THE ISOPERIMETRIC PROBLEM IN CARNOT-CARATHÉODORY SPACES
IL PROBLEMA ISOPERIMETRICO IN SPAZI DI CARNOT-CARATHÉODORY

VALENTINA FRANCESCHI

Abstract. We present some recent results obtained in [13] and [14] on the isoperimetric problem in a class of Carnot-Carathéodory spaces, related to the Heisenberg group. This is the framework of Pansu's conjecture about the shape of isoperimetric sets. Two different approaches are considered. On one hand we describe the isoperimetric problem in Grushin spaces, under a symmetry assumption that depends on the dimension and we provide a classification of isoperimetric sets for special dimensions. On the other hand, we present some results about the isoperimetric problem in a family of Riemannian manifolds approximating the Heisenberg group. In this context we study constant mean curvature surfaces. Inspired by Abresch and Rosenberg techniques on holomorphic quadratic differentials, we classify isoperimetric sets under a topological assumption.

Sunto. Presentiamo risultati recenti ottenuti in [13] e [14] sul problema isoperimetrico in spazi di Carnot-Carathéodory legati al gruppo di Heisenberg. Questo è il contesto della congettura di Pansu sulla forma degli insiemi isoperimetrici. Presentiamo due diversi approcci. Da un lato, descriviamo il problema isoperimetrico negli spazi di Grushin, sotto un'ipotesi di simmetria che dipende dalla dimensione. In questo contesto, forniamo una classificazione degli insiemi isoperimetrici, valida per specifiche dimensioni. Dall'altro lato, presentiamo alcuni risultati sul problema isoperimetrico in una famiglia di varietà Riemanniane che approssimano il gruppo di Heisenberg. In questo contesto, studiamo superfici a curvatura media costante. Ispirati dalle tecniche di Abresch and Rosenberg sui differenziali quadratici olomorfi, classifichiamo gli insiemi isoperimetrici sotto un'ipotesi topologica.

2010 MSC. 53C17, 49Q20.

Keywords. isoperimetric problem, Heisenberg group, H-type groups, Grushin spaces.

Bruno Pini Mathematical Analysis Seminar, Vol. 8 (2017) pp. 102–120

Dipartimento di Matematica, Università di Bologna

ISSN 2240-2829.

1. Introduction

Let M be a n-dimensional manifold, V a volume and P a perimeter on M. The *isoperimetric problem* relative to V and P consists in studying existence, qualitative properties and, if possible, classifying the minimizers of

$$\inf \{P(E),\ E \subset M \text{ is } V\text{-measureable and } V(E) = v\}, \tag{1}$$

for a given volume $v > 0$. Minimizers of (1) are called *isoperimetric sets*, and their existence and classification is equivalent to determine the best constant $C > 0$ in the *isoperimetric inequality*. This consists in finding a number $\nu = \nu(M, P, V) \in \mathbb{N}$ and a constant $C = C(M, P, V) > 0$ such that, for any $E \subset M$ with $V(E) < \infty$, it holds

$$P(E)^\nu \geq CV(E). \tag{2}$$

Isoperimetric inequalities turn out to be useful in a number of problems in geometry and analysis on manifolds (and on more general metric spaces), that ranges from characterizing isometries (see Ahlfors [3]), to the study of heat kernels (see Chavel [7]).

If M is a space form (Euclidean space $\mathbb{R}^n$, n-sphere and n-hyperbolic space), endowed with the Riemannian volume and perimeter, isoperimetric sets are metric balls. We refer *e.g.* to Hurwitz [20] for the first proof in the 2-dimensional Euclidean space, De Giorgi [9] for the most general formulation in $M = \mathbb{R}^n$, and Schmidt [32] for the first proof in the other cases. Moreover, the isoperimetric inequality holds for $\nu = \frac{n}{n-1}$. As we shall see in what follows (see Section 1.1), the isoperimetric property of metric balls fails to hold in more general settings and the exponent ν in the isoperimetric inequality may change.

In this note, we resume some recent results obtained in [13] and [14] on the isoperimetric problem in a class of Carnot-Carathéodory spaces. These are metric structures on $\mathbb{R}^n$, arising from the study of hypoelliptic operators and from control theory. Let $\mathcal{X} = \{X_1, \dots, X_r\}$ be a family of self-adjoint vector fields defined in an open set $\Omega \subset \mathbb{R}^n$. Associated with $\mathcal{X}$, we define the *Carnot-Carathéodory distance* $d_{\mathcal{X}}$ between two points in Ω as the shortest length of *horizontal* curves connecting them, *i.e.*, absolutely continuous curves that are almost everywhere tangent to the distribution of planes generated by $X_1, \dots, X_r$, called the *horizontal distribution*. Under the so-called *bracket generating* *condition*, any two points can be connected by means of horizontal curves (see Chow [8] and Rashevsky [28]), and $d_{\mathcal{X}}$ defines a metric structure on Ω. In this case we say that $\mathcal{X}$ defines a *Carnot-Carathéodory structure* on $\mathbb{R}^n$. The natural notion of perimeter on such a structure, endowed with the Lebesgue measure $\mathcal{L}^n$, has been introduced by Capogna, Danielli and Garofalo in [5], following the De Giorgi definition of perimeter (see also [16], [17]): the *$\mathcal{X}$-perimeter* of a set $E \subset \mathbb{R}^n$, satisfying $\mathcal{L}^n(E) < \infty$ is

$$P_{\mathcal{X}}(E) = \sup\Big\{ \int_E \sum_{i=1}^{r} X_i\varphi_i(x)\,dx \;:\; \varphi \in C^1_c(\mathbb{R}^n;\mathbb{R}^r),\ \max_{x\in\mathbb{R}^n} |\varphi(x)| \le 1\Big\}. \tag{3}$$

Notice that the $\mathcal{X}$-perimeter corresponds to the "metric" perimeter introduced by Miranda in [21] for more general metric spaces.

An important example is given by the Heisenberg perimeter, for which the isoperimetric problem is still unsolved and the famous Pansu's conjecture has not been proved (see Section 1.1 below). In this paper, we study perimeters that are related to the Heisenberg one. Two different approaches are presented. On one hand we present the results obtained in [14] about the isoperimetric problem in *Grushin spaces*, under a symmetry assumption that depends on the dimension, and allows the approach via *rearrangements techniques*, see [24]. On the other hand, we present the results obtained in [13] for a Riemannian approximation of the Heisenberg group. In this context we study constant mean curvature surfaces, inspired by Abresch and Rosenberg techniques on *holomorphic quadratic differentials* [1, 2].

