TIME FRACTIONAL DERIVATIVES AND EVOLUTION EQUATIONS

DAVIDE GUIDETTI

ABSTRACT. In this seminar we introduce the fractional derivatives of Riemann-Liouville and Caputo, with some of their main properties. We conclude by illustrating certain results of maximal regularity for mixed initial-boundary value problems, evolving them.

SUNTO. In questo seminario introduciamo le derivate frazionarie di Riemann-Liouville e di Caputo, con alcune delle loro principali proprietà. Concludiamo illustrando alcuni risultati di regolarità massimale per problemi misti al contorno, in cui compaiono tali derivate.

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The main aim of this seminar is to introduce the two most well known notions of "fractional time derivative" (the Riemann-Liouville derivative and the Caputo derivative). In the final part, we shall also present some results of maximal regularity for mixed initial-boundary value problem where the time derivatives may be fractional (see [4]). Some of these results are due to the author and seem to be new. We take also the occasion to illustrate some (essentially well known) definitions and results concerning fractional powers of positive operators.

In order to illustrate the basic idea of the definition of "fractional power", we begin with a very elementary example: let B be a complex number, not belonging to the set of nonpositive real numbers $(-\infty, 0]$. We set

$$\log(B) := \ln(|B|) + iArg(B),$$

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Dipartimento di Matematica, Università di Bologna

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with $Arg(B)$ belonging to the argument set of B and $-\pi < Arg(B) < \pi$. So, for any $\beta \in \mathbb{C}$, we can define

$$B^{\beta} := \exp(\beta \log(B)).$$

where we have indicated with exp the standard exponential function in $\mathbb{C}$.

Now we introduce the following notation: let $R \in \mathbb{R}^{+}$, $\theta \in (0, \pi)$. We indicate with $\gamma(\theta, R)$ a piecewise C^1 path, describing

$$\{\lambda \in \mathbb{C} \setminus (-\infty, 0] : |\lambda| = R, |Arg(\lambda)| \leq \theta\} \cup \{\lambda \in \mathbb{C} \setminus (-\infty, 0] : |\lambda| \geq R, |Arg(\lambda)| = \theta\},$$

oriented from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$. Then a simple application of the residue theorem shows that, if $Re(\alpha) > 0$, $R < |B|$ and $\theta > |Arg(B)|$,

$$B^{-\alpha} = -\frac{1}{2\pi i} \int_{\gamma(\theta,R)} \lambda^{-\alpha} (\lambda - B)^{-1} d\lambda. \tag{1}$$

Now we introduce the notion of positive operator.

Definition 1. *Let X be a complex Banach space and let $B : D(B) \subseteq X \to X$ be a linear (unbounded) operator. We shall say that it is positive if*

(a) $(-\infty, 0] \subseteq \rho(B)$*;*

(b) there exists M in $\mathbb{R}^{+}$ such that, $\forall \lambda \geq 0$,

$$\|(\lambda + B)^{-1}\|_{\mathcal{L}(X)} \leq \frac{M}{1 + |\lambda|}.$$

One can show (by a simple perturbation argument) that, if B is positive, there exist $\omega_0 \in (0, \pi)$, M' in $\mathbb{R}^{+}$, such that

$$\{\mu \in \mathbb{C} \setminus (-\infty, 0] : |Arg(\mu)| \geq \omega_0\} \subseteq \rho(B).$$

Moreover, if μ is in this set,

$$\|(\mu - B)^{-1}\|_{\mathcal{L}(X)} \leq \frac{M'}{1 + |\mu|}.$$

Definition 2. *Let B be a positive operator in X and let $\omega \in (0, \pi)$. We shall say that B is of type ω if $\{\mu \in \mathbb{C} \setminus (-\infty, 0] : |Arg(\mu)| > \omega\} \subseteq \rho(B)$ and if for every $\omega' \in (\omega, \pi)$ there exists $M(\omega') > 0$ such that $\|(1 + |\mu|)(\mu - B)^{-1}\|_{\mathcal{L}(X)} \leq M(\omega')$ in case $|Arg(\mu)| \geq \omega'$.*

