ON THE SUBELLIPTIC EIKONAL EQUATION
SULL'EQUAZIONE ICONALE SUBELLITTICA

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Abstract. On a bounded smooth domain, we consider the viscosity solution of the homogeneous Dirichlet problem for the eikonal equation associated with a system of Hörmander's vector fields. We present some results on the regularity and the structure of the singular support of such a function.

Sunto. In un dominio con frontiera regolare, consideriamo la soluzione di viscosità del problema di Dirichlet omogeneo per l'equazione iconale associata ad un sistema di campi vettoriali di Hörmander. Presentiamo alcuni risultati sulla regolarità e sulla struttura del supporto singolare di tale funzione.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and assume that the boundary of Ω, Γ, is a smooth manifold of dimension $n-1$. (Hereafter smooth means of class C^∞.)

Let $X_1, \ldots, X_N$ be a family of smooth vector fields defined in a neighborhood of Ω, Ω'. We say that $\{X_1, \ldots, X_N\}$ is *a system of Hörmander's vector fields* on $\overline{\Omega}$ if the following bracket generating condition holds:

$$\text{Lie}\{X_1, \ldots, X_N\}(x) = \mathbb{R}^n \qquad \forall x \in \Omega'. \tag{1}$$

Let us point out that here we just assume $n, N \geq 2$, in other words our analysis is not restricted to the case of (vectorial) distributions of constant rank.

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The following hypotheses **(H)** will be assumed throughout:

$(H1)$ $\Omega \subset \mathbb{R}^n$ *is a bounded open set with boundary* Γ *of class* C^∞,

$(H2)$ $\{X_1, \ldots, X_N\}$ *is a system of* C^∞ *Hörmander's vector fields on* $\overline{\Omega}$.

Let $T : \overline{\Omega} \longrightarrow \mathbb{R}$ be the continuous viscosity solution of the homogeneous Dirichlet problem for the following eikonal equation:

$$\begin{cases} \sum_{j=1}^N (X_j T)^2(x) = 1 \quad \text{in} \quad \Omega, \\ T = 0 \quad \text{on} \quad \Gamma. \end{cases} \tag{2}$$

Remark 1.1. (1) *We adopt the notion of viscosity solution related with the elliptic regularization:*

$$-\epsilon(\partial_{x_1}^2 + \ldots + \partial_{x_n}^2)T(x) + \sum_{j=1}^N (X_j T)^2(x) = 1$$

(i.e. the concavity of the solution is privileged w.r.t. the convexity).

(2) *It is well-known that equation* (2) *admits a unique viscosity solution* T *(see Subsection 2.1.1). Furthermore,* T *is not a classical solution of* (2).

We define the Hamiltonian as

$$h(x,p) = \sum_{j=1}^N X_j(x,p)^2, \quad (x,p) \in \overline{\Omega} \times \mathbb{R}^n, \tag{3}$$

where X_j is the symbol of the vector field X_j, namely, $X_j(x,p) := \langle X_j(x), p \rangle$, for any $(x,p) \in \overline{\Omega} \times \mathbb{R}^n$ and $j = 1, \ldots, N$. Then, the characteristic set is defined as

$$\text{Char } (X_1, \ldots X_N) = \{(x,p) \in \overline{\Omega} \times (\mathbb{R}^n \setminus \{0\}) \mid h(x,p) = 0\}. \tag{4}$$

From the PDE point of view, the above Dirichlet problem has a typical feature: the Hamiltonian $h(x,p)$ is not strictly convex in p. Thus, characteristic (boundary) points may appear. We recall that a point $x \in \Gamma$ is *characteristic* if the linear space generated by $X_1(x), \ldots, X_N(x)$ is contained in the tangent space to Γ at x. We denote by $E \subset \Gamma$ the set of all characteristic points. Let us recall a result by Derridj [14].

Theorem 1.1. *Under assumption* (H), E *is a closed set of* $(n-1)$*-dimensional Hausdorff measure zero.*

Remark 1.2. (1) *We observe that if span* $\{X_1, \ldots, X_N\}(x) = \mathbb{R}^n$, *for every* x *in* $\overline{\Omega}$, *then the eikonal equation is nondegenerate. In particular, we have that the characteristic set is empty and* T *is locally Lipschitz continuous in* $\overline{\Omega}$. *For the regularity theory, in the case of the nondegenerate eikonal equation, we refer the reader to the papers* [2] *and* [3].