1.1. The Heisenberg isoperimetric problem.

The *Heisenberg group* $\mathbb{H}^1$ is $M = \mathbb{R}^3$ endowed with the following non-commutative Lie group operation:

$$(x,y,t) * (x',y',t') = (x+x', y+y', t+t'+2(x'y-xy'))\,, \quad (x,y,t),\ (x'y't') \in \mathbb{R}^3. \tag{4}$$

We call *left-translations* the mappings $(x,y,t) \mapsto \tau_{(x',y',t')} = (x,y,t) * (x',y',t')$, for $(x',y',t') \in \mathbb{H}^1$. The Heisenberg Lie algebra is $\mathfrak{h} = \mathrm{span}\{X,Y,T\}$, where

$$X(x,y,t) = \partial_x + 2y\partial_t, \quad Y(x,y,t) = \partial_y - 2x\partial_t, \quad T = \partial_t. \tag{5}$$

Since the only nonzero commutator is $[X,Y] = -4T$, the family $\mathcal{X}_H = \{X,Y\}$ satisfies the bracket generating condition at step 2, defining a Carnot-Carathéodory structure on

$\mathbb{R}^3$. We call *H-perimeter* the $\mathcal{X}$-perimeter associated with $\mathcal{X}_H$, and we denote it by P_H. Motivated by the study of quasiregular mappings from $\mathbb{R}^3$ into $\mathbb{H}^1$, Pansu proved in [27] the validity of the isoperimetric inequality (2) for P_H and the Lebesgue measure $\mathcal{L}^3$, with $\nu = \frac{4}{3}$. Notice that $\mathcal{L}^3$ and P_H are homogeneous with respect to the anisotropic dilations

$$(x, y, t) \mapsto \delta_\lambda^H(x, y, t) = (\lambda x, \lambda y, \lambda^2 t), \quad \lambda > 0. \tag{6}$$

In fact, for any measurable set $E \subset \mathbb{R}^3$ and for all $\lambda > 0$ we have $\mathcal{L}^3(\delta_\lambda^H(E)) = \lambda^Q \mathcal{L}^3(E)$ and $P_H(\delta_\lambda^H(E)) = \lambda^{Q-1} P_H(E)$, where $Q = 4$ is the Hausdorff dimension of $\mathbb{H}^1$ computed with respect to the Carnot-Carathéodory distance. Still in [27], Pansu formulated a conjecture about the best constant in (2): *Pansu's conjecture* claims that isoperimetric sets are obtained by rotating a Carnot-Carathéodory geodesic around the vertical axis $\{(0, 0, t), t \in \mathbb{R}\}$. The conjectured isoperimetric sets are then obtained by left-translations and anisotropic dilations of the set

$$\begin{aligned} &E_{\mathrm{isop}}^H = \left\{(z, t) \in \mathbb{H}^1,\ |t| < \varphi_H(|z|),\ |z| < 1\right\}, \\ &\varphi_H(r) = \arccos r + r\sqrt{1 - r^2},\ r \in [0, 1], \end{aligned} \tag{7}$$

where we identify a point $(x, y) \in \mathbb{R}^2$ with $z = x + iy \in \mathbb{C}$ and $|z|$ (resp. $|t|$) is the standard norm of $z \in \mathbb{C}$ (resp. $t \in \mathbb{R}$). Notice that the set E_{isop}^H is not a Carnot-Carathéodory ball in $\mathbb{H}^1$: Monti proved indeed in [22] that metric balls in the Heisenberg group are not isoperimetric sets. Pansu's conjecture has been proved only assuming some regularity (C^2-regularity, convexity) or symmetry, see [6, 12, 13, 14, 23, 26, 29, 30, 31].

1.2. **Grushin spaces.** The isoperimetric problem in Grushin spaces is the subject of Section 2. To define a Grushin space, let $\mathbb{R}^n = \mathbb{R}^h \times \mathbb{R}^k$, where $h, k \geq 1$ are integers and $n = h + k$. Given $\alpha > 0$, we define the following family of vector fields in $\mathbb{R}^n$:

$$\mathcal{X}_\alpha = \{X_1, \ldots, X_h, Y_1, \ldots, Y_k\}, \qquad \begin{aligned} &X_i = \partial_{x_i}, \quad i = 1, \ldots, h, \\ &Y_j = |x|^\alpha \partial_{y_j}, \quad j = 1, \ldots, k, \end{aligned} \tag{8}$$

where $(x, y) \in \mathbb{R}^h \times \mathbb{R}^k$ and $|x|$ is the standard norm of $x \in \mathbb{R}^h$. The family $\mathcal{X}_\alpha$ defines a Carnot-Carathéodory structure on $\mathbb{R}^n$ (even though for non-integers $\alpha > 0$ the bracket generating condition cannot be verified). For $\alpha > 0$, we call *α-perimeter* the $\mathcal{X}$-perimeter associated with the family $\mathcal{X}_\alpha$, and we denote it by P_α. Notice that $\mathcal{L}^n$ and P_α are homogeneous with respect to the anisotropic dilations

$$(x,y) \mapsto \delta_\lambda^\alpha(x,y) = (\lambda x, \lambda^{\alpha+1} y), \quad \lambda > 0. \tag{9}$$

Namely, for any $\lambda > 0$ and any measurable set $E \subset \mathbb{R}^2$ we have $\mathcal{L}^n(\delta_\lambda^\alpha(E)) = \lambda^Q \mathcal{L}^n(E)$ and $P_\alpha(\delta_\lambda^\alpha(E)) = \lambda^{Q-1}$, where $Q = h + (\alpha+1)k$ is the Hausdorff dimension of the Grushin structure, computed with respect to the Carnot-Carathéodory distance.

We study the isoperimetric problem in Grushin spaces under a symmetry assumption that depends on the dimension. We say that a set $E \subset \mathbb{R}^h \times \mathbb{R}^k$ is *x-spherically symmetric* if there exists a set $F \subset \mathbb{R}^+ \times \mathbb{R}^k$, called *generating set of E*, such that

$$E = \left\{(x,y) \in \mathbb{R}^n : (|x|, y) \in F\right\}. \tag{10}$$

We denote by $\mathcal{S}_x$ the class of $\mathcal{L}^n$-measurable, x-spherically symmetric sets. If $h \geq 2$, we consider the isoperimetric problem in the class of x-spherically symmetric sets, see (11) below. In the sequel, by a *vertical translation* we mean a mapping of the form $(x,y) \mapsto (x, y + y_0)$ for some $y_0 \in \mathbb{R}^k$. One of the main results in [14] is the following.

