D(L)) \times C([0,\tau];\mathbb{C})$ such that

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$$
\begin{aligned}
&(1) \qquad M^*\tfrac{d}{dt}(My(t)) &&= Ly(t) + f(t)z, \quad 0 \le t \le \tau,\\
&(2) \qquad (My)(0) &&= My_0,\\
&(3) \qquad \Phi[(My)(t)] &&= g(t), \quad 0 \le t \le \tau,
\end{aligned}
$$

when $M \in \mathcal{L}(Y)$ and L is a closed linear operator in Y satisfying

$$
\begin{aligned}
&Re(Lu,u)_y \le \beta\|Mu\|^2, \quad \beta \in \mathbb{R}, y \in D(L),\\
&range(\lambda_0 M^*M - L) = Y, \text{and}\\
&(\lambda_0 M^*M - L)^{-1} \in \mathcal{L}(Y), \quad \lambda_0 > \beta,
\end{aligned}
$$

z is a fixed element in Y, $y_0 \in D(L)$, $g \in C([0,\tau];\mathbb{C})$.

In the paper [2], p.1515, Favini and Marinoschi considered such a problem taking into account that Y can be represented as a direct sum of the kernel $N(T)$ and the closure of the range $R(T)$ of operator $T = ML^{-1}M^* \in \mathcal{L}(Y)$, i.e.

$$Y = N(T) \oplus \overline{R(T)},$$

in view of the ergodic theorem. We denote the projector operator on $N(T)$ along $\overline{R(T)}$ by P.

Because L has a bounded inverse, if an argument change $y = x - f(t)L^{-1}z$ is performed, system (1)-(3) becomes

$$
\begin{aligned}
&(4) \qquad M^*\tfrac{d}{dt}(Mx) &&= Lx + f'(t)M^*ML^{-1}z, \quad 0 \le t \le \tau,\\
&(5) \qquad (Mx)(0) &&= w_0 = My_0 + f(0)ML^{-1}z,\\
&(6) \qquad \Phi[(Mx)(t)] &&= g(t) + f(t)\Phi[ML^{-1}z],
\end{aligned}
$$

provided that the compatibility relation

$$g(0) = \Phi[w_0] \tag{7}$$

holds.

As a first step, we note that (4) translates into the inclusion

$$\frac{d}{dt}(Mx) - f'(t)ML^{-1}z \in M^{*^{-1}}Lx(t). \tag{8}$$

Then, changing the variable to $x = L^{-1}M^*$ in (8), we get the equivalent problem

$$\frac{d}{dt}T\xi(t) = \xi(t) + f'(t)M^*ML^{-1}z, \quad 0 \le t \le \tau, \tag{9}$$

$$T\xi(0) = w_0 + f(0)ML^{-1}z, \tag{10}$$

$$\Phi[T\xi(t)] = g(t) + f(t)\Phi[ML^{-1}z]. \tag{11}$$

Since we know (see Favini and Marinoschi [2]) that the restriction $\tilde{T}$ of T to $\overline{R(T)}$ is invertible, i.e.

$$\tilde{T}^{-1} : \overline{R(T)} \to \overline{R(T)}$$

and it generates a C_0-semigroup in $\overline{R(T)}$, (9) reads equivalently

$$\frac{d}{dt}\tilde{T}(1-P)\xi = (1-P)\xi + f'(t)(1-P)ML^{-1}z, \quad 0 \le t \le \tau, \tag{12}$$

and analogously

$$\tilde{T}(1-P)\xi(0) = (1-P)w_0 + f(0)(1-P)ML^{-1}z, \tag{13}$$

$$\Phi[\tilde{T}(1-P)\xi(t)] = g(t) + f(t)\Phi[ML^{-1}z], \tag{14}$$

together with

$$0 = Pw_0 + f(0)PML^{-1}z, \tag{15}$$

$$0 = P\xi(t) + f'(t)PML^{-1}z. \tag{16}$$

The case where $PML^{-1}z = Pw_0 = 0$ was discussed in Favini-Marinoschi [3]. Here we investigate what happens if $\Phi[PML^{-1}z] \neq 0$.

In Section 2 we study problem (1)-(3) by using multivalued linear operators. Section 3 is devoted to studying (12)-(14), (16) characterizing $(1-P)\xi(t)$. In Section 4 we consider equations of the second order and in Section 5 various applications to PDEs are described. The forthcoming paper [4] by Favini, Marinoschi, Tanabe, Yakubov provides a more complete and detailed state of the art on the subject.