(2) *It is clear that if Char* $(X_1, \ldots, X_N) = \emptyset$ *then* $E = \emptyset$ *but, as shown in the next example, the implication "*$E = \emptyset \implies$ *Char* $(X_1, \ldots, X_N) \neq \emptyset$*" is false.*

Example 1.1. *In* $\mathbb{R}^2$ *consider* $X_1 = \partial_{x_1}$, $X_2 = x_1\partial_{x_2}$ *and* $\Omega = \{(x_1-1)^2 + x_2^2 < 4\}$. *Then we have that* $E = \emptyset$ *and*

$$\mathit{Char}\ (X_1, X_2) = \{(x_1, x_2, p_1, p_2)\ :\ p_1 = x_1 p_2 = 0\} = \{(0, x_2, 0, p_2)\ :\ p_2 \neq 0\}.$$

In this paper, we describe some results, on the regularity and the structure of the singular support of the viscosity solution of (2). These results are mainly part of a joint project with Piermarco Cannarsa and Teresa Scarinci (see [10] and [11]).

2. Regularity results in Hölder spaces and semiconcavity

It is well–known that a lower bound for the Hölder exponent of T is given by the Hörmander condition (see e.g. [18]). We recall that the length of a commutator is the number of the Lie brackets involved in the commutator plus one (for instance, given vector fields X, Y, and Z, the length of $[X,[X,[Y,Z]]]$ is 4), let us give the following definition.

Definition 2.1. *For any* $x \in \overline{\Omega}$ *we call* $r(x)$ *the maximal length of a Lie bracket which is needed to generate* $\mathrm{Lie}\{X_1, \ldots, X_N\}(x)$. *Furthermore, we define*

$$r = \max_{x \in \overline{\Omega}} r(x).$$

We begin by recalling a consequence of a result due to Evans and James (see [15]).

Theorem 2.1. *Assume* (H) *and let* T *be the viscosity solution of Equation* (2). *Then* T *is locally Hölder continuous of exponent* r.

We observe that, in the case of the nondegenerate eikonal equation, a better regularity result holds: T is locally semiconcave in $\overline{\Omega}$ (see e.g. [2]). We recall that a function is locally semiconcave if it can be locally represented as the sum of a concave with a smooth function. (In particular, if a function is locally semiconcave on a set then it is locally Lipschitz in such a set.)

Then a natural question arises: is Theorem 2.1 the best regularity result one can hope for?

In order to answer to the previous question let us consider the following example.

Let $M > 0$ and let k be a positive integer. Consider the (unbounded) set

$$\Omega = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \, : \, y > M|x|^{k+1}\}$$

and the eikonal equation

$$\begin{cases} |\nabla_x T(x, y)|^2 + |x|^{2k}(\partial_y T(x, y))^2 = 1 & \text{in } \Omega, \\ T = 0 \qquad \text{on } \partial\Omega. \end{cases}$$

Then we have the following

Theorem 2.2 ([5]). *The nonnegative viscosity solution of the Dirichlet problem above is locally Lipschitz continuous in* Ω. *Furthermore,* T *is Hölder continuous of the exponent* $1/(k+1)$ *at* $(0, 0)$. *Finally,* T *is real analytic in the set* $\Omega \setminus \{(0, y) \, : \, y \geq 0\}$.

In other words, in general, the Evans-James theorem is optimal at the characteristic boundary points only.

In order to improve Theorem 2.1 we use ideas and methods from Control Theory. The first step is a representation formula for the solution of (2): T is characterized as the minimum time function of a certain optimal control problem. (In essence, this is a natural, global, generalization of the method of the characteristics.)