Theorem 1.1. *Let $\alpha > 0$, $h, k \geq 1$, and $n = h + k$. The isoperimetric problem*

$$\inf\left\{P_\alpha(E) : E \subset \mathbb{R}^n \ \mathcal{L}^n\text{-measurable with } 0 < \mathcal{L}^n(E) < \infty \text{ and } E \in \mathcal{S}_x, \text{ if } h \geq 2\right\} \tag{11}$$

has solutions and, up to a vertical translation and a null set, any isoperimetric set $E \subset \mathbb{R}^n$ is of the form

$$E = \left\{(x,y) \in \mathbb{R}^n : |y| < f(|x|)\right\}, \tag{12}$$

for a function $f \in C([0,r_0]) \cap C^1([0,r_0)) \cap C^\infty(0,r_0)$, with $0 < r_0 < \infty$, satisfying $f(r_0) = 0$, $f' \leq 0$ on $(0,r_0)$, and solving the following equation

$$\frac{f'(r)}{\sqrt{r^{2\alpha} + f'(r)^2}} = \frac{k-1}{r^{h-1}} \int_0^r \frac{s^{2\alpha+h-1}}{f(s)\sqrt{s^{2\alpha} + f'(s)^2}} ds - \frac{\kappa}{h} r, \quad \text{for } r \in (0, r_0), \tag{13}$$

with $\kappa = \frac{(Q-1)P_\alpha(E)}{Q\mathcal{L}^n(E)}$.

Notice that, when $h = 1$ no symmetry assumptions are considered and we deduce from formula (12) that isoperimetric sets are x-symmetric. Moreover, in the special case $k = 1$, equation (13) can be integrated (see Remark 2.1), leading to the classification of all isoperimetric sets as vertical translations and dilations of

$$\begin{aligned} E^\alpha_{\text{isop}} &= \{(x,y) \in \mathbb{R}^n : |y| < \varphi_\alpha(|x|)\}\,, \\ \varphi_\alpha(r) &= \int_{\arcsin r}^{\pi/2} \sin^{\alpha+1}(s)\,ds,\ \ r \in [0,1]. \end{aligned} \tag{14}$$

Formula (14) generalizes to dimensions $h \geq 2$ the results of [25]. When $k = 1$ and $\alpha = 1$, the profile function satisfying the final condition $\varphi_1(1) = 0$ is $\varphi_1(r) = \frac{1}{2}\big(\arccos(r) + r\sqrt{1-r^2}\big)$, $r \in [0,1]$. This is the profile function φ_H of the conjectured isoperimetric set in the Heisenberg group, defined in (7). In fact, in Section 2.4 we show that the isoperimetric problem (11) for $\alpha = 1$ is equivalent to the isoperimetric problem in the class of spherically symmetric sets in H-type groups, that are Carnot groups generalizing the Heisenberg group. Hence Theorem 1.1 is a generalization of [23, Theorem 1.2].

1.3. **The Riemannian Heisenberg isoperimetric problem.** In Section 3 we study the isoperimetric problem in a family of Riemannian approximations of $\mathbb{H}^1$. Given two parameters $\varepsilon > 0$ and $\sigma \neq 0$, consider the 3-dimensional Lie algebra $\mathfrak{h}_\varepsilon$ generated by the following family of vector fields defined on $\mathbb{R}^3$: $\mathcal{X}_\varepsilon = \{X_\varepsilon, Y_\varepsilon, T_\varepsilon\}$, with

$$X_\varepsilon = \frac{1}{\varepsilon}\left(\partial_x + \sigma y \partial_t\right), \quad Y_\varepsilon = \frac{1}{\varepsilon}\left(\partial_y - \sigma x \partial_t\right), \quad \text{and} \quad T_\varepsilon = \varepsilon^2 \partial_t, \tag{15}$$

where $(x,y) \in \mathbb{R}^2$ is identified with $z = x + iy \in \mathbb{C}$. We endow $\mathbb{R}^3$ with a metric structure by defining a scalar product $\langle\cdot,\cdot\rangle_\varepsilon$ on $\mathfrak{h}_\varepsilon$ that makes $X_\varepsilon, Y_\varepsilon, T_\varepsilon$ orthonormal, and we extend it to a Riemannian metric $g_\varepsilon = \langle\cdot,\cdot\rangle$ in $\mathbb{R}^3$, which is left-invariant with respect to the operation of the underlying Lie group $\mathbb{H}^1_\varepsilon$. When $\varepsilon = 1$ and $\sigma \to 0$, $\mathbb{H}^1_\varepsilon$ converges to the Euclidean space. When $\sigma \neq 0$ and $\varepsilon \to 0^+$, then $\mathbb{H}^1_\varepsilon$ endowed with the distance function induced by the rescaled metric $\varepsilon^{-2}\langle\cdot,\cdot\rangle$ converges to the Heisenberg group $\mathbb{H}^1$. The Riemannian volume of $(\mathbb{H}^1_\varepsilon, g_\varepsilon)$ is the Lebesgue measure $\mathcal{L}^3$, independently of ε and σ. The Riemannian perimeter of $(\mathbb{H}^1_\varepsilon, g_\varepsilon)$ is the $\mathcal{X}$-perimeter associated with the family $\mathcal{X}_\varepsilon$, and we denote it by P_ε. For $\sigma \neq 0$ and $\varepsilon \to 0^+$ the rescaled functional $\varepsilon P_\varepsilon$ Γ-converges to the H-perimeter. This implies that, for $\sigma \neq 0$ and $\varepsilon \to 0^+$, solutions to the isoperimetric problem

$$\inf \left\{ P_\varepsilon(E) : E \subset \mathbb{H}^1_\varepsilon,\ \mathcal{L}^3\text{-measurable, with } 0 < \mathcal{L}^3(E) < \infty \right\}, \tag{16}$$

converge to solutions of the Heisenberg isoperimetric problem. We study the isoperimetric problem (16) under a *topological assumption* on minimizers.

Theorem 1.2. *If $\Sigma \subset \mathbb{H}^1_\varepsilon$ is an embedded topological sphere and it is the boundary of an isoperimetric set for* (16)*, then there exists $R > 0$ such that, up to left-translations, $\Sigma = \Sigma_R$, where*

$$\Sigma_R = \{(z,t) \in \mathbb{H}^1_\varepsilon : |t| = \varphi_\varepsilon(|z|; R),\ |z| < R\} \quad \textit{and,} \quad \textit{for } r \in [0, R] \tag{17}$$

$$\varphi_\varepsilon(r; R) = \frac{1}{2\sigma\varepsilon^4}\left[(\varepsilon^2 + \sigma^2 R^2)\arctan\left(\frac{\sigma}{\varepsilon}\sqrt{\frac{R^2 - r^2}{\varepsilon^4 + \sigma^2 r^2}}\right) + \varepsilon\sigma\sqrt{(R^2 - r^2)(\varepsilon^4 + \sigma^2 r^2)}\right].$$

When $\varepsilon = 1$ and $\sigma \to 0$, the spheres Σ_R converge to the standard spheres of the Euclidean space. When $\sigma \neq 0$ and $\varepsilon \to 0^+$, the spheres Σ_R converge to the boundary of the Pansu's bubble (7). We conjecture that, within its volume class and up to left translations, the sphere Σ_R is the unique solution of the isoperimetric problem in $\mathbb{H}^1_\varepsilon$.

