ABOUT RAYS, DREADLOCKS AND PERIODIC POINTS IN TRANSCENDENTAL DYNAMICS

RAGGI DINAMICI E PUNTI PERIODICI IN DINAMICA COMPLESSA TRASCENDENTE

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Abstract. We study some aspects of the iteration of an entire map f over the complex plane $\mathbb{C}$. In many settings in complex dynamics one can define periodic curves (called dynamic rays) in the dynamical plane and study their relation with periodic points. The most famous example of this kind of results is the *Douady-Hubbard landing theorem* for polynomial dynamics. We describe an analogous statements for transcendental maps which satisfy some growth conditions and a further generalization to general transcendental maps with bounded postsingular set, without any growth assumption. We also describe some implications for rigidity. The results described here are from a joint work with Lasse Rempe-Gillen.

Sunto. Studiamo alcuni aspetti della iterazione di una funzione olomorfa f sul piano complesso $\mathbb{C}$. Nello studio della dinamica complessa in una variabile in molti casi si può costruire nel piano dinamico una famiglia di curve (chiamate raggi dinamici) dotate di una dinamica simbolica. Queste curve possono essere messe in relazione con i punti periodici. Il risultato maggiormente noto è il teorema di Douady e Hubbard nel caso in cui f è un polinomio. In questa nota descriviamo i risultati ottenuti di recente dall'autore con Lasse Rempe-Gillen. Consistono in una generalizzazione del teorema di Douady e Hubbard per funzioni trascendenti per cui esistono i raggi dinamici, e successivamente per una classe più ampia di mappe trascendenti, per le quali il ruolo dei raggi dinamici viene svolto da insiemi più generali chiamati dreadlocks.

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1. Introduction

One-dimensional complex dynamics studies the iterates $f^n := f \circ \ldots \circ f$ of a holomorphic function f over a Riemann surface. Excellent introductory texts are [24], [12], [2]. Riemann surfaces are classified into *parabolic* (if their universal covering is the complex plane $\mathbb{C}$), *elliptic* (if their universal covering is the Riemann sphere $\hat{\mathbb{C}}$) or *hyperbolic* (if their universal covering is the unit disk is $\mathbb{D}$). By the Schwarz-Ahlfors-Pick Lemma, a holomorphic map from a hyperbolic Riemann surface into itself either contracts the hyperbolic metric or is an isometry, so this case is usually considered as completely understood.

The natural holomorphic maps which act on the Riemann sphere are rational maps, while the natural map which act on the complex plane are entire maps, either transcendental or polynomials. When studying holomorphic maps of $\mathbb{C}$, since the group of automorphisms of $\mathbb{C}$ is just the set of affine maps, it is usually assumed that the topological degree of f is at least two. Being transcendental simply means that infinity is an essential singularity. Polynomials are both entire functions and a special case of rational maps, so depending on the context one considers them as maps acting on $\mathbb{C}$ or on $\hat{\mathbb{C}}$. The *orbit* of a point $z \in \mathbb{C}$ is the infinite set $O(z) = \{z, f(z), f^2(z), \ldots\}$. A very important role in all of dynamical system is played by *invariant* sets: a set X is called *forward invariant* if $f(X) \subset X$, *backward invariant* if $f^{-1}(X) \subset X$, and *completely invariant* if both conditions hold. For a given set X, its preimage $f^{-1}(X)$ is the set of all points whose image belongs to X.

A point z is said to belong to the *Fatou set* $F(f)$ if there exists a neighborhood U of z such that the family of iterates $f|U^n$ is *normal* (that is, every sequence either diverges locally uniformly or has a subsequence which converges uniformly on compact sets). The complement of the Fatou set is called the *Julia set* $J(f)$ and is the set of points near which the map exhibits chaotic behaviour. For example, using a theorem by Montel it is easy to show that the union of the forward iterates of any open set intersecting $J(f)$ covers the entire plane $\mathbb{C}$ minus at most one exceptional point.