2. The multivalued operators approach

In this section we recall previous results by A.Favini and A.Yagi [5] concerning differential problems in a Hilbert space Y. We begin with the easier problem

$$\text{(DE)} \qquad \begin{cases} M^* \frac{d}{dt}(My(t)) = Ly(t) + M^* f(t), \quad 0 \leq t \leq \tau, \\ My(0) = w_0, \end{cases}$$

where M is a bounded linear operator in Y and L is a closed single-valued linear operator in Y, w_0 is a given element in Y, $f \in C([0,\tau]; H)$. We assume

$$\Re\langle Lu, u\rangle_H \leq \beta \|Mu\|_H^2, \quad \exists \beta \in \mathbb{R}, \forall u \in D(L), \tag{17}$$

$$R(\lambda_0 M^* M - L) \supseteq R(M^*), \tag{18}$$

and $(\lambda_0 M^* M - L)^{-1}$ is single-valued on $R(M^*)$, for some $\lambda_0 > \beta$.

The following lemma holds. It is Theorem 2.10 in [5].

Lemma 2.1. *Under assumptions (17) and (18), for any $f \in C^1([0,\tau]; H)$ and any w_0, satisfying*

$$w_0 = My_0, \quad Ly_0 \in R(M^*), \tag{19}$$

for some $y_0 \in D(L)$, problem (DE) possesses a unique solution y such that

$$My \in C^1([0,\tau], Y), \;\; Ly \in C([0,\tau]; Y). \tag{20}$$

The key step consists in noting that, under the present assumptions, $A = (M^*)^{-1} L M^{-1}$ is maximal dissipative in H. Notice that this also implies that $H = N(T) \oplus \overline{R(T)}$, where $T = A^{-1} = ML^{-1}M^*$.

Using multivalued operators, problem (1)-(3) becomes a differential inclusion; precisely, if $M^* z$ substitutes z:

$$\frac{d}{dt}(My(t)) - f(t) z \in (M^*)^{-1} Ly(t), \quad 0 \leq t \leq z.$$

A further change of variable $My(t) = x(t)$ implies

$$\frac{dx(t)}{dt} - f(t)z \in Ax(t),$$
$$x(0) = w_0,$$
$$\Phi[x(t)] = g(t),$$

where necessarily $\Phi[w_0] = g(0)$ and $A = (M^*)^{-1}LM^{-1}$.

From [5] we know that

$$x(t) = e^{tA}w_0 + \int_0^t e^{(t-s)A}zf(s)ds.$$

Thus, condition $\Phi[x(t)] = g(t)$ becomes the integral equation

$$\Phi[e^{tA}w_0] + \int_0^t \Phi[e^{(t-s)A}z]f(s)ds = g(t). \tag{21}$$

Notice that (21) must admit a unique solution $f \in C^1([0,\tau];\mathbb{C})$. Thus, we have to solve an integral equation of the second kind. This is accomplished in the following Theorem.

Theorem 2.1. *Under the assumptions (17), (18) on the operators M, M^*, L, problem (1)-(3), with $M^*f(t)z$ instead of $f(t)z$, admits a unique solution $(y, f) \in C([0,\tau];Y) \times C^1([0,\tau];\mathbb{C})$ provided that $\Phi[z] \neq 0$, $w_0 = A^{-1}w_1$, $w_1 = A^{-1}w_2$, $z = A^{-1}z_1$, $g \in C^2([0,\tau];\mathbb{C})$.*

Proof. See [4]. □

In order to solve the general problem (1)-(3) by using the trick in [4], we perform the change of variable $y = x - f(t)L^{-1}z$. Then system (1)-(3) becomes

$$M^*\frac{d}{dt}(Mx(t)) - f'(t)M^*ML^{-1}z = Lx(t),$$

and then $Mx(t) = \xi(t)$ satisfies

$$\frac{d\xi(t)}{dt} - f'(t)ML^{-1}z \in A\xi(t), \quad 0 \le t \le \tau,$$
$$\xi(0) = w_0 + f(0)ML^{-1}z,$$
$$\Phi[\xi(t)] = g(t) + f(t)\Phi[ML^{-1}z], \quad 0 \le t \le \tau.$$