Let now B be a positive operator in X. We suppose that it is of type ω, for some $\omega \in (0, \pi)$. We fix θ in (ω, π) and R in $\mathbb{R}^+$ such that $\{\mu \in \mathbb{C} : |\mu| \leq R\} \subseteq \rho(B)$. Then, if $\alpha > 0$ (so that $-\alpha < 0$), we set (inspired by (1),

$$B^{-\alpha} := -\frac{1}{2\pi i} \int_{\gamma(\theta,R)} \lambda^{-\alpha}(\lambda - B)^{-1} d\lambda. \tag{2}$$

The complex integral is convergent in the Banach space $\mathcal{L}(X)$. The following facts can be checked (see, for example, [7]):

(a) (2) is consistent with the usual definition of $B^{-\alpha}$ in case $\alpha \in \mathbb{N}$;

(b) if $\alpha, \beta \in \mathbb{R}^+$, $B^{-\alpha}B^{-\beta} = B^{-(\alpha+\beta)}$;

(c) $\forall \alpha$ in $\mathbb{R}^+$ $B^{-\alpha}$ is injective;

(d) in case B is positive and self-adjoint in the Hilbert space X, (2) is equivalent with the definition of fractional power obtained employing the spectral resolution.

(c) is very easy to show. In fact, assume that (for example) $0 < \alpha < 1$. Then, if $x \in X$ and $B^{-\alpha}x = 0$,

$$B^{-1}x = B^{-(1-\alpha)}B^{-\alpha}x = B^{-(1-\alpha)}0 = 0,$$

so that $x = 0$ by (a). So we can define, for $\alpha \in \mathbb{R}^+$,

$$B^{\alpha} := (B^{-\alpha})^{-1}. \tag{3}$$

Of course, the domain $D(B^\alpha)$ of B^α is the range of $B^{-\alpha}$. We observe, that, if B is unbounded and $\beta \in \mathbb{R}$, B^β is bounded only if $\beta \leq 0$. Moreover, employing (a)-(c), it is easy to show that, if $0 < \alpha < \beta$, $D(B^\beta) \subseteq D(B^\alpha)$,

$$D(B^\beta) = \{x \in D(B^\alpha) : B^\alpha x \in D(B^{\beta-\alpha})\}$$

and, if $x \in D(B^\beta)$,

$$B^\beta x = B^{\beta-\alpha}B^\alpha x.$$

Now we show an example: let X be a complex Banach space and $T \in \mathbb{R}^+$. We consider the Banach space $C([0,T];X)$ of continuous functions with values in X. We introduce the following operator B in $C([0,T];X)$:

$$\left\{\begin{array}{l} D(B) := \{u \in C^1([0,T];X) : u(0) = 0\}, \\ \\ Bu(t) = u'(t). \end{array}\right. \tag{4}$$

Then it is easy to see that $\rho(B) = \mathbb{C}$. Moreover, $\forall \lambda \in \mathbb{C}$, $\forall f \in C([0,T];X)$,

$$(\lambda - B)^{-1} f(t) = -\int_0^t e^{\lambda(t-s)} f(s) ds. \tag{5}$$

It is easily seen that B is positive of type $\frac{\pi}{2}$. Following the previous construction, we can define $B^{-\alpha}$, with $\alpha \in \mathbb{R}^+$: we fix θ in $(\frac{\pi}{2}, \pi)$ and consider (2). We have, on account of (5), in case $0 < \alpha < 1$, $f \in C([0,T];X)$, $t \in [0,T]$,

$$B^{-\alpha} f(t) = \frac{1}{2\pi i} \int_0^t \Big(\int_{\gamma(\theta,1)} \lambda^{-\alpha} e^{\lambda(t-s)} d\lambda\Big) f(s) ds.$$