There are two main classes of problems that one may want to consider in this setting. The first class of problems is related to the dynamics of a specific function, or a specific class of functions which behave in a dynamically similar way. The second class of problems is to fix a naturally defined family of functions $\{f_\lambda\}_{\lambda\in\Lambda}$ with Λ a complex manifold and to try to understand how the dynamics change when the parameter λ varies in Λ. Natural families of functions are for example the family of unicritical polynomials $\{z^2+\lambda\}_{\lambda\in\mathbb{C}}$ of fixed degree d, the family of rational maps of degree d, or finite dimensional families of transcendental maps as defined in [18].

Natural questions among the first classes of problems include the following:

- Understand the statistical properties of orbits;
- Understand the structure of invariant sets and their properties (geometric, topological, measure theoretical...);
- Understand the asymptotic behaviour of orbits, for example study the set of their accumulation points.

The invariant sets that are most relevant for this note are periodic orbits and the set of escaping points. A *periodic orbit* of period p is simply a (minimal) finite set $\{z_0 \dots z_{p-1}\}$ of p points which are permuted by f; in the case in which $p=1$, we talk about a *fixed point.* The *set of escaping points* $I(f)$ is the set of points whose orbits converge to infinity under iteration by f. For example points on the real line for the map e^z belong to $I(e^z)$. While a periodic orbit is forward invariant- but not backward invariant, since every point has more than one preimage under f- the set $I(f)$ is completely invariant.

Both periodic points and the escaping set have deep relations to the Julia set. Indeed, for an entire map f the Julia set $J(f)$ equals both the closure of the set of repelling periodic points (see the definition later on) and the boundary of the set of escaping points ([24], [19]).

A natural notion which arises when working on problems related to parameter spaces is the notion of conjugacy. Two maps f,g in the same family are *topologically (resp. quasi-conformally, resp. conformally conjugate)* if there exists a homeomorphism (resp. a quasiconformal, or a conformal map) h such that $f\circ h = h\circ g$. Being conjugate essentially means that the two maps have similar dynamical features; for example, an f-invariant set X gives a g-invariant set $h(X)$. The main problem is to understand the structure of the parameter space, in many different aspects like for example studying the different classes of conjugacies, the density of classes of maps with well understood dynamical behaviour, or topological and geometric aspects of sets which appear naturally in the parameter space.

The goal of this note is to state some results on the relation between the set of escaping points and the set of periodic points for transcendental maps. The new results cited in this note are from a joint paper of the author with Lasse Rempe-Gillen, to appear [8]. The original motivation for this work was that the relation between escaping and periodic points is very well understood for polynomials, and has led to fundamental results for the study not only of the dynamics of the individual maps but also for the understanding of the parameter spaces of unicritical polynomials. Before being able to state the results, though, we will need some background on the theory of iteration.

1.1. **Singular Values and postsingular set.** The set of *singular values* $S(f)$ is the closure of all asymptotic and critical values for f. Critical values are images of critical points (for example, c for z^d+c) while an asymptotic value a is a point such that there exists a curve $\gamma(t):[0,\infty)\to\mathbb{C}$ such that $|\gamma(t)|\to\infty$ and $f(\gamma(t))\to a$ as $t\to\infty$ (for example, 0 is an asymptotic value for e^z, and γ can be chosen to be any curve with $\operatorname{Re}\gamma(t)\to-\infty$ as $t\to\infty$). The main characteristic of a singular value $s\in S(f)$ is that for any arbitrary small neighborhood of s there exists an inverse branch of f which is not well defined and univalent. On the other side,

$$f:\mathbb{C}\setminus f^{-1}(S(f))\to\mathbb{C}\setminus S(f)$$

is an infinite degree unbranched covering.

The *postsingular set*

$$P(f):=\overline{\bigcup_{n\in\mathbb{N},s\in S(f)} f^n(s)}$$

is the set of iterates of all singular values. Observe that on any simply connected open set U not intersecting $P(f)$ all branches of f^{-n} are well defined and univalent for all n. Because of this, asssuming that the postsingular set is bounded gives a (non simply connected) backward invariant neighborhood of infinity on which all univalent inverse branches are locally well defined and univalent. When all singular values are critical, like for polynomials, $S(f)$ is called the *set of critical values*, and $P(f)$ is called the *postcritical set*.