2.1. **The minimum time problem.**

2.1.1. *Well-posedness of the Dirichlet problem.* It is well known that problem (2) admits a unique viscosity solution. Indeed, taking the boundary of Ω, Γ, as target set, the minimum time function associated with a system of Hörmander's vector fields is a solution of the eikonal equation. In order to recall the definition of such a function, given $x \in \overline{\Omega}$, let us consider the controlled dynamical system

$$(5) \qquad \begin{cases} y'(t) = \sum_{j=1}^{N} u_j(t) X_j(y(t)) \quad (t \geq 0) \\ y(0) = x, \end{cases}$$

where $u = (u_1, \dots, u_N) : [0, +\infty[\to \mathbb{R}^N$ is a control, that is, a measurable map taking values in the unit ball of $\mathbb{R}^N$. Denoting by $y^{x,u}(\cdot)$ the solution of the above equation, we define the *transfer time* to Γ as

$$\tau_\Gamma(x, u) = \inf\{t \geq 0 \mid y^{x,u}(t) \in \Gamma\}.$$

Then the *Minimum Time Problem* with target Γ is as follows:

(MTP) To minimise $\tau_\Gamma(x, u)$ over all controls $u : [0, +\infty[\to \overline{B}_1(0)$.

The *minimum time function* T is defined as

$$T(x) = \inf_{u(\cdot)} \tau_\Gamma(x, u) \quad (x \in \overline{\Omega})$$

turns out to be the unique viscosity solution of the Dirichlet problem (2).

Remark 2.1. We recall that a $u(\cdot)$ is called an *optimal control* relative to the point $x \in \Omega$ if $T(x) = \tau_\Gamma(x, u)$. The corresponding solution of (5), $y^{x,u}$, is called the *time-optimal trajectory* at x associated with u.

It is well-known that Hörmander's bracket generating condition implies that T is finite and continuous. Suppose that we want to show that T is locally Lipschitz continuous on Ω. For this purpose, let $K \subset \Omega$ be a compact set properly inclused in Ω. Then our goal is to show that T is Lipschitz continuous on K. Let us consider all the time–optimal trajectories intersecting K. If for every time–optimal trajectory from K the terminal point is a non-characteristic point (i.e. it is in $\Gamma \setminus E$), then, by estimating T along these trajectories, we deduce that T is locally Lipschitz continuous on K (see e.g. [13]). Indeed, near a non–characteristic boundary point, by the method of the characteristics (see also Subsection 2.4 below), we have that T is smooth, then the Lipschitz estimates "propagate" from the boundary towards the interior of Ω. We observe that no higher regularity is expected (e.g. C^1 regularity): an interior point may be the starting point of several time–optimal trajectories.

The above arguments suggest that T may fail to be Lipschitz continuous due to the presence of a time–optimal trajectory ending at a characteristic boundary point. In order to analyze this phenomenon we need the notion of singular time–optimal trajectory.

2.2. **Singular time–optimal trajectories.** For any boundary point $z \in \Gamma$ we denote by $\nu(z)$ the outward unit normal to Γ at z.

Definition 2.2. *Let $x \in \Omega$ and let $y(\cdot) = y^{x,u}(\cdot)$ be the time-optimal trajectory at x associated with $u : [0, T(x)] \longrightarrow \overline{B}_1(0)$. We say that $y(\cdot)$ is a* singular time-optimal trajectory *if there exists an absolutely continuous arc $p : [0, T(x)] \to \mathbb{R}^n \setminus \{0\}$ such that, for a.e. $t \in [0, T(x)]$,*

$$p'_k(t) = -\sum_{j=1}^{N} u_j(t)\langle \partial_{x_k} X_j(y(t)), p(t)\rangle, \quad \langle X_k(y(t)), p(t)\rangle = 0, \tag{6}$$

for every $k = 1, \dots, N$, and

$$p(T(x)) = \lambda \nu(y(T(x))), \tag{7}$$

for a suitable $\lambda \geq 0$.

Remark 2.2. (i) We observe that, introducing the Control Theory Hamiltonian

$$h(x, p, u) = \sum_{j=1}^{N} u_j X_j(x, p), \tag{8}$$

the optimal triple (y, u, p) arising from Definition 2.2 satisfies, for a.e. $t \in [0, T(x)]$, the Hamiltonian system

$$\begin{cases} y'(t) = D_p h(y(t), p(t), u(t)) \\ p'(t) = -D_x h(y(t), p(t), u(t)) \end{cases} \tag{9}$$

In other words, a time-optimal trajectory is singular if it can be lifted in the phase space in such a way that the lifted trajectory:

- satisfies the Hamiltonian system (9) (with Hamiltonian given by (8)) as well as the transversality condition (7), and

- lies in the characteristic set Char $(X_1, \dots, X_N)$.