As $-\alpha > -1$, we can deform the path of integration, replacing $\gamma(\theta,1)$ with another path $\gamma(\theta,0)$, describing $\{\lambda \in \mathbb{C} \setminus (-\infty, 0] : |Arg(\lambda)| = \theta\}$, oriented from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$. So we obtain, if $r \in \mathbb{R}^+$,

$$\tfrac{1}{2\pi i} \int_{\gamma(\theta,1)} \lambda^{-\alpha} e^{\lambda r} d\lambda = \int_{\gamma(\theta,0)} \lambda^{-\alpha} e^{\lambda r} d\lambda$$

$$= \tfrac{1}{2\pi i} \Big(\int_0^\infty (\rho e^{i\theta})^{-\alpha} e^{r\rho e^{i\theta}} e^{i\theta} d\rho - \int_0^\infty (\rho e^{-i\theta})^{-\alpha} e^{r\rho e^{-i\theta}} e^{-i\theta} d\rho\Big).$$

The value of the integral is independent of $\theta \in (\frac{\pi}{2}, \pi)$. Passing to the limit as θ goes to π, we get

$$\tfrac{1}{2\pi i}\Big(-\int_0^\infty \rho^{-\alpha} e^{-i\alpha\pi} e^{-r\rho} d\rho + \int_0^\infty \rho^{-\alpha} e^{i\alpha\pi} e^{-r\rho} d\rho\Big)$$

$$= \tfrac{\sin(\alpha\pi)}{\pi} \int_0^\infty e^{-r\rho} \rho^{-\alpha} d\rho = \tfrac{\sin(\alpha\pi)\Gamma(1-\alpha)}{\pi} r^{\alpha-1} = \tfrac{r^{\alpha-1}}{\Gamma(\alpha)},$$

on account of the classical formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \quad (z \in \mathbb{C} \setminus \mathbb{Z}).$$

We conclude that, in case $\alpha \in (0,1)$,

$$B^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \tag{6}$$

and, in general, employing (5), (7) can be extended to $\alpha \in \mathbb{R}^+$.

Now we are able to define the "Riemann-Liouville derivative of order $\alpha \in \mathbb{R}^+$" (see [6]):

Definition 3. *Let $f \in C([0,T];X)$, $m \in \mathbb{N}_0$ and $m < \alpha < m+1$. Suppose that $B^{\alpha-m-1}f \in C^{m+1}[0,T];X)$. Then we define the Riemann-Liouville derivative of order α ${}_aD^\alpha f$ as*

$$(7) \qquad {}_aD^\alpha f(t) := D_t^{m+1}(B^{\alpha-m-1}f)(t) = D_t^{m+1}(\frac{1}{\Gamma(m+1-\alpha)}\int_0^t (t-s)^{m-\alpha}f(s)ds).$$

Remark 1. In case $0 < \alpha < 1$, ${}_aD^\alpha = B^\alpha$. In fact, ${}_aD^\alpha f$ is defined if $B^{\alpha-1}f \in C^1([0,T];X)$. From (7) it is clear that $B^{\alpha-1}f(0) = 0$. So

$$D({}_aD^\alpha) = \{f \in C([0,T];X) : B^{\alpha-1}f \in D(B)\}$$

and it is not difficult to see that this coincides with $D(B^\alpha)$. The identity between ${}_aD^\alpha$ and B^α does not hold if $\alpha > 1$. Consider, for example, the case $1 < \alpha < 2$. Then

$${}_aD^\alpha f = D_t^2(B^{\alpha-2}f),$$

so that

$$D({}_aD^\alpha) = \{f \in C([0,T];X) : B^{\alpha-2}f \in C^2([0,T];X)\}$$

As $B^{\alpha-2}f(0) = 0$, again $B^{\alpha-2}f \in D(B)$. So

$$D({}_aD^\alpha) = \{f \in C([0,T];X) : B^{\alpha-1}f \in C^1([0,T];X)\}$$

and

$${}_aD^\alpha f = D_t(B^{\alpha-1}f) = B[B^{\alpha-1}f - B^{\alpha-1}f(0)],$$

while

$$D(B^\alpha) = \{f \in C([0,T];X) : B^{\alpha-1}f \in D(B)\}$$

$$= \{f \in C([0,T];X) : B^{\alpha-1}f \in C^1([0,T];X), B^{\alpha-1}f(0) = 0\}.$$

A second type of fractional derivative is the Caputo derivative of order α ${}_a^\alpha Du$, again in case $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$. Let $m < \alpha < m+1$, with $m \in \mathbb{N}_0$. If $u \in C^{m+1}([0,T];X)$, we set

$${}_a^\alpha Du(t) := B^{\alpha-(m+1)}(D^{m+1}u)(t) = \frac{1}{\Gamma(m+1-\alpha)}\int_0^t (t-s)^{m-\alpha}D^{m+1}u(s)ds.$$