Therefore, f must belong to $C^2([0,\tau];\mathbb{C})$ to obtain the strict solution

$$\xi(t) = e^{tA}[w_0 + f(0)ML^{-1}z] + \int_0^t e^{(t-s)A}ML^{-1}zf'(s)ds.$$

After an integration by parts we get that

$$\begin{aligned}\xi(t) &= e^{tA}w_0 + f(0)e^{tA}ML^{-1}z - f(0)e^{tA}ML^{-1}z \\ &+ ML^{-1}zf(t) + \int_0^t e^{(t-s)A}(M^*)^{-1}zf(s)ds \\ &= e^{tA}w_0 + ML^{-1}zf(t) + \int_0^t e^{(t-s)A}(M^*)^{-1}zf(s)ds\end{aligned}$$

that can be handled only if $z = M^*z_1$, just the assumption that we want to avoid. Therefore, another approach must be introduced to this end.

3. Equations of the first order

The first step consists in studying (12)-(14). Taking into account that $\tilde{T}^{-1}$ generates a C_0-semigroup in $\overline{R(T)}$, in order to have a unique solution to (12),(13) we must assume that $f \in C^2([0,\tau];\mathbb{C})$ and $(1-P)ML^{-1}z$ and $(1-P)w_0$ belong to $D(\tilde{T}^{-1}) = R(\tilde{T}) = R(T)$. Now we proceed formally to get to the mean theorems. We necessarily have

$$\begin{aligned}\tilde{T}(1-P)\xi(t) &= e^{t\tilde{T}^{-1}}[(1-P)w_0 + f(0)(1-P)ML^{-1}z] \\ &+ \int_0^t e^{(t-s)\tilde{T}^{-1}}(1-P)ML^{-1}zf'(s)ds.\end{aligned}$$

After an integration by parts, we get

$$\begin{aligned}\tilde{T}(1-P)\xi(t) &= e^{t\tilde{T}^{-1}}[(1-P)w_0 + f(0)(1-P)ML^{-1}z] \\ &+ f(t)(1-P)ML^{-1}z - f(0)e^{t\tilde{T}^{-1}}(1-P)ML^{-1}z \\ &+ \int_0^t \tilde{T}^{-1}e^{(t-s)\tilde{T}^{-1}}(1-P)ML^{-1}zf(s)ds \\ &= e^{t\tilde{T}^{-1}}(1-P)w_0 + f(t)(1-P)ML^{-1}z \\ &+ \int_0^t \tilde{T}^{-1}e^{(t-s)\tilde{T}^{-1}}(1-P)ML^{-1}zf(s)ds.\end{aligned} \tag{22}$$

To this end, we are compelled to assume that

$$\sup_{t>0} \|\tilde{T}^{-1}e^{t\tilde{T}^{-1}}(1-P)ML^{-1}z\| < \infty, \tag{23}$$

i.e. in the case where $\tilde{T}^{-1}$ generates an analytic semigroup in $\overline{R(T)}$, $(1-P)ML^{-1}z$ belongs to the Favard space F_1 for the operator $\tilde{T}^{-1}$, according to the Engel-Nagel monograph [1]. However, if we take $(1-P)ML^{-1}z \in R(\tilde{T}) = R(T)$, (23) surely holds. Applying Φ to both members of equality (22), we get

$$\begin{aligned}\Phi[e^{t\tilde{T}^{-1}}(1-P)w_0] \quad &+ f(t)\Phi[(1-P)ML^{-1}z] + \\ &+ \int_0^t \Phi[\tilde{T}^{-1}e^{(t-s)\tilde{T}^{-1}}(1-P)ML^{-1}z]f(s)ds = g(t) + f(t)\Phi[ML^{-1}z],\end{aligned}$$

i.e.

$$\begin{aligned}f(t)\Phi[PML^{-1}z] \quad &= \int_0^t \Phi[\tilde{T}^{-1}e^{(t-s)\tilde{T}^{-1}}(1-P)ML^{-1}z]f(s)ds \\ &+ \Phi[e^{t\tilde{T}^{-1}}(1-P)w_0] - g(t),\ 0 \le t \le \tau. \end{aligned} \tag{24}$$

If $\Phi[PML^{-1}z] \neq 0$, a classical integral equation of the first type is obtained. Using Lorenzi's monograph [6], we deduce that (24) admits a unique global continuous solution f on $[0,\tau]$. However, we need some more regularity for $f(t)$, precisely $f \in C^{(2)}([0,\tau])$.