(ii) The notion of singular time–optimal trajectory is well–known in the Geometric Optimal Control Theory (in such a context these trajectories are also called abnormal minimizers). We point out that there is a difference between Definition 2.2 and the usual one: since we are interested in a boundary value problem we incorporate in our definition the transversality condition (7). A consequence of this fact is that, using the language of the Geometric Optimal Control Theory, in the present context minimizers are either normal or abnormal.

2.2.1. *Properties of singular time–optimal trajectories.* The first result provides a positive answer to the following question: is it possible to verify whether a time–optimal trajectory is singular without lifting it to the characteristic set?

Theorem 2.3 ([10])**.** *Assume (H) and let $y^{x,u}$ be a time-optimal trajectory with $x \in \Omega$. Then, $y^{x,u}$ is singular if and only if $y^{x,u}(T(x)) \in E$.*

In particular, a singular time-optimal trajectory is tangent to Γ at the terminal point.

Remark 2.3. *The proof of Theorem 2.3 is a direct consequence of the Pontryagin Maximum Principle. We recall that such a principle provides a set of necessary optimality conditions (loosely speaking, it can be viewed as a generalization of the classical Euler–Lagrange equations). For a detailed analysis of the Pontryagin Maximum Principle, we refer the interested reader to* [24]*.*

In order to relate the singular time–optimal trajectories with the regularity of T, we need to introduce the following point-wise notion of Lipschitz continuity.

Definition 2.3. *We say that a function $f : \overline{\Omega} \to \mathbb{R}$ is Lipschitz continuous at a point $x_0 \in \overline{\Omega}$ if there exists a neighbourhood U of x_0 and a constant $L \geq 0$ such that*

$$|f(x) - f(x_0)| \leq L|x - x_0| \qquad \forall x \in U \cap \overline{\Omega}.$$

In the next result we show that the presence of singular time–optimal trajectories is related with a non-Lipschitz behaviour of the minimum time function T.

Theorem 2.4 ([10]). *Assume* (H), *let* $x_0 \in \Omega$. *Then,* T *fails to be Lipschitz continuous at* x_0 *if and only if there exists a singular time–optimal trajectory* $y^{x_0,u}$.

The next result is a straightforward consequence of Theorem 2.4 and the Dynamic Programming Principle.

Corollary 2.1 ([10]). *Assume* (H), *let* $x_0 \in \Omega$, *and let* $y^{x_0,u}$ *be a singular time-optimal trajectory. Then, for any* $t \in [0, T(x_0)[$, T *fails to be Lipschitz continuous at* $y^{x_0,u}(t)$.

2.3. **Interior regularity.** We have the following characterization.

Theorem 2.5 ([10]). *Under assumption* (H), *the following properties are equivalent:*

(1) (MTP) *admits no singular time-optimal trajectory;*
(2) T *is locally semiconcave in* Ω*;*
(3) T *is locally Lipschitz continuous in* Ω.

Remark 2.4. The fact that the existence of singular optimal trajectories may destroy the regularity of a solution of a first order Hamilton-Jacobi equation was already observed by Sussmann (in an implicit form) in [23] and (explicitly) by Agrachev in [1]. The regularity these authors consider is subanalyticity of the point-to-point distance function associated with real analytic distributions.

Let us give a sketch of the proof of Theorem 2.5.