In this paper we mostly consider maps with bounded postsingular set in order to simplify the exposition. However, for polynomials the situation is quite well understood also when the postsingular set is not bounded. Indeed, in this case any critical value whose orbit is not bounded converges to infinity under iteration, while in the transcendental case singular values with non-bounded orbits can have any possible behaviour, including having orbits which are dense in $\mathbb{C}$.

1.2. **Periodic points.** Let f be an entire map. Periodic points of period p for f are classified according to their *multiplier*, that is the derivative of f^p calculated at any periodic point in the orbit. Given that any periodic point of period p is a fixed point for f^p, it is sufficient to state the classification for a fixed point z_0. We have the following cases:

- $|f'(z_0)| > 1$: z_0 is called *repelling* and there is a neighborhood $U(z_0)$ such that $f(U) \supset U$, and f is conjugated to the linear map $z \to \lambda z$ on U (with $\lambda = f'(z_0)$);
- $0 < |f'(z_0)| < 1$: z_0 is called *attracting* and there is a neighborhood $U(z_0)$ such that $f(U) \subset U$, and f is conjugated to the linear map $z \to \lambda z$ on U (with $\lambda = f'(z_0)$);
- $|f'(z_0)| = 0$: z_0 is called *superattracting* and there is a neighborhood $U(z_0)$ such that $f(U) \subset U$, and f is conjugated to the map $z \to z^d$ on U (for some $d > 1$);
- $f'(z_0) = e^{2\pi i p/q}$ with $p/q \in \mathbb{Q}$: z_0 is called *parabolic* and there are a certain number of alternated attracting and repelling directions;
- $f'(z_0) = e^{2\pi i\theta}$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and f linearizable: z_0 is called *Siegel* and there is a neighborhood $U(z_0)$ such that $f(U) = U$, and f is conjugate to the linear map $z \to \lambda z$ on U (with $\lambda = f'(z_0)$);
- $f'(z_0) = e^{2\pi i\theta}$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and f non-linearizable: z_0 is called *Cremer* and there is a non-locally connected, completely invariant compact set $K \ni z_0$, called hedgehog, on which the dynamics keep some of the features of an irrational rotation.

The classification above and the linearization results can be found in [24]; The hedgehog case is described in [25]. Since we are interested in the relation between periodic points and escaping points, let us point out that there cannot be escaping points near attracting, superattracting or Siegel periodic points, since in these cases there is a neighborhood of the periodic point in which the dynamics are bounded.

It is not hard to show that Siegel and (super)attracting periodic points are in the Fatou set, while Cremer, repelling and parabolic periodic points are in the Julia set.

1.3. **Symbolic Dynamics.** Let $\mathbb{Z}^{\mathbb{N}}$ be the space of infinite sequences over the integers. A natural operator acting on $\mathbb{Z}^{\mathbb{N}}$ is the left sided shift map σ which sends a sequence $\underline{s} = s_0 s_1 s_2 \ldots \in \mathbb{Z}^{\mathbb{N}}$ to the sequence $\sigma\underline{s} = s_1 s_2 s_3 \ldots$. There is a natural order on $\mathbb{Z}^{\mathbb{N}}$: $\underline{s}^1 > \underline{s}^2$ if and only if the first entry in which the two sequences are different is larger for $\underline{s}^1$ than for $\underline{s}^2$. The map σ is locally order preserving, that is, if $s_0^1 = s_0^2$ and $\underline{s}^1 > \underline{s}^2$ it follows that $\sigma\underline{s}^1 > \sigma\underline{s}^2$. The preimages $\sigma^{-1}\underline{s}$ of a sequence $\underline{s} = s_0 s_1 s_2 \ldots$ are given by all sequences of the form $a\underline{s} := a s_0 s_1 s_2 \ldots$ where $a \in \mathbb{Z}$. In particular σ is an infinity-to-1 map.

Each compact subspace $\mathcal{S}_d := \{0, 1, \ldots d-1\}^{\mathbb{N}} \subset \mathbb{Z}^{\mathbb{N}}$ of sequences over d symbols is forward invariant under σ, and each sequence $\underline{s} \in \mathcal{S}_d$ has exactly d preimages under σ^{-1} which are contained in $\mathcal{S}_d$. So $\sigma|\mathcal{S}_d$ is a d-to-1 map.