2. Isoperimetrc problem in Grushin spaces

The aim of this Section is to describe the steps followed in [14] to prove Theorem 1.1.

2.1. **Rearrangements for the α-perimeter.** In this Section we describe how to rearrange sets in the minimization class for the isoperimetric problem (11), in order to decrease the α-perimeter gaining additional symmetries. This will allow us to prove existence of isoperimetric sets and deduce their *a-priori* regularity.

Theorem 2.1 (Rearrangement for the α-perimeter)**.** *Let $h, k \geq 1$, $n = h + k$, $\alpha > 0$. Let $E \subset \mathbb{R}^n$ be $\mathcal{L}^n$-measurable, and assume $E \in \mathcal{S}_x$ if $h \geq 2$. Then there exists a decreasing function $f : I \to [0, +\infty)$ defined on a real interval $I = [0, r_0)$ such that the set*

$$E^* = \{(x, y) \in \mathbb{R}^n : |y| < f(|x|)\} \tag{18}$$

satisfies

$$P_\alpha(E^*) \le P_\alpha(E), \quad \mathcal{L}^n(E^*) = \mathcal{L}^n(E). \tag{19}$$

Moreover, if E is isoperimetric and $P_\alpha(E^*) = P_\alpha(E)$, then $E = E^*$.

The proof makes use of two different techniques depending on whether $h = 1$ or $h \ge 2$.

2.1.1. **The case $h = 1$**. We use Steiner and Schwarz rearrangements, combined with a convenient change of coordinates that we describe below and that transforms the α-perimeter into the standard perimeter (see [25] for the planar case $h = k = 1$). Let $n = 1 + k$ and consider the mappings

$$\Phi, \Psi : \mathbb{R}^n \to \mathbb{R}^n, \quad \Psi(x,y) = \Big(\operatorname{sgn}(x)\frac{|x|^{\alpha+1}}{\alpha+1}, y\Big), \quad \Phi(\xi,\eta) = \Big(\operatorname{sgn}(\xi)|(\alpha+1)\xi|^{\frac{1}{\alpha+1}}, \eta\Big).$$

Then we have $\Phi \circ \Psi = \Psi \circ \Phi = \mathrm{Id}_{\mathbb{R}^n}$. In [14, Proposition 2.5] we prove that, for any measurable set $E \subset \mathbb{R}^n$, we have

$$P_\alpha(E) = P(\Psi(E)), \tag{20}$$

where P denotes the standard Euclidean perimeter.

Sketch of the proof of Theorem 2.1 in the case $h = 1$. By (20) the set $F = \Psi(E) \subset \mathbb{R}^n$ satisfies $P(F) = P_\alpha(E)$. Moreover, the measure μ on $\mathbb{R}^n$ defined by

$$\mu(F) = \int_F |(\alpha+1)\xi|^{-\frac{\alpha}{\alpha+1}}\, d\xi d\eta$$

satisfies $\mu(F) = \mathcal{L}^n(E)$. First, we rearrange the set F using Steiner symmetrization in direction ξ. Namely, we let $F_1 = \{(\xi,\eta) \in \mathbb{R}^n : |\xi| < \mathcal{L}^1(F^\eta)/2\}$, where $F^\eta = \{\xi \in \mathbb{R} : (\xi,\eta) \in F\}$. We have $P(F_1) \le P(F)$ and the equality $P(F_1) = P(F)$ implies that a.e. section F^η is equivalent to an interval. Moreover, it can be proved that $\mu(F_1) \ge \mu(F)$ and $\mu(F_1) = \mu(F)$ implies that a.e. section F^η is a symmetric interval centered at 0. Then, we let F_2 be the Schwarz symmetrization of F_1 in $\mathbb{R}^k$, obtaining $P(F_2) \le P(F_1)$. Equality implies that a.e. section $F_1^\xi = \{\eta \in \mathbb{R}^k : (\xi,\eta) \in F^1\}$ is an Euclidean ball. Moreover $\mu(F_2) = \mu(F_1)$. In conclusion, the set $E^* = \delta^\alpha_\lambda(\Phi(F_2))$ with $\lambda > 0$ such that $\mathcal{L}^n(E^*) = \mathcal{L}^n(E)$, ($\delta^\alpha_\lambda$ defined in (9)) satisfies (19).

For the second part of the statement, notice that from the equality $P_\alpha(E^*) = P_\alpha(E)$ and the characterization of the equality cases that we recorded above, we deduce that existence of two functions $f : [0,\infty) \to [0,\infty]$ and $c : [0,\infty) \to \mathbb{R}^k$ such that

$$E = \{(x, y) \in \mathbb{R}^n : |y - c(|x|)| < f(|x|)\}.$$

In the case that E is isoperimetric, it can be also deduced that c is a constant function, concluding the proof (see [14, Proposition 5.4]). $\square$

2.1.2. **The case** $h \geq 2$. We describe how to prove Theorem 2.1 in the case when $h \geq 2$ and $E \subset \mathbb{R}^n$ is already x-spherically symmetric. The reasoning starts from the validity of the reduction formula (21) below, that allows us to use a rearrangement introduced in [24] to obtain formula (18) for isoperimetric sets.

Proposition 2.1 (Reduction formula for the α-perimeter of x-spherically symmetric sets)**.** *Let $E \subset \mathbb{R}^n$ be a bounded open set with finite α-perimeter that is x-spherically symmetric with generating set $F \subset \mathbb{R}^+ \times \mathbb{R}^k$. Then we have:*

$$P_\alpha(E) = h\omega_h \sup_{\psi \in \mathcal{F}_{1+k}(\mathbb{R}^+\times\mathbb{R}^k)} \int_F \Big(\partial_r\big(r^{h-1}\psi_1\big) + r^{h-1+\alpha} \sum_{j=1}^k \partial_{y_j}\psi_{1+j}\Big) dr dy, \tag{21}$$

where $\omega_h = \mathcal{L}^h(\{x \in \mathbb{R}^h : |x| < 1\})$, and

$$\mathcal{F}_{1+k}(\mathbb{R}^+ \times \mathbb{R}^k) = \left\{\psi \in C_c^1(\mathbb{R}^+ \times \mathbb{R}^k; \mathbb{R}^{1+k}) : \max_{(x,y)\in\mathbb{R}^+\times\mathbb{R}^k} |\psi(x, y)| \leq 1\right\}.$$

With this tool in hand, we are ready to complete the proof of Theorem 2.1.