Proof. The implication (1) $\implies$ (2) is a consequence of the following remark. Let K be a compact set properly inclused in Ω. Then, by (1) and Theorem 2.3, every time–optimal trajectory intersecting K has the terminal point in $\Gamma \setminus E$. Since $E \subset \Gamma$ is a closed set the set of all the terminal points of time–optimal trajectories intersecting K, F, has positive distance from the set of all the boundary characteristic points E. Then the local semiconcavity can be proved by "propagating", along the time–optimal trajectories, the semiconcavity estimates from a neighborhood of the set F towards K. The implication (2) $\implies$ (3) is a consequence of the fact that a concave function is locally Lipschitz continuous. Finally, (3) $\implies$ (1) is the content of Theorem 2.4. □

The next example shows that the function T may exhibit a non-Lipschitz behaviour. In $\mathbb{R}^3$, we consider a system of vector fields introduced by Liu and Sussmann:

$$X_1 = \partial_{x_1}, \qquad X_2 = (1 - x_1)\partial_{x_2} + x_1^2 \partial_{x_3}.$$

Theorem 2.6 ([10])**.** *There exists an open bounded set with C^∞ boundary such that the solution of the equation*

$$\begin{cases} (X_1 T)^2 + (X_2 T)^2 = 1 & \text{in } \Omega, \\ T|_\Gamma = 0, \end{cases}$$

is not locally Lipschitz continuous in Ω.

We recall also the following "companion" result.

Theorem 2.7 ([10])**.** *Let Ω be a bounded convex open set with smooth boundary. Then the solution of the equation*

$$\begin{cases} (X_1 T)^2 + (X_2 T)^2 = 1 & \text{in } \Omega, \\ T|_\Gamma = 0, \end{cases}$$

is locally Lipschitz continuous on Ω.

In other words, the geometry of the boundary Γ may exclude the presence of singular time-optimal trajectories.

2.4. **Boundary regularity.** We complete this section on the regularity of T, with a result on its boundary behaviour. We point out that the boundary regularity for the solution of (2) is, in essence, well-known.

Theorem 2.8. (1) *For every $x \in \Gamma \setminus E$, T is smooth on a neighborhood of x.*
(2) *For every $x \in E$, T is Hölder continuous at x of exponent $1/r(x)$.*

As shown in [5], in general, Theorem 2.8 is optimal.

3. Sufficient conditions for the regularity of T

In this section, we give conditions to prevent the appearance of singular time-optimal trajectories. Let us recall that, in canonical coordinates[1], the symplectic form in $T^*\Omega$ is the 2-form

$$\sigma = \sum_{k=1}^{n} dp_k \wedge dx_k. \tag{10}$$

Finally, we say that a manifold $M \subset T^*\Omega$ is *symplectic* if the restriction of σ to M is nondegenerate. Let $W \subset T^*\Omega$ be a smooth manifold, we denote by $(T_\rho W)^\sigma$, for $\rho \in W$, the orthogonal w.r.t. the symplectic form σ of the linear space $T_\rho W$. We have the following result.

Theorem 3.1 ([10])**.** *If $E = \emptyset$ or Char $(X_1, \dots, X_N)$ is a symplectic manifold then T is locally semiconcave in Ω.*

Proof. The implication $E = \emptyset \implies T$ is locally semiconcave in Ω is a direct consequence of the fact that $E = \emptyset$ implies that (MTP) admits no singular time–optimal trajectories (by Theorem 2.3). Then, by the interior regularity result Theorem 2.5, the local semiconcavity of T follows. Let us show that if Char $(X_1, \dots, X_N)$ is a symplectic manifold then T is locally semiconcave in Ω. Due to the argument above, it suffices to show that if Char $(X_1, \dots, X_N)$ is a symplectic manifold then (MTP) admits no singular time–optimal trajectories. For this purpose, let us suppose that there exists a singular time–optimal trajectory. Then, by definition, it can be lifted to the characteristic manifold, let

$$\rho : I \longrightarrow \text{ Char } (X_1, \dots, X_N)$$

be the lifted trajectory. (Here $I \subset [0, +\infty[$ is a suitable closed and connected interval.) Then, we have that

$$\dot\rho(t) \in T_{\rho(t)} \text{ Char } (X_1, \dots, X_N), \quad \text{ for a.e. } t \in I. \tag{11}$$

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[1]More in general a symplectic form in a C^∞ manifold is a non-degenerate, closed C^∞ two form.

In order to complete our proof it suffices to show that $\rho(t)$ is constant in I, for a projection on the base of $\rho(\cdot)$ would be a single point (i.e. (MTP) admits no singular time–optimal trajectories).