Symbolic dynamics can be used to study more general dynamical systems. A classical way of doing so it to find a (possibly only partial) partition of the phase space X into sets X_s, each of which is labeled by an integer s. One then studies the set of points which share a common itinerary $\underline{s} = s_0 s_1 s_2 \ldots$ under this partition, that is, $G_{\underline{s}}$ is the set of points x in the phase space such that $f^i(x) \in X_{s_i}$. Note that itineraries are not necessarily well defined for all points in X. If one consider the family $\mathcal{G}$ consisting of all sets $G_{\underline{s}} \neq \emptyset$, the dynamics on the family $\mathcal{G}$ is naturally conjugate to the dynamics of the shift map on sequences, that is,

$$f(G_{\underline{s}}) = G_{\sigma\underline{s}}.$$

An example which is very relevant for the sequel is the example of the map $z \to z^d : \mathbb{S}^1 \to \mathbb{S}^1$, where $\mathbb{S}^1$ is the unit circle. Up to rescaling, one can see this map as the map $\theta \to d\theta : \mathbb{T}^1 \to \mathbb{T}^1$, where $\mathbb{T}^1$ denotes the circle of length 1 given by $\mathbb{R}/\mathbb{Z}$. When $\theta \in \mathbb{T}^1$ is written in d-adic expansion (one has to recall that if $\theta = 1/d^n$, it has two different d-adyc expansions), the map $\theta \to d\theta$ is just the left-sided shift map acting on $\mathcal{S}_d$.

This gives a semi-conjugacy between $\sigma : \mathcal{S}_d \to \mathcal{S}_d$ and $z^d : \mathbb{S}^1 \to \mathbb{S}^1$. In this case it is a semiconjugacy and not a conjugacy because the d-adyc expansion is not unique for $\theta = 1/d^n$. The same conjugacy can be constucted on $\mathbb{T}^1 \setminus \{\theta = 1/d^n\}$ by using the following partition: assign the symbol 0 to the open arc between 0 and 1/2, and the symbol 1 to the arc between 1/2 and $1 = 0 \mod \mathbb{Z}$. Then itineraries with respect to this partition are defined for all $\theta \notin \{\theta = 1/d^n\}$. For more on symbolic dynamics see [11].

1.4. **Relations between escaping points and periodic points for polynomials.** For polynomials of degree d with bounded postsingular set, the set of escaping points consists of an uncountable family $\{G_{\underline{s}}\}_{\mathcal{S}_d}$ of injective curves, called *dynamic rays* (originally, *external rays*), equipped with symbolic dynamics in the sense that $f(G_{\underline{s}}) = G_{\sigma\underline{s}}$.

Dynamic rays for polynomials are constructed in the following way [14]. Under the assumption that f has bounded postsingular set, its filled Julia set (the set of points with bounde orbits) is compact, connected and full, so its complement (which coincides with the set of escaping points) has a unique unbounded connected and simply connected component U which is biholomorphic to $\mathbb{C} \setminus \mathbb{D}$ by the Riemann Uniformization Theorem. A theorem by Bottcher [24] ensures that the inverse of the Riemann map conjugates f on U to the map $z \to z^d$. The *dynamic ray* G_θ for f is defined as the image under the Riemann map of the straight ray $re^{2\pi i\theta}$, and then θ can be written in d-adyc expansion to get the dynamics on $\mathcal{S}_d$.

By definition, dynamic rays are curves $G_{\underline{s}} : (0, \infty) \to \mathbb{C}$, and $G_{\underline{s}}(t) \to \infty$ as $t \to \infty$. A dynamic ray is periodic of period p if and only if $\underline{s}$ is a periodic sequence of period p.

One can relatively easily see that $J(f) = \partial U$. Since the shift map induces a symbolic dynamics on U via dynamic rays, it is natural to ask under which hypothesis it is possible to associate symbolic dynamics to points in $J(f)$ by seeing them as limit points of dynamic rays. This motivates the definition of the concept of landing.

We say that a dynamic ray $G_{\underline{s}}$ *lands* at a point $z_0 \in \mathbb{C}$ if we have that $\overline{G_{\underline{s}}} = G_{\underline{s}} \cup z_0$ with $z_0 \in \mathbb{C}$, or equivalently $\lim_{t\to 0} G_{\underline{s}}(t) = z_0$.