Operating the change of variable $t - s = \tau$ in the integral entry in (24), we obtain

$$\int_0^t \Phi[\tilde{T}^{-1}e^{(t-s)\tilde{T}^{-1}}(1-P)ML^{-1}z]f(s)ds = \int_0^t \Phi[\tilde{T}^{-1}e^{\tau\tilde{T}^{-1}}(1-P)ML^{-1}z]f(t-\tau)d\tau,$$

so that

$$\begin{aligned}&\frac{d}{dt}\int_0^t \Phi[\tilde{T}^{-1}e^{(t-s)\tilde{T}^{-1}}(1-P)ML^{-1}z]f(s)ds = \\ &= \Phi[\tilde{T}^{-1}e^{(t\tilde{T}^{-1}}(1-P)ML^{-1}z]f(0) + \\ &+ \int_0^t \Phi[\tilde{T}^{-1}e^{(t-s)\tilde{T}^{-1}}(1-P)ML^{-1}z]f'(s)ds.\end{aligned}$$

Therefore, deriving both members of (24), we obtain

$$\begin{aligned}&f'(t)\Phi[PML^{-1}z] = \int_0^t \Phi[\tilde{T}^{-1}e^{(t-s)\tilde{T}^{-1}}(1-P)ML^{-1}z]f'(s)ds \\ &+ \Phi[\tilde{T}^{-1}e^{(t\tilde{T}^{-1}}(1-P)ML^{-1}z]f(0) + \Phi[\tilde{T}^{-1}e^{t\tilde{T}^{-1}}(1-P)w_0] - g'(t).\end{aligned}$$

By using the same trick, we get

$$\begin{aligned} f''(t)\Phi[PML^{-1}z] &= \Phi[\tilde{T}^{-1}e^{(t\tilde{T}^{-1}}(1-P)ML^{-1}z]f'(0) \\ &+\int_0^t \Phi[\tilde{T}^{-1}e^{(t-s)\tilde{T}^{-1}}(1-P)ML^{-1}z]f''(s)ds \\ &+\Phi[\tilde{T}^{-1}e^{t\tilde{T}^{-1}}\tilde{T}^{-1}(1-P)ML^{-1}z]f(0)+\Phi[\tilde{T}^{-1}e^{t\tilde{T}^{-1}}\tilde{T}^{-1}(1-P)w_0]-g''(t). \end{aligned}$$

It follows that if

$$\sup_{t>0}\|-\tilde{T}^{-1}e^{t\tilde{T}^{-1}}\tilde{T}^{-1}(1-P)ML^{-1}z\|<\infty,$$
$$\sup_{t>0}\|-\tilde{T}^{-1}e^{t\tilde{T}^{-1}}\tilde{T}^{-1}(1-P)w_0\|<\infty,$$

then $f''(t)$ solves a classical integral equation and then it is continuous on $[0,\tau]$. The following remark is by Hiroki Tanabe.

Remark 3.1. *If* $(1-P)ML^{-1}z$ *belongs to Favard space* F_1*, so that* $\tilde{T}^{-1}e^{t\tilde{T}^{-1}}(1-P)ML^{-1}z$ *is bounded on* $(0,\infty)$*, due to the reflexivity of the ambient space, there exists a sequence* $t_j\to 0$ *such that* $\tilde{T}^{-1}e^{t_j\tilde{T}^{-1}}(1-P)ML^{-1}z\to x$ *weakly, for some* $x\in\overline{R(T)}$*.*

Since $e^{t_j\tilde{T}^{-1}}(1-P)ML^{-1}z\to(1-P)ML^{-1}z$ *as* $t_j\to 0$*, the closedness of the operator* $\tilde{T}^{-1}$ *yields that* $(1-P)ML^{-1}z\in D(\tilde{T}^{-1})=R(\tilde{T})=R(T)$ *and* $\tilde{T}^{-1}(1-P)ML^{-1}z=x$*, i.e.* $(1-P)ML^{-1}z=\tilde{T}x\in R(T)$*.*

But this implies that

$$\sup_{t>0}\|\tilde{T}^{-2}e^{t\tilde{T}^{-1}}(1-P)ML^{-1}z\|=\sup_{t>0}\|\tilde{T}^{-1}e^{t\tilde{T}^{-1}}\tilde{T}^{-1}Tx\|$$
$$=\sup_{t>0}\|\tilde{T}^{-1}e^{t\tilde{T}^{-1}}x\|.$$