We have that, for every $\rho \in$ Char $(X_1, \dots, X_N)$,

$$\text{span } \{dX_1(\rho), \dots, dX_N(\rho)\} \perp T_\rho \text{ Char } (X_1, \dots, X_N)$$

(here "$\perp$" stands for orthogonality w.r.t. the Euclidean scalar product). Hence, we find

$$\text{span } \{H_{X_1}(\rho), \dots, H_{X_N}(\rho)\} = \\ (\text{ span } \{dX_1(\rho), \dots, dX_N(\rho)\})^\sigma \subset (T_\rho \text{ Char } (X_1, \dots, X_N))^\sigma.$$

(Here H_{X_j} is the Hamiltonian vector field associated with the symbol X_j, i.e.

$$dX_j(\rho)t = \sigma(t, H_{X_j}(\rho)), \quad \text{for every } t \in T_\rho \text{ Char } (X_1, \dots, X_N).)$$

We recall that $\rho(\cdot)$ satisfies the following broken Hamiltonian system

$$\dot\rho(t) = \sum_{j=1}^{N} u_j(t) H_{X_j}(\rho(t)), \quad \text{for a.e. } t \in I. \tag{12}$$

On the other hand, since Char $(X_1, \dots, X_N)$ is a symplectic manifold, using the identity

$$T_\rho \text{ Char } (X_1, \dots, X_N) \cap (T_\rho \text{ Char } (X_1, \dots, X_N))^\sigma = \{0\},$$

we conclude that the broken Hamiltonian flow (12) is a stationary flow. This completes our proof. □

3.1. **On the strongly bracket generating assumption.** Let us suppose that the vector fields $X_1, \dots, X_N$ are linearly independent (and that $N < n$). We need a

Definition 3.1. *The system* $\{X_1, \dots, X_N\}$ *is strongly bracket generating if for every* $(\alpha_1, \dots, \alpha_N) \neq 0$ *and for every* $x \in \Omega$,

$$\dim \left(\textit{span } \{X_j\}_{j=1,\dots,N}(x) + \textit{ span } \left\{ \sum_{h=1}^{N} \alpha_h [X_h, X_j] \right\}_{j=1,\dots,N} (x) \right) = n.$$

A direct consequence of the definition above is that if the system $\{X_1, \ldots, X_N\}$ is strongly bracket generating then $r = 2$ (i.e. in Hörmander's condition, in order to generate $\mathbb{R}^n$, it suffices to use commutators of length 2).

We recall that, in the sub-Riemannian geometry, the strongly bracket generating assumption is used to exclude the presence of abnormal minimizers (see e.g. [21] and [22]). Then we have the following

Theorem 3.2. *Assume* (H) *and let*

$$\textit{dim span}\ \{X_1, \ldots, X_N\}(x) = N, \quad \forall x \in \Omega.$$

Then the following assertions are equivalent

(1) *the system* $\{X_1, \ldots, X_N\}$ *is strongly bracket generating;*
(2) *Char* $(X_1, \ldots, X_N)$ *is a symplectic manifold.*

Proof. Let us assume that the system $\{X_1, \ldots, X_N\}$ is not strongly bracket generating. Then there exist $x \in \Omega$, $p \in \mathbb{R}^n \setminus \{0\}$ and $\alpha = (\alpha_1, \ldots, \alpha_N) \neq 0$ such that

$$\left\langle p, \left(\text{ span } \{X_j(x)\}_{j=1,\ldots,N} + \text{ span } \{\sum_{j=1}^{N} \alpha_j [X_j, X_\ell](x)\}_{\ell=1,k} \right) \right\rangle = 0. \tag{13}$$

We observe that (13) is equivalent to the following conditions

$$\begin{cases} (x,p) \in \text{ Char } (X_1, \ldots, X_N) \\ \\ \sum_{j,\ell=1}^{N} \langle p, [X_j, X_\ell](x) \rangle \alpha_j \beta_\ell = 0 \end{cases}$$

for every $\beta = (\beta_1, \ldots, \beta_N) \neq 0$. In other words, we have that

$$\begin{cases} (x,p) \in \text{ Char } (X_1, \ldots, X_N), \\ \\ \text{rank } (\{X_j, X_\ell\}(x,p))_{1 \leq j,\ell \leq N} < N. \end{cases}$$