By Carathéodory-Torhorst's theorem, the Riemann map extend continuously to $\partial\mathbb{D}$ if and only if $\partial U = J(f)$ is locally connected, so all dynamic rays land (and the landing point depends continuously on the argument) if and only if $J(f)$ is locally connected. This gives a complete combinatorial description of the Julia set in the locally connected case (observe that there are several known examples of non-locally connected Julia sets). Much more general results can be obtained when one only restricts attention to the landing of periodic rays. The landing of periodic rays, and the relation between periodic rays and periodic points is very well understood in the polynomial case. Indeed we have the following theorem, which has been a keystone for the study of polynomial dynamics for the past thirty years.

Douady-Hubbard landing theorem. [14], [21], [17] *Let f be a polynomial whose post-critical set $P(f)$ is bounded. Then every periodic ray of f lands at a repelling or parabolic periodic point, and conversely every repelling or parabolic periodic point of f is the landing point of at least one periodic dynamic ray, and at most finitely may dynamic rays, all of which are periodic with the same period.*

Douady-Hubbard landing theorem is at the base of the puzzle techniques pioneered by Yoccoz [21], which are the core of the past and current work around the conjecture that the Mandelbrot set is locally connected.

Observe that if a periodic ray of period p lands it can only land at periodic points of period dividing p, since by continuity of f we have that $f^p(G_{\underline{s}}) = G_{\underline{s}}$ implies $f^p(\overline{G_{\underline{s}}}) = \overline{G_{\underline{s}}}$. It is known (see the snail lemma in [24]) that periodic rays cannot land at Cremer points.

The first part of the theorem essentially relies on an hyperbolic contraction argument on U, while the second part is more delicate and uses in an essential way the existence of an attracting basin for infinity.

For generalizations to the case in which $P(f)$ is not bounded, as well as for a statement of rays landing at points in expansive sets, see [17] and [26].

1.5. **Relations between escaping points and periodic points for transcendental maps.** For all of this section we restrict our attention to the class of transcendental entire functions with bounded set of singular values. This is called the Eremenko-Lyubich class and will be denoted as class $\mathcal{B}$. For entire transcendental functions even in class $\mathcal{B}$ the theory is not nearly as complete as for polynomials. In particular, there are examples of entire transcendental functions in class $\mathcal{B}$ for which the path connected components of the set of escaping points are points ([32]).

On the other side, curves in the escaping set of transncendental functions have been observed as far as by Fatou in 1926 [20]. The main methods for constructing such curves in the escaping set (which were originally called hairs) and investigation of the symbolic dynamics associated to these curves where further developed in [15], [16]. The idea that they should be considered as an analogue of dynamic rays for polynomials probably started taking shape in [10]. For the exponential family it was proven in [33] that the entire escaping sets consists of dynamic rays. Despite the counterexample from [32], one can put conditions on the order of a function to ensure the existence of curves in the escaping set endowed with symbolic dynamics (and more, to ensure that all escaping points are connected to infinity by such a curve) [32]. We recall that the order of a function f is defined as $\rho(f) = \limsup_{r\to\infty} \frac{\ln\ln M(r,f)}{\ln r}$, where $M(r,f) = \max_{|z|=r} |f(z)|$. For example, e^{z^q} has order q.

The most general theorem available about existence of dynamic rays is the following.

Theorem 1.1. *Let f be an entire transcendental function with $S(f)$ bounded and which is a finite composition of functions of finite order with bounded set of singular values. Then the set of escaping points consists of injective curves $\{G_{\underline{s}}\}_{\underline{s}\in\mathcal{N}\subset\mathbb{Z}^{\mathbb{N}}}$ satisfying the relation $f(G_{\underline{s}}) = G_{\sigma\underline{s}}$, together with their landing points.*

The set $\mathcal{N}$ is not completely characterized in the transcendental case, but it always contains sequences associated to finitely many symbols, in particular periodic sequences ([1], [29], see also [6]). So the definition of periodic rays makes sense.