We observe that

$$D^{m+1}u = D^{m+1}[u - \sum_{k=0}^{m} \frac{t^k}{k!} D^k u(0)]$$

and that, if $u \in C^{m+1}([0,T];X)$, $u - \sum_{k=0}^{m} \frac{t^k}{k!} u^{(k)}(0) \in D(B^{m+1})$. So

$${}_a^\alpha Du = B^{\alpha-(m+1)} B^{m+1}[u - \sum_{k=0}^{m} \frac{t^k}{k!} u^{(k)}(0)] = B^\alpha [u - \sum_{k=0}^{m} \frac{t^k}{k!} u^{(k)}(0)].$$

as $u - \sum_{k=0}^{m} \frac{t^k}{k!} u^{(k)}(0) \in D(B^{m+1})$. This suggests the following

Definition 4. *Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$, $m < \alpha < m+1$, with $m \in \mathbb{N}_0$. We shall say that $u \in D({}_a^\alpha D)$ if $u \in C^m([0,T];X)$ and $u - \sum_{k=0}^{m} \frac{t^k}{k!} u^{(k)}(0) \in D(B^\alpha)$. In this case, we set*

$${}_a^\alpha Du := B^\alpha (u - \sum_{k=0}^{m} \frac{t^k}{k!} u^{(k)}(0)).$$

Remark 2. In case $0 < \alpha < 1$, $D({}_a^\alpha D) = \{u \in C([0,T];X) : u - u(0) \in D(B^\alpha)\}$, which contains $D(B^\alpha) = D({}_aD^\alpha)$. If $\alpha > 1$ there is no inclusion relation between $D({}_a^\alpha D)$ and $D({}_aD^\alpha)$. Consider, for example, the case $\alpha = \frac{3}{2}$. Let $u \equiv c$, with $c \in X \setminus \{0\}$. Then

$${}_a^{3/2} Du = B^{-1/2} u'' = B^{-1/2} 0 = 0.$$

From

$$\frac{1}{\Gamma(\frac{1}{2})} \int_0^t (t-s)^{-1/2} ds = \frac{2}{\sqrt{\pi}} t^{1/2},$$

we deduce, setting $v(t) := \frac{2}{\sqrt{\pi}} t^{1/2} c$,

$$B^{1/2}(v) = u.$$

and $B^{-1/2}u = v$. So ${}_aD^{3/2}u$ is not defined, because $v \notin C^2([0,T];X)$ (unless $c = 0$).

On the other hand,

$${}_aD^{3/2} v = (B^{-1/2} v)'' = (B^{-1} u)'' = 0,$$

because $B^{-1}u(t) = tc$. On the contrary, ${}_a^{3/2} Dv$ is not defined, because $v \notin C^1([0,T];X)$.

It is of interest to characterize $D(B^\alpha)$, with $\alpha \in \mathbb{R}^+$. Of course, if $\alpha \in \mathbb{N}$,

$$D(B^\alpha) = \{u \in C^\alpha([0,T];X) : u(0) = \cdots = u^{(\alpha-1)}(0) = 0\}.$$

If $\alpha = m + \beta$, with $\beta \in (0,1)$,

$$D(B^\alpha) = \{u \in C^m([0,T];X) : u(0) = \cdots = u^{(m)}(0) = 0, D^m u \in D(B^\beta)\},$$

so it remains to consider the case $\alpha \in (0,1)$. In this case, it is possible to show the following double inclusion: $\forall \epsilon \in (0,\alpha]$,

$$\overset{o}{C}^{\alpha+\epsilon}([0,T];X) \subseteq D(B^\alpha) \subseteq \overset{o}{C}^{\alpha}([0,T];X),$$

with

$$\overset{o}{C}^{\beta}([0,T];X) = \{f \in C^\beta([0,T];X) : f(0) = 0\}.$$

In case we look for conditions assuring that $B^\alpha u$ belongs to some space of Hölder continuous functions $C^\beta([0,T];X)$, with $\beta \in \mathbb{R}^+ \setminus \mathbb{N}$, it is possible to give a very precise answer. To this aim, we introduce some notations: we begin by considering the Zygmund class $B^1_{\infty,\infty}([0,T];X)$, defined as

$$\begin{aligned} B^1_{\infty,\infty}([0,T];X) &:= \{f \in C([0,T];X) : \\ sup_{0<h<T/2} h^{-1} \|f(.+h) - 2f + f(.-h)\|_{C([h,T-h];X)} &< \infty\}. \end{aligned} \tag{8}$$