(Here $\{X_j, X_\ell\}$ is the Poisson bracket, which in local coordinates reads as

$$\{X_j, X_\ell\}(x,p) = \sum_{i=1}^{n} \left(\partial_{p_i} X_j(x,p) \partial_{x_i} X_\ell(x,p) - \partial_{x_i} X_j(x,p) \partial_{p_i} X_\ell(x,p) \right).)$$

This means that the symplectic form is degenerate, i.e. Char $(X_1, \ldots, X_N)$ is a manifold but it is not symplectic. This completes our proof. □

We observe that, as a consequence of the result above, we have that if the system $\{X_1, \ldots, X_N\}$ is stronlgy bracket generating then N is an even number.

We point out that the assumption Char $(X_1, \ldots, X_N)$ is a symplectic manifold is more general than the strongly bracket condition: singular vector fields are also admitted as well as no upper bound on r, the length of the commutators needed to generate $\mathbb{R}^n$, is imposed.

4. Examples

Example 4.1 (Heisenberg vector fields)**.** *In $\mathbb{R}^3$ consider vector fields*

$$X_1 = \partial_{x_1}, \quad X_2 = \partial_{x_2} + x_1 \partial_{x_3}$$

and let Ω be a bounded open set with C^∞ boundary. We have that

$$\textit{Char}\ (X_1, X_2) = \big\{(x_1, x_2, x_3, 0, -x_1 p_3, p_3) : (x_1, x_2, x_3) \in \Omega,\ p_3 \neq 0\big\}$$

is a smooth submanifold of $\mathbb{R}^6$ of codimension 2. Furthermore, the restriction of σ to Char (X_1, X_2) is nondegenerate, i.e. Char (X_1, X_2) is a symplectic manifold. Then, (MTP) has no singular time-optimal trajectory.

Example 4.2 (Oleinik vector fields)**.** Consider in $\mathbb{R}^3$ the vector fields

$$X_1 = \partial_{x_1}, \quad X_2 = x_1^{p-1} \partial_{x_2} \text{ and } X_3 = x_1^{q-1} \partial_{x_3}.$$

(Here $2 \leq p \leq q$ with p and q positive integers.) Let $\Omega \subset \mathbb{R}^3$ be an (arbitrary) open bounded set with smooth boundary. Then, (MTP) admits no singular time–optimal trajectories. Indeed, in this case

$$\text{Char}\ (X_1, X_2, X_3) = \{(0, x_2, x_3, 0, p_2, p_3)\ :\ x_2, x_3 \in \mathbb{R},\ (p_2, p_3) \neq (0, 0)\}$$

is a symplectic manifold.

Theorem 3.1 can be easily generalized: in the presence of changes of rank of the symplectic form it suffices to assume that Char $(X_1, \dots, X_N)$ can be decomposed as a locally finite union of smooth symplectic manifolds. For instance, let us consider the following

Example 4.3. Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with smooth boundary and consider, in $\mathbb{R}^3$, vector fields $X_1 = \partial_{x_1} - x_2^{2k+1}\partial_{x_3}$ and $X_2 = \partial_{x_2} + x_1^{2k+1}\partial_{x_3}$, where k is a positive integer. We have that the characteristic set,

$$\text{Char }(X_1, X_2) = \big\{(x_1, x_2, x_3, x_2^{2k+1}p_3, -x_1^{2k+1}p_3, p_3) \ : \ x_1, x_2, x_3 \in \mathbb{R},\ p_3 \neq 0\big\},$$

is a manifold of condimension 2 in $\mathbb{R}^3$ but the rank of the symplectic form is not constant, i.e. Char (X_1, X_2) is not a symplectic manifold. On the other hand, the characteristic set can be splitted into four connected submanifolds

$$\Sigma_{1,\pm} = \big\{(x_1, x_2, x_3, x_2^{2k+1}p_3, -x_1^{2k+1}p_3, p_3) \ : x_1, x_2, x_3 \in \mathbb{R},\ (x_1, x_2) \neq (0,0),\ \pm p_3 > 0\big\}$$

and

$$\Sigma_{2,\pm} = \big\{(0, 0, x_3, 0, 0, p_3) \ : \ x_3 \in \mathbb{R},\ \pm p_3 > 0\big\}.$$

We point out that all these submanifolds are symplectic (the rank of the symplectic form is constant and the symplectic form is nondegenerate on these sets). So, also in this case there are no singular time-optimal trajectories.