Analogously $(1-P)w_0=\tilde{T}x_1\in R(T)$ *and thus*

$$\sup_{t>0}\|\tilde{T}^{-2}e^{t\tilde{T}}(1-P)w_0\|=\sup_{t>0}\|\tilde{T}^{-1}e^{t\tilde{T}^{-1}}x_1\|.\#$$

It remains to verify (15). We know that

$$f(0)=\frac{\Phi[(1-P)w_0]-g(0)}{\Phi[PML^{-1}z]}=-\frac{\phi[Pw_0]}{\Phi[PML^{-1}z]}.$$

Then (15) holds if and only if

(25) $$\Phi[PML^{-1}z]Pw_0=\Phi[Pw_0]PML^{-1}z.$$

Hence, we can establish our main result as follows.

Theorem 3.1. *Suppose that the Hilbert space* $Y = N(T) \oplus \overline{R(T)}$, *where* $T = ML^{-1}M^*$, L, M *having the properties described above, and let* P *be the projection on* $N(T)$ *along* $\overline{R(T)}$. *Let* $\Phi \in Y^*$ *and* $\Phi[PML^{-1}z] \neq 0$, *where* z *is fixed in* Y, $0 \in \rho(L)$,

$$(1-P)ML^{-1}z = \tilde{T}x, \quad (1-P)w_0 = \tilde{T}x_1,$$

with

$$\sup_{t>0} \|\tilde{T}^{-1}e^{t\tilde{T}^{-1}}x\|_{\overline{R(T)}} < \infty,$$
$$\sup_{t>0} \|\tilde{T}^{-1}e^{t\tilde{T}^{-1}}x_1\|_{\overline{R(T)}} < \infty.$$

If, in addition, (25) holds, then the inverse problem (1)-(3), i.e.

$$\begin{aligned} M^*\tfrac{d}{dt}My(t) &= Ly(t) + f(t)z, \quad 0 \le t \le \tau, \\ (My)(0) &= w_0, \\ \Phi[My(t)] &= g(t), \quad 0 \le t \le \tau, \end{aligned}$$

together with the compatibility relation $g(0) = \Phi[w_0]$, *admits a unique solution* $(y, f) \in C([0,\tau]; D(L)) \times C^2([0,\tau]; \mathbb{C})$.

If $PML^{-1}z = Pw_0 = 0$, more regularity for the data is needed. Indeed, it is shown in Favini and Marinoschi [3] that the following result holds.

Theorem 3.2. *Let* $Y = N(T) \oplus \overline{R(T)}$, $T = ML^{-1}M^*$, $P =$ *projection on* $N(T)$ *along* $\overline{R(T)}$. *Suppose* $PML^{-1}z = Pw_0 = 0$, $z \in Y$, $w_0, ML^{-1}z \in R(T^2)$:

$$\Phi \in Y^*, \ \Phi[\tilde{T}^{-1}ML^{-1}z] \neq 0, \ 0 \in \rho(L), \ g \in C^{(3)}([0,\tau]; \mathbb{C})$$

$$\Phi[w_0] = g(0),$$

$$\sup_{t>0} \|\tilde{T}^{-1}e^{t\tilde{T}^{-1}}\tilde{T}^{-2}ML^{-1}z\| < \infty, \quad \sup_{t>0} \|\tilde{T}^{-1}e^{t\tilde{T}^{-1}}\tilde{T}^{-2}w_0\| < \infty.$$

Then the inverse problem

$$M^* \tfrac{d}{dt} My(t) = Ly(t) + f(t)z, \quad 0 \le t \le \tau, \tag{26}$$

$$(My)(0) = w_0, \tag{27}$$

$$\Phi[My(t)] = g(t), \quad 0 \le t \le \tau \tag{28}$$

admits a unique solution $(y, f) \in C([0,\tau]; D(L)) \times C^2([0,\tau]; \mathbb{C})$.