Sketch of the proof of Theorem 2.1 in the case $h \geq 2$. Let $F \subset \mathbb{R}^+ \times \mathbb{R}$ be the generating set of E. Then $P_\alpha(E) = h\omega_h\mathcal{Q}(F)$, where $\mathcal{Q}$ is the perimeter functional defined by (21). In [14, Theorem 3.2] we refine [24, Theorem 1.5] to define a set

$$F^\sharp = \big\{(r, y) \in \mathbb{R}^+ \times \mathbb{R}^k : 0 < r < g(y)\big\},$$

for a suitable function g such that $\mathcal{Q}(F^\sharp) \leq Q(F)$ and $V(F^\sharp) \geq V(F)$, with equality $V(F^\sharp) = V(F)$ holding if and only if $F^\sharp = F$ (up to a negligible set), where V

$$V(F) = \omega_h \int_F r^{h-1} dr dy = \mathcal{L}^n(E).$$

We then consider the set generated by $F^\sharp$ in $\mathbb{R}^n$, $E_1^\sharp = \big\{(x,y) \in \mathbb{R}^n : (|x|, y) \in F^\sharp\big\}$, and define $E^\sharp = \delta_\lambda^\alpha(E_1^\sharp)$ where $\lambda > 0$ is chosen to get $\mathcal{L}^n(E^\sharp) = \mathcal{L}^n(E)$. Then the statements of theorem follow by applying a Schwarz rearrangement in the variable $y \in \mathbb{R}^k$ to $E^\sharp$. □

2.2. **Existence of isoperimetric sets.** As a consequence of [17], the isoperimetric inequality (2) is valid for $\nu = \frac{Q}{Q-1}$, where $Q = h + (\alpha+1)k$ is the homogeneous dimension of the Grushin space. By the homogeneity properties of Lebesgue measure and α-perimeter, we can thus define the best constant

$$C_I = \inf\{P(E) : \mathcal{L}^n(E) = 1 \text{ and } E \in \mathcal{S}_x, \text{ if } h \geq 2\} > 0. \tag{22}$$

To prove existence of isoperimetric sets we are then reduced to prove existence of minimizers to (22).

Theorem 2.2. *Let $h, k \geq 1$ and $n = h + k$. There exists $E \subset \mathbb{R}^n$ realizing the infimum in* (22) *such that $E = \{(x,y) \in \mathbb{R}^n : |y| < f(|x|)\}$ for a decreasing function $f : I \to [0, \infty]$ defined on a bounded interval $I \subset R$.*

The theorem is proved by the *direct method of the Calculus of Variations*, using the lower semi-continuity of the α-perimeter and the compactness Theorem for sets of finite $\mathcal{X}$-perimeter, see [16]. In the setting of Carnot groups, existence of isoperimetric sets has been proved by Leonardi and Rigot in [19], exploiting the invariance under left translations of the $\mathcal{X}$-perimeter. Their argument cannot be applied to Grushin spaces, due to the lack of invariance under translations of P_α (Grushin spaces are not Carnot groups, see [11, Remark 1.1.12] and references therein) and we introduce a suitably adapted *concentration-compactness* type argument. The idea is to take a minimizing sequence $\{E_m\}_{m\in\mathbb{N}}$ for (22), assumed to satisfy (18) by Theorem 2.1, and perform on it the following type of "cuts" $E_m' = \{(x,y) \in E_m : |x| < x_0\}$ or $E_m'' = \{(x,y) \in E_m : |y| < y_0\}$. Together with a suitable choice of dilations, this will give us a bounded sequence $\{\tilde{E}_m\}_{m\in\mathbb{N}}$ with $P_\alpha(\tilde{E}_m) \leq P_\alpha(E_m)$ to which we apply the direct method.

2.3. **Profile of isoperimetric sets.** By Theorem 2.2 and 2.1, isoperimetric sets exist and can be always written in the form (18), *i.e.,*

$$E = \{(x,y) \in \mathbb{R}^n : |y| < f(|x|)\}, \tag{23}$$

for a decreasing function f. The monotonicity of the profile function $f : [0, r_0] \to [0, \infty)$ is enough to deduce first its Lipschitz continuity, and then its smoothness. In particular, ∂E is rectifiable and $P(E)$ can be computed using the following representation formula (24).

Proposition 2.2 (Representation formula for P_α)**.** *Let $E \subset \mathbb{R}^n$ be a bounded open set with rectifiable boundary. For any $\alpha > 0$, define the α-normal to ∂E to be the mapping $N_\alpha^E : \partial E \to \mathbb{R}^n$, $N_\alpha^E = (N_x^E, |x|^\alpha N_y^E)$, where the Euclidean outer unit normal $N^E : \partial E \to \mathbb{R}^n$ has been split as $N^E = (N_x^E, N_y^E)$ with $N_x^E \in \mathbb{R}^h$ and $N_y^E \in \mathbb{R}^k$. Then the α-perimeter of E in $\mathbb{R}^n$ is*

$$P_\alpha(E) = \int_{\partial E} |N_\alpha^E(x,y)|\, d\mathcal{H}^{n-1}, \tag{24}$$

where $\mathcal{H}^{n-1}$ denotes the standard $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$.

Thanks to (24), we can then perform a first-variation argument on the isoperimetric set $E = \{(x,y) \in \mathbb{R}^n : |y| < f(|x|)\}$: for any $\varepsilon > 0$ and $\psi \in C_c^\infty(0, r_0)$, let $E_\varepsilon = \{(x,y) \in \mathbb{R}^n : |y| < f(|x|) + \varepsilon\psi(|x|)\}$. Then, by minimality of E and homogeneity of P_α and $\mathcal{L}^n$, we have

$$\frac{d}{d\varepsilon}\left(\frac{P_\alpha(E_\varepsilon)^Q}{\mathcal{L}^n(E_\varepsilon)^{Q-1}}\right)\bigg|_{\varepsilon=0} = 0.$$

This leads to the differential equation (13) for $f \in C^\infty(0, r_0)$:

$$\frac{f'(r)}{\sqrt{r^{2\alpha} + f'(r)^2}} = \frac{k-1}{r^{h-1}} \int_0^r \frac{s^{2\alpha+h-1}}{f(s)\sqrt{s^{2\alpha} + f'(s)^2}} ds - \frac{\kappa}{h} r, \quad \text{for } r \in (0, r_0),$$

with $\kappa = \frac{(Q-1)P_\alpha(E)}{Q\mathcal{L}^n(E)}$.