5. Partial regularity

We recall that the singular support of T is the closed set where T is not smooth. In [11] it is proved the following

Theorem 5.1. *Assume (H). Then the singular support of T is a closed set of measure zero.*

In other words, except for a closed set of measure zero, the solution of (2) inherits the regularity of the data of the Dirichlet problem (2).

We point out that, in Theorem 5.1, no condition is required on the time–optimal trajectories.

Remark 5.1. *We note that Theorem 5.1 is related to the so called Minimizing Sard Conjecture (see e.g.* [20]*). We recall that such a conjecture claims the almost everywhere differentiability of the subriemannian distance to a point. Theorem 5.1 above yields the smoothness of the subriemannian distance function to a smooth submanifold of codimension* 1 *off a set of Lebesgue measure zero.*

The proof of Theorem 5.1 is based on the following two steps:

(1) the set where T is not C^∞ coincides with the set where T is not $C^{1,1}$. (This fact can be proved arguing as in [8] and using the fact that if x_0 is not in the $C^{1,1}$ singular support of T and $y^{x_0,u}$ is a time–optimal trajectory then T is $C^{1,1}$ along such a trajectory, see [12] and [11].) This sort of regularization is not unexpected: the viscosity solution of the Dirichlet problem (2) can be seen as the limit of the solutions of a family of regularized equations of the form

$$\begin{cases} -\epsilon \sum_{j=1}^N X_j^2 T_\epsilon + \sum_{j=1}^N (X_j T_\epsilon)^2 = 1, & \text{in } \Omega, \\ T_\epsilon = 0, & \text{on } \Gamma. \end{cases}$$

Defining $u_\epsilon = e^{-\frac{T_\epsilon}{\epsilon}}$, we find that u_ϵ solves the equation

$$\begin{cases} \epsilon^2 \sum_{j=1}^N X_j^2 u_\epsilon - u_\epsilon = 0, & \text{in } \Omega, \\ u_\epsilon = 1, & \text{on } \Gamma, \end{cases}$$

and, due to the C^∞ regularity result in [16], we find that $u_\epsilon \in C^\infty(\Omega)$.

(2) The $C^{1,1}$ singular support of T is a set of measure zero. (In the case of the nondegenerate eikonal equation this fact was proved in [7].) The fact that the Lipschitz singular support is a set of measure zero was proved in [19].

Assuming that the (MTP) admits no singular time–optimal trajectories one can give a result on the topological structure of the singular support of T.

Theorem 5.2. *If T is locally semiconcave in Ω, then the singular support of T has the same homotopy type as the set Ω.*

As a consequence of the above result, one can deduce the existence of a one-to-one correspondence between the connected components of the singular support of T and those of Ω. In particular, the singular support of T is path-wise connected if and only if so is Ω.

Remark 5.2. *(i) To our knowledge, a first result of this kind was proved in* [17] *for the solution of the Euclidean eikonal equation on an arbitrary open bounded subset of* $\mathbb{R}^n$*. In this case,* T *is the Euclidean distance function from the boundary of the open set under exam. In* [17]*, it is shown that the set where the distance is not differentiable, the singular set, has the same homotopy as the set* Ω*. In* [4]*, it is shown that* Ω*, an open bounded subset of a Riemannian manifold, has the same homotopy type as the singular set of the Riemannian distance from the boundary of* Ω*. (See also* [9]*.) Finally, the singular support, i.e. the closure of the singular set, is studied in* [6]*.*

(ii) The above result can be proved arguing as in [6]*: it suffices to use Lemma 2.1. in* [11] *instead of Theorem 1.2 in* [6]*.)*

(iii) Even replacing the singular support of T with the real analytic singular support of T, Theorems 5.1 and 5.2 hold true.

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