Basically in all known examples dynamic rays are constructed using the following strategy. One can find find a (partial) partition of the plane $\mathbb{C}$ into infinitely many sets, called *fundamental domains*, each of which can be put in correspondence with an integer. The construction of fundamental domains is relatively canonical and dates back to [18], although they were using it for different purposes. One can then see that for any escaping point z its iterates $f^n(z)$ belong to the set of fundamental domains for $n \geq n(z)$ large enough. In particular, for large enough iterates, itineraries of escaping points with respect to the partition induced by fundamental domains are well defined. Roughly speaking, one can then define the set $G_{\underline{s}}$ as the set of (escaping) points sharing the same itinerary. The challenge is to show that at least some of such sets are non empty and that they are indeed curves; this is where the condition of being a finite composition of functions of finite order comes into play in [32]. Simpler constructions work for the exponential map and the sine family, although the underlying idea is the same.

The relation between periodic dynamic rays and periodic points is more subtle. A novelty about landing of dynamic rays in the transcendental case is that, for some specific types of itineraries, dynamic rays can land at points which are themselves escaping.

The first part of Douady-Hubbard Landing Theorem is relatively easier to address than the second part. Indeed, it has been known for some time that if $f \in \mathcal{B}$ and $P(f)$ is bounded than periodic rays land ([29]; see also [13]). The proof uses hyperbolic contraction on the complement of the postsingular set. A more general result holds for the exponential family [27] and for polynomials [24]: indeed, every periodic dynamic ray lands unless possibly if its forward orbit intersects the orbit of a singular value. The proof of the latter theorem for exponentials uses detailed information of the parameter space for the exponential family, which seems out of reach for more general transcendental maps at the current state of knowledge.

The second part of Douady-Hubbard Landing Theorem , concerning the question whether repelling and parabolic periodic points where landing points of periodic rays, is harder to deal with, even for the exponential family.

Indeed, it has been shown only in [7] that, if f is an exponential map and $P(f)$ is bounded, then every repelling periodic point is the landing point of at least one and at most finitely many periodic rays. This type of results has implication for questions related to rigidity in this family ([4]). The proof in [7] does not need the existence of an attracting basin of infinity and so opened up the possibility of generalizing Douady-Hubbard's theorem to more general transcendental functions in class $\mathcal{B}$. Let us also mention that it follows from the main theorem in [6] that parabolic periodic points are always landing points of periodic rays, under a weaker assumption than having bounded postsingular set (but still under the assumption from Theorem 1.1 which implies existence of dynamic rays).

The approach in [8] is different in that it tries to go beyond the concept of dynamic rays. Indeed, following [29], it is proven that for all functions in class $\mathcal{B}$, with no assumptions on being composition of functions of finite order, the escaping set consists of unbounded connected sets, equipped with symbolic dynamics (see also [29]). These sets are called *dreadlocks* and, when the function satisfies the growth assumption from Theorem 1.1, they coincide with the dynamic rays which had been previously defined. As before, a dreadlock $G_{\underline{s}}$ is periodic if $\underline{s}$ is a periodic sequence. If $\underline{s}$ is periodic, we say that $G_{\underline{s}}$ lands if $\overline{G_{\underline{s}}} \setminus G_{\underline{s}} = \{z_0\}$ (a more precise, though more technical definition of landing via accumulation sets is given in [8]).

The construction of dreadlocks is quite natural and follows the strategy described before. The partial partition of the plane given by fundamental domains is used, with respect to which itineraries of escaping points are well defined. Then, dreadlocks are essentially sets of points with the same itinerary, so that the symbolic dynamics comes for free. When f satisfies the conditions of Theorem 1.1, it has good geometric properties that one can use to ensure that dreadlocks are curves. When these geometric properties fail, as in the counterexample in [32], one has to do some additional work to show that (at least for periodic itineraries) dreadlocks, while not being curves, are still unbounded connected sets satisfying several useful properties.