Remark 3.2. *One may suppose that the general problem (26)-(28) could be handled by means of multivalued linear operators using the change of variable* $y = x - f(t)L^{-1}z$. *It is trivial to verify that after the change of variable* $Mx = \xi$ *one obtains*

$$\frac{d\xi(t)}{dt} - f'(t)ML^{-1}z \in {M^*}^{-1}LM^{-1}\xi(t) = \mathcal{A}\xi(t), \; 0 \le t \le \tau,$$
$$\xi(0) = w_0 + f(0)ML^{-1}z,$$
$$\Phi[\xi(t)] = g(t) + f(t)\Phi[ML^{-1}z], \quad 0 \le t \le \tau,$$

whose solution reads

$$\begin{aligned} \xi(t) &= e^{t\mathcal{A}}[w_0 + f(0)ML^{-1}z] + \int_0^t e^{(t-s)\mathcal{A}} ML^{-1}zf'(s)ds \\ &= e^{t\mathcal{A}}w_0 + f(t)ML^{-1}z + \int_0^t e^{(t-s)\mathcal{A}}\mathcal{A}ML^{-1}zf(s)ds, \end{aligned}$$

but the last integral term makes sense only if z *has the particular form* $z = M^*\bar{z}$ *and we already considered such a case in* [2].

Remark 3.3. *Clearly* $N(M^*) \subseteq N(ML^{-1}M^*)$. *On the other hand, if* $ML^{-1}M^*z = 0$, *then* $0 = \langle ML^{-1}M^*z, z\rangle = \langle L^{-1}M^*z, M^*z\rangle$ *and so* $\Re e\langle L^{-1}M^*z, M^*z\rangle = 0$. *Suppose that* $\Re e\langle Ly, y\rangle \ne 0$ *for all* $y \ne 0$. *Then* $\Re e\langle L^{-1}w, w\rangle \ne 0$ *for any* $w \ne 0$. *Hence* $\Re e\langle L^{-1}M^*z, M^*z\rangle = 0$ *if and only if* $M^*z = 0$. *This implies* $N(ML^{-1}M^*) = N(M^*)$.

Remark 3.4. *Suppose* L *self-adjoint and* $\langle Lx, x\rangle \le 0$ *for all* $x \in D(L)$. *If* $ML^{-1}M^*x = 0$, *then* $\langle (ML^{-1}M^*)x, x\rangle = -\langle M(-L)^{1/2}(-L)^{1/2}M^*x, x\rangle = -\langle (-L)^{1/2}M^*x, (-L)^{1/2}M^*x\rangle = 0$. *Therefore,* $N(ML^{-1}M^*)$ *and* $N(M^*)$ *coincide.*

Remark 3.5. *Suppose* $N(ML^{-1}M^*) = N(M^*)$. *Then* $M^*PML^{-1}z = 0$, *so that* $0 = \langle M^*PML^{-1}z, L^{-1}z\rangle = \langle PML^{-1}z, ML^{-1}z\rangle = \|PML^{-1}z\|_x^2 + \langle PML^{-1}z, (1-P)ML^{-1}z\rangle$. *Therefore, if* L *is also self-adjoint, with* $\langle Lx, x\rangle \le 0$ *for all* $x \in D(L)$, *since operator* P *is self-adjoint,* $\langle Px, (1-P)x\rangle = \langle Px, x\rangle - \langle Px, Px\rangle = \langle Px, x\rangle - \langle P^2x, x\rangle = \langle Px, x\rangle - \langle Px, x\rangle = 0$. *Hence* $0 = \|PML^{-1}z\|^2$, *i.e.* $PML^{-1}z = 0$.

4. Equations of the second order

Our aim is the identification problem in the Hilbert space H with inner product $(\cdot,\cdot)_H$

$$
\begin{aligned}
&C^{1/2}\frac{d}{dt}(C^{1/2}y'(t)) + B\frac{dy(t)}{dt} + A_H y(t) = f(t)z, \\
&y(0) = y_0, \quad C^{1/2}y'(0) = C^{1/2}y_1, \quad 0 \le t \le \tau, \\
&\Phi[C^{1/2}y(t)] = g(t), \quad 0 \le t \le \tau,
\end{aligned}
$$

where C is a bounded self-adjoint operator in H, $C \ge 0$, B is a closed linear operator in H, A is a linear bounded operator from another Hilbert space V continuously and densely embedded in H to V', precisely

$$
\begin{aligned}
&\langle Au, v\rangle = \langle u, Av\rangle \text{ for all } u, v \in V, \\
&\langle Au, v\rangle \ge w_0\|u\|_V^2 \text{ for all } u, v \in V, \ \langle,\rangle \text{ is the duality between } V \text{ and } V', \\
&w_0 > 0, \\
&D(A_H) = \{v \in V; Av \in H\}, \quad A_H v = Av, v \in D(A_H).
\end{aligned}
$$

One has $D(A^{1/2}) = V$, $D(B) \supseteq D(A^{1/2})$, $\Re e\langle Bu, u\rangle \ge 0$ for all $u \in V$, $\Phi \in H^*$, $g \in C^1([0,\tau];\mathbb{C})$ at least, $\Phi[C^{1/2}y_0] = g(0)$, $\Phi[C^{1/2}y_1] = g'(0)$.