Remark 2.1 (Computation of the solution when $k = 1$)**.** *When $k = 1$, equation* (13) *reads*

$$\frac{f'}{\sqrt{r^{2\alpha} + f'^2}} = -\frac{\kappa}{h} r,$$

and this is equivalent to

$$f'(r) = -\frac{\kappa r^{\alpha+1}}{\sqrt{h^2 - C_{hk\alpha}^2 r^2}}, \quad r \in [0, r_0). \tag{25}$$

Without loss of generality we assume $r_0 = 1$*. Integrating* (25) *with* $f(1) = 0$ *we obtain the solution*

$$f(r) = \int_r^1 \frac{s^{\alpha+1}}{\sqrt{1-s^2}}\, ds = \int_{\arcsin r}^{\pi/2} \sin^{\alpha+1}(s)\, ds.$$

This is the profile function for the isoperimetric set when $k = 1$ *in* (14)*.*

We also obtain the following initial and final conditions for the profile function of isoperimetric sets. Conditions (26) imply that the profile function touches the y-axis with infinite derivative, while (27) gives an asymptotic behavior of f around 0 that in turn implies global C^1-smoothness of ∂E and concavity of f around $r = 0$, see [11, Section 2.6.1].

Proposition 2.3. *Let* $E = \{(x, y) \in \mathbb{R}^n : |y| < f(|x|)\}$ *be an isoperimetric set, with* $f \in C^\infty(0, r_0)$ *for* $0 < r_0 < \infty$ *and* f *decreasing. Then*

$$f(r_0) = 0, \quad \lim_{r \to r_0^-} f'(r) = -\infty \tag{26}$$

and

$$\lim_{r \to 0^+} \frac{f'(r)}{r^{\alpha+1}} = -\frac{C_{hk\alpha}}{h}. \tag{27}$$

2.4. **Relations with the Heisenberg isoperimetric problem.** Still in [14] we consider a class of Carnot groups that generalizes the Heisenberg group $\mathbb{H}^1$, called *H-type groups.* An H-type Lie algebra is a stratified nilpotent real Lie algebra $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ of dimension $n \geq 3$ and step 2 (so that $\mathfrak{h}_2 = [\mathfrak{h}_1, \mathfrak{h}_1]$), satisfying some properties that in particular imply $\mathfrak{h}_1$ to be even dimensional (for a precise description see [4, Ch. 18]). We can thus identify the underlying Lie group with $\mathbb{R}^h \times \mathbb{R}^k$, where $h = \dim(\mathfrak{h}_1) \geq 2$ and $k = \dim(\mathfrak{h}_2) \geq 1$. As an example, the N-dimensional Heisenberg group is the only H-type group for the parameters $h = 2N$ and $k = 1$. The perimeter on an H-type group (still denoted P_H) can be defined as the $\mathcal{X}$-perimeter associated with an orthonormal basis of $\mathfrak{h}_1$.

Using the representation formula (24) and the analogous one for the H-perimeter in H-type groups (see [14, Proposition 2.1]), the following statement can be proved:

Proposition 2.4. *Let $h \geq 2, k \geq 1$ be the dimensions of the two layers of an H-type group. Let $n = h + k$ and denote points as $(x, y) \in \mathbb{R}^h \times \mathbb{R}^k$. Then, for any x-spherically symmetric set $E \in \mathcal{S}_x$ there holds $P_H(E) = P_\alpha(E)$ with $\alpha = 1$.*

Thanks to Proposition 2.4, the study of the isoperimetric problem in H-type groups in the class of x-spherically symmetric sets is included in the study of the isoperimetric problem (11). In particular, Remark 2.1 can be thought of as a generalization of [23].

3. Isoperimetric problem in the Riemannian Heisenberg group

In this Section we first describe the spheres Σ_R introduced in Theorem 1.2, and then present the main steps to prove it.

3.1. **The spheres Σ_R.** For $\varepsilon > 0$ and $\sigma \neq 0$, we consider $\mathbb{R}^3$ with the Riemannian metric g_ε defined in Section 1.3, left invariant with respect to the translations of the Lie group $\mathbb{H}^1_\varepsilon$. We denote points in $\mathbb{H}^1_\varepsilon$ by $(z, t) \in \mathbb{C} \times \mathbb{R}$. Given $R > 0$, the sphere Σ_R defined in (17) can be obtained as the boundary of the unique symmetric minimizer for the Riemannian perimeter P_ε under volume constraint, and such that $\Sigma_R \cap \{t = 0\} = \{|z| = R\}$. We resume it in the next proposition. Formula (28) has been first computed by Tomter [33].

Proposition 3.1. *For any $R > 0$ there exists a unique compact smooth embedded surface $\Sigma_R \subset \mathbb{H}^1$ that is area stationary under volume constraint and such that*

$$\Sigma_R = \{(z, t) \in \mathbb{H}^1 : |t| = f(|z|; R)\}$$

for a function $f(\cdot; R) \in C^\infty([0, R))$ continuous at $r = R$ with $f(R) = 0$. Namely, for any $0 \leq r \leq R$ the function is given by

$$\varphi_\varepsilon(r; R) = \frac{1}{2\sigma\varepsilon^4}\Big[(\varepsilon^2 + \sigma^2 R^2)\arctan\Big(\frac{\sigma}{\varepsilon}\sqrt{\frac{R^2 - r^2}{\varepsilon^4 + \sigma^2 r^2}}\Big) + \varepsilon\sigma\sqrt{(R^2 - r^2)(\varepsilon^4 + \sigma^2 r^2)}\Big]. \tag{28}$$

If Σ is the boundary of an isoperimetric set in $\mathbb{H}^1_\varepsilon$, then it has constant mean curvature (CMC) with respect to g_ε. Computing the second fundamental form of Σ_R with respect to a suitable frame, it can be shown that the sphere Σ_R has indeed constant mean curvature

$$H = \frac{1}{\varepsilon R}.$$

In the following we present two nice properties of the spheres Σ_R that underline their relation both with the Euclidean isoperimetric problem and with Pansu's conjecture.

3.1.1. *Limits of* Σ_R. The function $\varphi_\varepsilon(\cdot; R) = \varphi_\varepsilon(\cdot; R; \sigma)$ depends also on the parameter σ, that we omitted from our notation. With $\varepsilon = 1$, we find

$$\lim_{\sigma\to 0} \varphi_1(r; R; \sigma) = \sqrt{R^2 - r^2}.$$

Namely, when $\sigma \to 0$, the spheres Σ_R converge to Euclidean spheres with radius $R > 0$ in the three-dimensional space. Moreover

$$\lim_{\varepsilon\to 0} \varphi_\varepsilon(r; R; \sigma) = \frac{\sigma}{2}\left[R^2 \arccos\left(\frac{r}{R}\right) + r\sqrt{R^2 - r^2}\right],$$

which gives the profile function of the Pansu's bubble (7), with $R = 1$ and $\sigma = 2$.

3.1.2. *Foliation by geodesics.* Recall that the Pansu's bubble is defined as the set obtained by rotating a sR geodesic around the t-axis. We present how to recover this property in the Riemannian approximation $\mathbb{H}^1_\varepsilon$. First of all the following foliation property is needed.

Proposition 3.2. *For any nonzero* $(z, t) \in \mathbb{H}^1$ *there exists a unique* $R > 0$ *such that* $(z, t) \in \Sigma_R$.