We set also

$$\overset{o}{B}{}^1_{\infty,\infty}([0,T];X) := \{f \in B^1_{\infty,\infty}([0,T];X) : \sup_{0<t\le T} t^{-1}\|f(t)\| < \infty\}. \tag{9}$$

More generally, we set, if $m \in \mathbb{N}$,

$$B^m_{\infty,\infty}([0,T];X) := \{f \in C^{m-1}([0,T];X) : f^{(m-1)} \in B^1_{\infty,\infty}([0,T];X)\}, \tag{10}$$

$$\overset{o}{B}{}^m_{\infty,\infty}([0,T];X) := \{f \in C^{m-1}([0,T];X) : f(0) = \dots f^{(m-1)}(0) = 0, f^{(m-1)} \in \overset{o}{B}{}^1_{\infty,\infty}([0,T];X)\}. \tag{11}$$

It is also convenient to set, $\forall \gamma \in \mathbb{R}^+ \setminus \mathbb{N}$,

$$\overset{o}{C}^{\gamma}([0,T];X) := \{f \in C^\gamma([0,T];X) : u^{(k)}(0) = 0 \quad \text{if} \quad k < \gamma\},$$

and, $\forall \gamma \in \mathbb{R}^+$,

$$\overset{o}{\mathcal{C}}^{\gamma}([0,T];X) := \begin{cases} \overset{o}{C}^{\gamma}([0,T];X) & \text{if} \quad \gamma \notin \mathbb{N}, \\ \\ \overset{o}{B}{}^{\gamma}_{\infty,\infty}([0,T];X) & \text{if} \quad \gamma \in \mathbb{N}. \end{cases} \tag{12}$$

Then one can show the following result:

Proposition 1. *Let* $\alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R}^+ \setminus \mathbb{N}$. *Then:*

$$\{u \in D(B^\alpha) : B^\alpha u \in C^\beta([0,T];X)\} \;=\; \overset{o}{\mathcal{C}}^{\alpha+\beta}([0,T];X) \oplus \{\textstyle\sum_{k=0}^{[\beta]} t^{k+\alpha} f_k : f_k \in X\}.$$

Finally, we consider evolution equations with the Caputo derivative of the form

$$ {}_a^\alpha D u(t) = Au(t) + f(t), \quad t \in [0,T]. \tag{13}$$

Here A is a closed (generally) unbounded operator in the complex Banach space X and $f \in C([0,T];X)$.

By definition, a **strict solution** of (15) will be a function u belonging to $D({}_a^\alpha D) \cap C([0,T];D(A))$. By the link of ${}_a^\alpha D$ with B^α, it is not surprising that our results depend on the spectral properties of B^α. More precisely, we employ the following

Proposition 2. *If* $\alpha \in (0,2)$, B^α *is a positive operator in the Banach space* $C([0,T];X)$ *of type* $\frac{\alpha\pi}{2}$.

The main tool will be Da Prato-Grisvard's theory, concerning abstract equations in the form

$$\mathcal{A}u + \mathcal{B}u = f, \quad f \in E, \tag{14}$$

Here $\mathcal{A}$ and $\mathcal{B}$ are assumed to be positive (unbounded) operators in E. The key conditions are:

(a) $\mathcal{A}$ and $\mathcal{B}$ commute in the sense of the resolvents, that is, given arbitrary elements $\lambda \in \rho(\mathcal{A})$, $\mu \in \rho(\mathcal{B})$,

$$(\lambda - \mathcal{A})^{-1}(\mu - \mathcal{B})^{-1} = (\mu - \mathcal{B})^{-1}(\lambda - \mathcal{A})^{-1};$$

(b) $\mathcal{A}$ and $\mathcal{B}$ are positive operators of type (respectively) $\omega_{\mathcal{A}}$ and $\omega_{\mathcal{B}}$, with

$$\omega_{\mathcal{A}} + \omega_{\mathcal{B}} < \pi.$$

If (a) and (b) hold (with some further details), it is possible to give quite precise results of existence and uniqueness of a solution to (14). It is also possible to establish theorems of maximal regularity in spaces of interpolation between E and $D(\mathcal{A})$ $(D(\mathcal{B}))$.