We recall that H is identified with its dual space, so that the embeddings $V \hookrightarrow H \hookrightarrow V'$ hold. Then the previous identification problem is written in the form

$$\begin{bmatrix} 1 & 0 \\ 0 & C^{1/2} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} 1 & 0 \\ 0 & C^{1/2} \end{bmatrix} \begin{bmatrix} y & \\ 0 & y' \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ A_H & B \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = f(t) \begin{bmatrix} 0 \\ z \end{bmatrix}, \quad 0 \le t \le \tau,$$

$$\begin{bmatrix} 1 & 0 \\ 0 & C^{1/2} \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & C^{1/2} \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix},$$

$$\tilde{\Phi} \left(\begin{bmatrix} 1 & 0 \\ 0 & C^{1/2} \end{bmatrix} \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} \right) := \Phi[C^{1/2} y'(t)] = g'(t), \quad 0 \le t \le \tau,$$

in the ambient space $Y = D(A^{1/2}) \times H$ with inner product

$$\langle (x, y), (x_1, y_1) \rangle_Y = (A^{1/2} x, A^{1/2} x_1)_H + (y, y_1)_H,$$
$$(x, y), \ (x_1, y_1) \in Y.$$

Of course, operators M and L from Y into itself are defined by

$$D(M) = Y, \quad M(y, x) = (y, C^{1/2} x), \quad (y, x) \in Y,$$
$$D(L) = D(A_H) \times D(A^{1/2}), \quad L(y, x) = (x, -A_H y - Bx)_H$$

for $(y, x) \in D(L)$.

It should be easy to translate Theorems 3.1 and 3.2 in this situation.

5. Examples and applications

Example 5.1. Consider this inverse problem to find $(\nu, f) \in C([0, \tau]; H^1(R)) \times C([0, \tau]; \mathbb{C})$ satisfying

$$\frac{\partial m(\cdot)\nu}{\partial t} = -\frac{\partial \nu}{\partial x} + f(t) z(x), \quad -\infty < x < \infty, \quad 0 \le t \le \tau, \tag{29}$$

$$m(x)\nu(x, 0) = u_0(\cdot), \quad -\infty < x < \infty, \tag{30}$$

$$\Phi[M\nu] = \int_{-\infty}^{\infty} \eta(x) m(\cdot) \nu(x, t) dx = g(t), \quad 0 \le t \le \tau, \tag{31}$$

with given $\eta \in L^2(R)$.

The direct problem (29)-(30) is considered in Favini and Yagi [5], pp. 40,41.

Here $m(\cdot)$ is the characteristic function of some measurable set $J \subset R$, $z(x)$ is a given function, $u_0(x)$ is the initial data, $\nu = \nu(x,t)$ is the desired solution. The problem is viewed in $X = L^2(R)$. If M is the multiplication operator by $m(x)$ acting in X, $M \in \mathcal{L}(X)$ and $M^* = M$. Therefore, problem (29),(30) is formulated in the form

$$\begin{aligned} &M^* \frac{\partial Mx}{\partial t} = L\nu + f(t)z, \quad 0 \le t \le \tau, \\ &M\nu(0) = u_0, \end{aligned}$$

where $L = -\frac{d}{dx}$, $D(L) = H^1(R)$. Clearly L is a closed linear operator in X. Consider two particular cases:

1) $J = (-\infty, a) \cup (b, \infty)$, $a < b$. Then $R(\lambda_0 M^*M - L) = X$ for some $\lambda_0 > 0$ and $(\lambda_0 M^*M - L)^{-1}$ is single-valued. Then, Theorem 2.1 applies with $z = M^*z_1, z_1 \in X$.

2) $J = (a, \infty)$. In this case only $R(\lambda_0 M^*M - L) \supset R(M^*)$ and $(\lambda_0 M^*M - L)^{-1}$ is single-valued on $R(M^*)$ for some $\lambda_0 > \beta$. However, by using Theorem 2.1 again, the inverse problems can be solved provided that $z = M^*z_1$, $z_1 \in X$, again. On the other hand it was noted previously that the ambient space is the direct sum of $N(T)$ and the closure of $R(T), T = ML^{-1}M^*$, cfr Favini, Marinoschi [2]. Then, we can apply both Theorem 3.1 and 3.2 for general $z \in X$.