The tools developed in [8] allow to prove the following analogue of Douady-Hubbard theorem which holds for all functions in class $\mathcal{B}$ with bounded postsingular set:

Theorem 1.2 (Douady-Hubbard Landing Theorem for Dreadlocks [8])**.** *Let f be an entire transcendental function with $P(f)$ bounded. Then the set of escaping points consists of unbounded disjoint connected sets $\{G_{\underline{s}}\}_{\underline{s}\in\mathcal{N}\subset\mathbb{Z}^{\mathbb{N}}}$, called dreadlocks, which satisfy the relation $f(G_{\underline{s}}) = G_{\sigma\underline{s}}$. Each periodic dreadlock lands at a repelling or parabolic periodic point, and each parabolic and repelling periodic points is the landing point of at least one and at most finitely many periodic dreadlocks, all of which have the same period.*

Since the dreadlocks from Theorem 1.2 coincide with the rays from Theorem 1.1 when f satisfies the required growth conditions, we get the following corollary for functions satisfying the hypothesis from Theorem 1.1.

Theorem 1.3 (Douady-Hubbard Landing Theorem for functions with rays)**.** *Let f be an entire transcendental function bounded set of singular values and which is a finite composition of functions of finite order with bounded set of singular values. Assume also that $P(f)$ is bounded. Then all periodic dynamic rays land, and all parabolic and repelling periodic points are landing points of at least one and at most finitely many periodic dynamic rays, all of the same period.*

As mentioned before, the part of landing of periodic rays goes back to [29].

Let us observe that for polynomials the orbit of a critical value is either bounded or belongs to the escaping set, that is, the set of dynamic rays. The case in which critical orbits belong to rays is well understood and one can write a corresponding version of the Douady-Hubbard landing theorem that takes into account the exceptions induced by escaping critical values. For transcendental maps, instead, $P(f)$ may well be unbounded (and indeed, even equal to $\mathbb{C}$) without the singular values involved been escaping. So Theorem 1.2 and 1.3 do not offer a picture which is as complete as the Douady-Hubbard landing theorem for polynomials. The case in which the postsingular set is unbounded seems to require drastically new insights into the problem.

Although the study of parameters spaces of transcendental functions with finitely many singular values is still out of reach from the current state of the art of the field, we hope that this kind of results will have some influence in the further development of this topic.

1.6. **Parameter spaces for one-dimensional families.** Consider a naturally defined family of holomorphic maps $\mathcal{F} = \{f_\lambda\}_{\lambda\in\Lambda}$- for example, the family of unicritical polynomials, or rational maps of a fixed degree, or families of functions with finitely many singular values as in [18]- whose parameter space is an analytic manifold Λ. There is a dichotomy between the set of *structurally stable parameters* and the *bifurcation locus.* A parameter λ is *structurally stable* if f_λ is topologically conjugate to all functions $f_{\lambda'}$ for all λ' in a neighborhood of λ, and belongs to the bifurcation locus otherwise. It is a remarkable result that the set of structurally stable parameters is dense in each family considered above (see [23],[18]). Parameters for which the orbit of all singular values converge to attracting cycles are called *hyperbolic*. Parameters which are hyperbolic are structurally stable. The famous Fatou Conjecture (also called Density of Hyperbolicity Conjecture) states that when $\mathcal{F}$ is a family of polynomials or rational maps of fixed degree d, then hyperbolic maps are dense. Since the set of structurally stable parameters is open, the negation of this would be that there are non-hyperbolic components of structurally stable parameters.

When studying the family of quadratic polynomials $\{z^2+c\}_{c\in\mathbb{C}}$, a special role is played by the Mandelbrot set, which is the set of parameters for which the postcritical set is bounded. The boundary of the Mandelbrot set is exactly the bifurcation locus for this family.

The most famous conjecture in one-dimensional holomorphic dynamics is whether the Mandelbrot set is locally connected. In addition to providing a very detailed understanding of the structure of this set, an affirmative answer would imply the Density of Hyperbolicity Conjecture.

The Douady-Hubbard landing theorem is very related to the theory of rigidity, in particular to many results about local connectivity of the Mandelbrot set at specific classes of parameters. In particular, together with the Pommerenke-Levin-Yoccoz inequality [21] it implies that there are no non-hyperbolic components attached to the boundaries of hyperbolic components. The analogue of Douady-Hubbard Theorem for the exponential family $\{e^z+\lambda\}_\lambda \in \mathbb{C}$ (see [29], [7]) has analogous implications for rigidity of parameters with bounded postsingular set ([4], [3]).

For more on the relation between rigidity and local connectivity in the case of the exponential family and of unicritical polynomials see [31].

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