We are then allowed to define a vector field $\mathcal{N}$ on $\mathbb{H}^1 \setminus \{0\}$ such that $\mathcal{N}(z, t)$ is the exterior unit normal to Σ_R at (z, t). The vector field

$$\mathcal{M}(z, t) = \operatorname{sgn}(t)\frac{\nabla_{\mathcal{N}}\mathcal{N}}{|\nabla_{\mathcal{N}}\mathcal{N}|} \tag{29}$$

is well-defined and smooth outside the center of $\mathbb{H}^1_\varepsilon$. The next theorem shows that the integral lines of $\mathcal{M}$ are Riemannian geodesics of Σ_R.

Theorem 3.1. *Let* $\Sigma_R \subset \mathbb{H}^1_\varepsilon$ *be the CMC sphere with mean curvature* $H > 0$*. Then the integral curves of* $\mathcal{M}$ *are Riemannian geodesics of* Σ_R *joining the north pole* N *to the south pole* S*.*

Remark 3.1 (Limits of the foliation of Theorem 3.1)**.** *If* $\varepsilon = 1$ *and* $\sigma \to 0$*,* $\mathcal{M}$ *tends to the vector field tangent to the* meridians *of the round sphere. On the other hand, for*

$\sigma \neq 0$, $\varepsilon \to 0^+$, let $\bar{\mathcal{M}} = \lim_{\varepsilon\to 0} \varepsilon\mathcal{M}$. Then, denoting by J the complex structure, we have

$$\nabla_{\bar{\mathcal{M}}}\bar{\mathcal{M}} = \frac{2}{R} J(\bar{\mathcal{M}}),$$

which is the equation for geodesics *in $\mathbb{H}^1$ for the Carnot-Carathéodory distance $d_{\mathcal{X}_H}$. Hence, the foliation by geodesics of the Pansu's sphere can be recovered in the Riemannian approximation $\mathbb{H}^1_\varepsilon$ by the integral lines of $\mathcal{M}$.*

3.2. **Proof of Theorem 1.2: a characterization of CMC topological spheres.**

Theorem 1.2 is a corollary of the next Theorem, see [13, Theorem 5.9].

Theorem 3.2. *If $\Sigma \subset \mathbb{H}^1_\varepsilon$ is an embedded topological sphere with CMC, then, up to left-translations, we have $\Sigma = \Sigma_R$, for some $R > 0$.*

In 3-dimensional homogeneous manifolds with 4-dimensional isometry group there is a theory of the so-called *holomorphic quadratic differentials*, see [1], [2], and [10]. One of the main outcomes of this theory is a result announced in [2]: CMC spheres have rotational symmetry. The ideas are based on the Hopf's proof of the following theorem, see [18].

Theorem 3.3 (Hopf, 1955)**.** *Let $\Sigma \subset \mathbb{R}^3$ be a closed surface of genus 0 with CMC with respect to the Euclidean metric. Then Σ is a Euclidean sphere.*

Proof of Theorem 3.3. The proof is divided into three main steps.
Step 1. First of all a differential characterization of Euclidean spheres is provided. Namely, Σ is a Euclidean sphere, if and only if $h_0 \equiv 0$, where h_0 is the traceless part of the second fundamental form

$$h = \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

Namely,

$$h_0 = \begin{pmatrix} \frac{L-N}{2} & M \\ M & -\frac{L-N}{2} \end{pmatrix}.$$

In fact, the complex function $\tilde{h}_0 = \dfrac{L-N}{2} - iM$ satisfies $2|\tilde{h}_0| = |k_1 - k_2|$, where k_1, k_2 are the principal curvatures.

Step 2. Assume $\Sigma = F(D)$, for an open set $D \subset \mathbb{C}$, and a smooth function $F : D \to \mathbb{R}^3$ that constitutes a (local) conformal parametrization of Σ. Then we write the second fundamental form of Σ as a function of the complex variable $z \in \mathbb{C}$, $h = h(z)$, $z = x + iy \in D \subset \mathbb{C}$. *Codazzi equations* for the derivatives of the mean curvature of Σ (which vanishes by assumption) and its second fundamental form, imply:

$$\partial_x \left(\frac{L - N}{2} \right)(z) + \partial_y M(z) = 0 \tag{30}$$

$$\partial_y \left(\frac{L - N}{2} \right)(z) - \partial_x M(z) = 0 \tag{31}$$

Notice that (30) and (31) are the Cauchy-Riemann (CR) equations for $\tilde{h}_0$, and imply its analiticity. This is the reason of the name "*holomorphic quadratic differential*", which is referred to the traceless part of the second fundamental form h_0.

Step 3. Since Σ is a topological sphere we can use two conformal parametrizations of Σ without the north pole and the south pole respectively. In this case, the analyticity ensured by Step 2, together with *Liouville's theorem* implies $h_0 \equiv 0$. By Step 1, this is equivalent to say that Σ is a Euclidean sphere. □

3.2.1. *Holomorphic quaratic differential for* $\mathbb{H}^1_\varepsilon$. We present a proof of Theorem 3.2 that follow by the definition of a suitable *holomorphic quadratic differential in* $\mathbb{H}^1_\varepsilon$, see formula (32). We start by noticing that in $\mathbb{H}^1_\varepsilon$ there are two main differences with respect to Hopf's proof in the Euclidean case:

- Σ_R are not umbilical, *i.e.*, there exist $p \in \Sigma_R$ such that $k_1(p) \neq k_2(p)$.
- Codazzi equations are more complicated and don't imply analyticity of h_0.

These two ingredients allow us to define the right object playing the role of h_0 in the proof of Hopf's Theorem 3.3.

Sketch of the proof of Theorem 3.2. We divide the proof in three steps.

Step 1. We characterize the spheres Σ_R as the only surfaces having CMC $H = 1/\varepsilon R$, and such that $k_0 \equiv 0$ where k is a quadratic form defined as a perturbation of the second fundamental form h. In particular,

$$k = h + b, \quad b = \frac{2\sigma^2}{\varepsilon^4\sqrt{\varepsilon^8 H^2 + \sigma^2}} q_H \circ (\theta \otimes \theta) \circ q_H^{-1}, \tag{32}$$

where θ is the contact form

$$\theta(V) = \langle V, T_\varepsilon \rangle \quad \text{for any } V \in \Gamma(T\mathbb{H}^1_\varepsilon),$$

and q_H is a rotation on every tangent plane of a given angle depending on H. This is the *holomorphic quadratic differential* of $\mathbb{H}^1_\varepsilon$. Formula (32) coincides, up to the sign, with the formula computed in [10] (page 5).

Step 2. Given a CMC surface Σ, we deduce analyticity of k_0 from the *Codazzi equations* written in this context. Indeed, as in the Euclidean case, they imply Cauchy-Riemann equations for k_0.