Example 5.2. (Maxwell's equations)

Assuming that the medium which fills space $\mathbb{R}^3$ is linear but it may be anisotropic and nonhomogeneous, we can write Maxwell's equations in the form

$$\frac{\partial}{\partial t}\begin{pmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{pmatrix}\begin{pmatrix} E \\ S \end{pmatrix} = \begin{pmatrix} 0 & rot \\ -rot & 0 \end{pmatrix}\begin{pmatrix} E \\ S \end{pmatrix} - \begin{pmatrix} \sigma(x) & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} E \\ S \end{pmatrix} - f(t)\begin{pmatrix} z(x) \\ 0 \end{pmatrix},$$

where $\varepsilon(x)$, $\mu(x)$, $\sigma(x)$, $x \in \mathbb{R}^3$, are some matrices such that

(i) $\varepsilon(x)$ are symmetric and $\varepsilon(x) \ge 0$ for all $x \in \mathbb{R}^3$;

(ii) $\mu(x)$ are symmetric and $\mu(x) \ge \delta$ with some $\delta > 0$ uniformly in $\mathbb{R}^3$;

(iii) $((\gamma\varepsilon(x) + \sigma(x))\xi, \xi)_{\mathbb{R}^3} \ge \delta|\xi|^2$, $\xi \in \mathbb{R}^3$, with some $\delta > 0$ and $\gamma \ge 0$ uniformly in $x \in \mathbb{R}^3$.

Then it is easy to verify that the previous problem can be written in the form

$$M^* \frac{dMy}{dt} = Ly + f(t) \begin{bmatrix} z \\ 0 \end{bmatrix},$$
$$Mv(0) = 0,$$

in the space $X = (L^2(\mathbb{R}^3))^6$, where M is the multiplication operator by $\sqrt{C(x)}$,

$$C(x) = \begin{pmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{pmatrix}$$

and L is a closed linear operator in X (see Favini and Yagi [5], p.43). Therefore, we can consider the related inverse problem to find (v, f) such that

$$\Phi[Mv] = g(t), \quad \Phi \in ((L^2(\mathbb{R}^3))^6)^*, \quad g \in C^{(3)}([0,\tau];\mathbb{C}).$$

Example 5.3. Consider the problem

$$\left(m(x)\frac{\partial}{\partial t}\right)^2 v = \Delta v + f(t)z, \quad (x,t) \in \Omega \times [0,\tau],$$
$$v = 0 \quad \text{on } (x,t) \in \partial\Omega \times [0,\tau],$$
$$v(x,0) = v_0(x), \quad m(x)\frac{\partial v}{\partial t}(x,0) = v_1(x), \quad x \in \Omega,$$
$$\int_\Omega \nu_1(x) v(x,t)dx + \int_\Omega \nu_2(x) m(x) \frac{\partial v}{\partial t}(x,t)dx = g(t),$$

where $\nu_1(x), \nu_2(x) \in L^2(\Omega)$, $g \in C^1([0,\tau];\mathbb{C})$ and the unknown is the pair (v, f). Our previous results apply to the Poisson-wave equation in a (bounded or unbounded) region $\Omega \subset \mathbb{R}^n$ with a smooth boundary, $m \in L^\infty(\Omega)$, $m(x) \geq 0$ and m is allowed to vanish in a bounded subset of Ω.

The equation is written as a system

$$\begin{pmatrix} 1 & 0 \\ 0 & m(x) \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} 1 & 0 \\ 0 & m(x) \end{pmatrix} \begin{pmatrix} v \\ v_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} v \\ v_t \end{pmatrix} + f(t) \begin{pmatrix} 0 \\ z \end{pmatrix},$$

i.e.

$$M^* \frac{dMV}{dt} = LV + f(t)Z,$$
$$MV(0) = V_0,$$

and all previous results apply. Also here under the present assumption $(M^*)^{-1}LM^{-1}$ is maximal dissipative. Hence, cfr Favini, Marinoschi [2] if $T = ML^{-1}M^*$, then the ambient space is the direct sum $N(T) \oplus \overline{R(T)}$.

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