In order to apply these results, it is convenient to study preliminarily the equation

$$B^\alpha u(t) - Au(t) = f(t), \quad t \in [0,T]. \tag{15}$$

We set $\mathcal{B} := B^\alpha$. We have already observed that it is positive of type $\frac{\alpha\pi}{2}$ in $E = C([0,T];X)$, if $0 < \alpha < 2$. We suppose that $-A$ is a positive operator in X of type less than $\frac{(2-\alpha)\pi}{2}$. We introduce the following operator $\mathcal{A}$:

$$\begin{cases} D(\mathcal{A}) = C([0,T];D(A)), \\ \mathcal{A}u(t) = -Au(t), \quad t \in [0,T]. \end{cases}$$

It is easy to check that $\mathcal{A}$ is positive of type less than $\frac{(2-\alpha)\pi}{2}$ and that $\mathcal{A}$ and $\mathcal{B}$ commute in the sense of resolvents. Then we can write (15) in the form (14). We omit the details and conclude with some specific applications, obtained by developing the previous suggested approach.

Let Ω be an open, bounded subset of $\mathbb{R}^n$, lying on one side of its boundary $\partial\Omega$, which is an submanifold of $\mathbb{R}^n$ of dimension $n-1$ and class C^3. We introduce in Ω the following operator $\tilde{A}$:

$$\begin{cases} D(\tilde{A}) = \{u \in \cap_{1\le p<\infty} W^{2,p}(\Omega) : \Delta u \in C(\overline{\Omega}), D_\nu u_{|\partial\Omega} = 0\}, \\ \\ \tilde{A}u = \Delta u. \end{cases} \tag{16}$$

We think of $\tilde{A}$ as an unbounded operator in

$$X := C(\overline{\Omega}). \tag{17}$$

The spectrum $\sigma(\tilde{A})$ of $\tilde{A}$ is contained in $(-\infty, 0]$. Moreover, $\forall\epsilon \in (0,\pi)$, $\forall\lambda_0 \in \mathbb{R}^+$ there exists $C(\epsilon)$ such that if $\lambda \in \mathbb{C}$ and $|\lambda| \ge \epsilon$ and $|Arg(\lambda)| \le \pi - \epsilon$,

$$\|(\lambda - \tilde{A})^{-1}\|_{\mathcal{L}(X)} \le C(\epsilon)|\lambda|^{-1}$$

We deduce that $\forall\delta > 0$ $-\tilde{A} + \delta$ is positive of type 0.

Now we set

$$A_\phi := e^{i\phi}\tilde{A}, \tag{18}$$

with $\phi \in (-\pi, \pi]$. It is not difficult to check that for every δ positive $\delta - A_\phi$ is positive of type $|\phi|$. We fix α in $(0, 2)$, $\phi \in (-\pi, \pi)$ and consider the mixed problem with the Caputo derivative

$$(19) \qquad \begin{cases} {}^{\alpha}_{a}Du(t,x) - e^{i\phi}\Delta_x u(t,x) = f(t,x), & (t,x) \in [0,T] \times \Omega, \\ D_\nu u(t,x') = 0, & (t,x') \in [0,T] \times \partial\Omega, \\ D_t^k u(0,x) = u_k(x), & k \in \mathbb{N}_0, k < \alpha, x \in \Omega. \end{cases}$$