Step 3. With the same reasoning of the Euclidean case, we conclude that k_0 vanishes identically on Σ thanks to the assumption that Σ is a topological sphere. By Step 1, this is equivalent to say that $\Sigma = \Sigma_R$ for some $R > 0$.

References

[1] U. Abresch, H. Rosenberg. *A Hopf differential for constant mean curvature surfaces in $S^2 \times R$ and $H^2 \times R$.* Acta Math. **193** (2004), no. 2, 141–174.

[2] U. Abresch, R. Rosenberg. *Generalized Hopf differential.* Mat. Contemp. **28**, (2005) 1–28.

[3] L. Ahlfors. *Zur Theorie der Überlagerungsflächen.* Acta Math. **65** (1935), no. 1, 157–194.

[4] A. Bonfiglioli, E. Lanconelli, F. Uguzzoni.Lie groups and potential theory for their sub-Laplacians. Springer Monographs in Mathematics. Springer, Berlin, 2007. xxvi+800 pp.

[5] L. Capogna, D. Danielli, N. Garofalo, *The geometric Sobolev embedding for vector fields and the isoperimetric inequality.* Comm. Anal. Geom. **2** (1994), no. 2, 203–215.

[6] L. Capogna, D. Danielli, S. Pauls, J. Tyson. An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem. Progress in Mathematics, 259. Birkhuser Verlag, Basel, 2007. xvi+223

[7] I. Chavel. Isoperimetric inequalities. Differential geometric and analytic perspectives. Cambridge Tracts in Mathematics, 145. Cambridge University Press, Cambridge, 2001. xii+268 pp.

[8] W. L. Chow. *Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung.* Math. Ann. **117** (1939), 98–105.

[9] E. De Giorgi. *Sulla proprietà isoperimetrica dell'ipersfera, nella classe degli insiemi aventi frontiera orientata di misura finita.* Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I (8) **5** (1958) 33–44.

[10] I. Fernandez, P. Mira. *A characterization of constant mean surfaces in homogeneous 3-manifolds.* Differential Geometry Appl. **15(3)** (2007), 281–289.

[11] V. Franceschi. *Sharp and quantitative isoperimetric inequalities in Carnot-Carathéodory spaces*, Ph.D. Thesis, (2016).

[12] V. Franceschi, G. P. Leonardi, R. Monti. *Quantitative isoperimetric inequalities in* $\mathbb{H}^n$. Calc. Var. Partial Differential Equations **54** (2015), no. 3, 3229–3239.

[13] V. Franceschi, F. Montefalcone, R. Monti. *CMC spheres in the Heisenberg group*, ArXiv e-prints 1611.09215.

[14] V. Franceschi, R. Monti. *Isoperimetric problem in* H*-type groups and Grushin spaces.* Rev. Mat. Iberoam. **32** (2016), no. 4, 1227–1258.

[15] B. Franchi, E. Lanconelli. *Une métrique associée à une classe d'opérateurs elliptiques dégénérés.* Conference on linear partial and pseudodifferential operators (Torino, 1982). Rend. Sem. Mat. Univ. Politec. Torino 1983, Special Issue, 105–114 (1984).

[16] B. Franchi, R. Serapioni, F. Serra Cassano. *Meyers-Serrin Type Theorems and Relaxation of Variational Integrals Depending Vector Fields.* Houston Journal of Mathematics **22** (1996), 859–889.

[17] N. Garofalo, D.-M. Nhieu. *Isoperimetric and Sobolev inequalities for Carnot–Carathéodory spaces and the existence of minimal surfaces.* Comm. Pure Appl. Math. **49** (1996), 1081–1144.

[18] H. Hopf. Differential geometry in the large. Notes taken by Peter Lax and John W. Gray. Lecture Notes in Mathematics, 1000. Springer-Verlag, Berlin, 1989.

[19] G. P. Leonardi, S. Rigot. *Isoperimetric sets on Carnot groups.* Houston J. Math. **29** (2003), no. 3, 609–637.

[20] A. Hurwitz. *Sur quelques applications gomtriques des sries de Fourier.* Ann. Sci. École Norm. Sup. (3) **19** (1902), 357–408.

[21] M. Miranda Jr. *Functions of bounded variation on "good" metric spaces.* J. Math. Pures Appl. (9) **82** (2003), no. 8, 975–1004.

[22] R. Monti. *Brunn–Minkowski and isoperimetric inequality in the Heisenberg group.* Ann. Acad. Sci. Fenn. Math. **28** (2003), 99–109.

[23] R. Monti. *Heisenberg isoperimetric problem. The axial case.* Adv. Calc. Var. **1** (2008), no. 1, 93–121.

[24] R. Monti. *Rearrangements in metric spaces and in the Heisenberg group.* J. Geom. Anal. **24** (2014), no. 4, 1673–1715.

[25] R. Monti, D. Morbidelli. *Isoperimetric Inequality in the Grushin Plane.* J. Geom. Anal., **14** (2004), no. 2, 355–368.

[26] R. Monti, M. Rickly. *Convex isoperimetric sets in the Heisenberg group.* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **8** (2009), no. 2, 391–415.

[27] P. Pansu. *An isoperimetric inequality on the Heisenberg group.* Conference on differential geometry on homogeneous spaces (Turin, 1983). Rend. Sem. Mat. Univ. Politec. Torino, Special Issue (1983), 159–174.

[28] P. K. Rashevsky. *Any two points of a totally nonholonomic space may be connected by an admissible line.* Uch. Zap. Ped. Inst. im. Liebknechta Sr. Phys. Math. **2** (1938), 83–94 (in Russian).

[29] M. Ritoré. *A proof by calibration of an isoperimetric inequality in the Heisenberg group* $\mathbb{H}^n$. Calc. Var. Partial Differential Equations, **44** (2012), no. 1-2, 47–60.

[30] M. Ritoré, C. Rosales. *Rotationally invariant hypersurfaces with constant mean curvature in the Heisenberg group* $\mathbb{H}^n$. J. Geom. Anal., **16** (2006), no. 4, 703–720.

[31] M. Ritoré, C. Rosales. *Area-stationary surfaces in the Heisenberg group* $\mathbb{H}^1$. Adv. Math. **219** (2008), no. 2, 633–671.

[32] E. Schmidt. *Beweis der isoperimetrischen Eigenschaft der Kugel im hyperbolischen und sphärischen Raum jeder Dimensionszahl*, Math. Z. **49** (1943), 1–109.

[33] P. Tomter. *Constant mean curvature surfaces in the Heisenberg group.* Differential geometry: partial differential equations on manifolds (Los Angeles, CA, (1990), 485–495, Proc. Sympos. Pure Math., **54**, Part 1, Amer. Math. Soc., 1993.

Inria Saclay (team GECO) and CMAP École Polytechnique

E-mail address: valentina.franceschi@inria.fr