Let $\beta, \gamma \in [0, \infty)$, $f : [0,T] \times \overline{\Omega} \to \mathbb{C}$. We shall write $f \in C^{\beta,\gamma}([0,T] \times \overline{\Omega})$ if $f \in C^\beta([0,T]; C(\overline{\Omega})) \cap B([0,T]; C^\gamma(\overline{\Omega}))$. Then the following result, which is well known in case $\alpha = 1$ (see [5]) and can be obtained as application of abstract results in [1], [2], [3], holds:

Theorem 1. *Consider system (19). Let $\alpha \in (0,2)$, $|\phi| < \frac{(2-\alpha)\pi}{2}$. Then, if $\gamma \in (0,1)$, $\gamma - \frac{2}{\alpha} \notin \mathbb{Z}$, the following conditions are necessary and sufficient, in order that there exist a unique strict solution u belonging to $D({}^{\alpha}_{a}D) \cap C([0,T]; D(\tilde{A}))$, with ${}^{\alpha}_{a}Du$ and $\tilde{A}u$ belonging to $C^{\frac{\alpha\gamma}{2},\gamma}([0,T] \times \overline{\Omega})$:*

(a) $u_0 \in C^{2+\gamma}(\overline{\Omega})$, $D_\nu u_0 = 0$;

(b) $f \in C^{\frac{\alpha\gamma}{2},\gamma}([0,T] \times \overline{\Omega})$;

(c) if $\alpha \in (1,2)$, $u_1 \in C^{\gamma+2-\frac{2}{\alpha}}(\overline{\Omega})$ and, if $\gamma + 2 - \frac{2}{\alpha} > 1$, $D_\nu u_1 = 0$.

We illustrate also a maximal regularity result for the Cauchy-Dirichlet problem (see [4]). Let Ω be, again, an open, bounded subset of $\mathbb{R}^n$, lying on one side of its boundary $\partial\Omega$, which is an submanifold of $\mathbb{R}^n$ of dimension $n-1$ and class C^3. We introduce in Ω the following operator $\tilde{A}$:

$$(20) \qquad \begin{cases} D(\tilde{A}) = \{u \in \bigcap_{1 \le p < \infty} W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) : \Delta u \in C(\overline{\Omega})\}, \\ \tilde{A}u = \Delta u. \end{cases}$$

We think of $\tilde{A}$ as an unbounded operator in

$$(21) \qquad X := C(\overline{\Omega}).$$

$-\tilde{A}$ is a positive operator of type 0. So, defining again A_ϕ as in (18), we deduce that $-A_\phi$ is positive of type $|\phi|$. We fix α in $(0,2)$, $\phi \in (-\pi, \pi)$ and consider the mixed problem

$$\text{(22)} \qquad \begin{cases} {}_a^\alpha Du(t,x) - e^{i\phi}\Delta_x u(t,x) = f(t,x), & (t,x) \in [0,T] \times \Omega, \\ \\ u(t,x') = 0, & (t,x') \in [0,T] \times \partial\Omega, \\ \\ D_t^k u(0,x) = u_k(x), & k \in \mathbb{N}_0, k < \alpha, x \in \Omega. \end{cases}$$

The following result holds (the case $\alpha = 1$ is exposed in [[5]]):

Theorem 2. *Consider system (22). Let $\alpha \in (0,2)$, $|\phi| < \frac{(2-\alpha)\pi}{2}$, $\gamma \in (0,2) \setminus \{1\}$, such that $\alpha\gamma < 2$. Then the following conditions are necessary and sufficient in order that there exists a unique strict solution u with ${}_a^\alpha Du$ and Δu belonging to $C^{\frac{\alpha\gamma}{2},\gamma}([0,T] \times \overline{\Omega}))$:*

(a) $f \in C^{\frac{\alpha\gamma}{2},\gamma}([0,T] \times \overline{\Omega}))$;

(b) $u_0 \in C^{2+\gamma}(\overline{\Omega})$, $(e^{i\phi}\Delta_x u_0 + f(0,\cdot))_{|\partial\Omega} = 0$;

(c) if $\alpha \in (1,2)$, $u_1 \in C^{\gamma+2-\frac{2}{\alpha}}(\overline{\Omega})$, $u_{1|\partial\Omega} = 0$.

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Dipartimento di matematica, Piazza di Porta S. Donato 5, 40126 Bologna, Italy.

E-mail address: davide.guidetti@unibo.it