RADIAL POSITIVE SOLUTIONS FOR p-LAPLACIAN SUPERCRITICAL NEUMANN PROBLEMS

SOLUZIONI RADIALI POSITIVE DI PROBLEMI DI NEUMANN SUPERCRITICI GOVERNATI DAL p-LAPLACIANO

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ABSTRACT. This paper deals with existence and multiplicity of positive solutions for a quasilinear problem with Neumann boundary conditions. The problem is set in a ball and admits at least one constant non-zero solution; moreover, it involves a nonlinearity that can be supercritical in the sense of Sobolev embeddings. The main tools used are variational techniques and the shooting method for ODE's. These results are contained in [6, 3].

SUNTO. In questo lavoro trattiamo l'esistenza e la molteplicità di soluzioni positive per un probelma quasilineare ambientato in una palla, con condizioni al bordo di Neumann. Il problema ammette almeno una soluzione costante non nulla e coinvolge una nonlinearità che può essere supercritica nel senso delle immersioni di Sobolev. I principali strumenti usati nello studio di tale problema sono tecniche variazionali e il metodo di shooting per le EDO. Questi risultati sono contenuti in [6, 3].

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1. Introduction

For $1 < p < \infty$, we consider the following quasilinear Neumann problem

$$\begin{cases} -\Delta_p u + u^{p-1} = u^{q-1} & \text{in } B_R, \\ u > 0 & \text{in } B_R, \\ \partial_\nu u = 0 & \text{on } \partial B_R, \end{cases} \tag{1}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplace operator, $B_R \subset \mathbb{R}^N$ is the ball of radius R centered at the origin, $N \geq 1$, and ν is the outer unit normal of ∂B_R. The nonlinearity u^{q-1} is supposed to be $(p-1)$-superlinear at infinity (i.e., $q > p$), but no conditions of subcriticality in the sense of Sobolev embeddings are required.

We point out that, regardless of the choice of $q > p$, even for q greater than the Sobolev critical exponent p^*, problem (1) admits the non-zero solution $u \equiv 1$. Hence, differently from Dirichlet supercritical problems, with Neumann boundary conditions there is not a Pohozaev-type obstruction to the existence of non-zero solutions. The natural question that arises in this setting is whether the problem admits any *non-constant* solutions.

Concerning existence, multiplicity, and oscillatory behavior (around the constant solution) of non-constant radial solutions of (1), we prove in [3] that the situation changes drastically depending on $p > 1$. More precisely, if $p > 2$ the problem has infinitely many solutions as soon as $q > p$. While, if $p = 2$, the problem admits at least k non-constant solutions provided that $q - 2$ is bigger than the $(k+1)$-th radial eigenvalue of the Neumann Laplacian[1]. Finally, for $1 < p < 2$ a surprising behavior appears, as non-constant solutions with the same oscillatory behavior come in couples as soon as the radius of the domain overcomes a certain threshold. The result obtained reads as follows.

Theorem 1.1 (Corollary 1.5 of [3]). *Let $q > p$ and λ_k^{rad} denote the k-th radial eigenvalue of the Neumann Laplacian.*

[1]For *radial eigenvalues of the Neumann Laplacian* we mean the eigenvalues of $-\Delta u = \lambda u$ in B_R with $\partial_\nu u = 0$ on ∂B_R, correspondinding to radial eigenfunctions. They are numbered starting from the zero eigenvalue: $0 = \lambda_1^{\mathrm{rad}} < \lambda_2^{\mathrm{rad}} < \lambda_3^{\mathrm{rad}} < \dots$.

(i) *If $p = 2$ and $q - 2 > \lambda_{k+1}^{\mathrm{rad}}$ for some $k \geq 1$, then* (1) *admits at least k different non-constant radial solutions $u_1, \dots, u_k$. Furthermore, $u_j - 1$ has exactly j zeros for any $j = 1, \dots, k$.*

(ii) *If $p > 2$, then* (1) *admits infinitely many non-constant radial solutions.*

(iii) *If $1 < p < 2$, then for every integer $k \geq 1$ there exist $R_*(k) > 0$ such that if $R > R_*(k)$, problem* (1) *admits at least $2k$ different non-constant radial solutions $u_1^{\pm}, \dots, u_k^{\pm}$. Furthermore, $u_j^{\pm} - 1$ has exactly j zeros for any $j = 1, \dots, k$.*

In [6, 3], we study p-Laplacian problems involving nonlinearities $g(u)$ possibly more general than the pure power u^{q-1}, namely

$$
\begin{cases}
-\Delta_p u + u^{p-1} = g(u) & \text{in } B_R, \\
u > 0 & \text{in } B_R, \\
\partial_\nu u = 0 & \text{on } \partial B_R.
\end{cases}
\tag{2}
$$

The main feature, in all the cases, is that we allow g to have Sobolev-supercritical growth.

This note is organized as follows. In Section 2, we show a variational approach to get a minimax solution of (2) when $p \geq 2$. In Section 3 we collect some comments, pre-existing results, and numerical simulations to get further insights into the features of the solutions of (2) when $p = 2$ and $p > 2$. Finally, in Section 4 we consider the general case $p > 1$ and obtain existence, multiplicity, and oscillatory behavior via shooting method.

2. A minimax solution for $p \geq 2$

In [6], we study the existence of non-constant solutions of problem (2) with $p \geq 2$. We assume that $g : [0, \infty) \to \mathbb{R}$ satisfies the following hypotheses [2]

[2]In [6], the hypothesis (g_0) requires the limit in 0 to belong to $[0, 1)$ instead of $(-\infty, 1)$. Nevertheless, that assumption can be weakened as stated here, because it is always possible to modify $g(s)$ into $\tilde{g}(s) := g(s) + ms^{p-1}$ for a suitable $m > 0$ such that $\lim_{s \to 0^+} \tilde{g}(s)/s^{p-1} \in [0, 1)$, and study the equivalent problem $-\Delta_p u + (m + 1)u^{p-1} = \tilde{g}(u)$. We observe in passing that the constant m can be also adjusted in such a way to deal with a non-negative and non-decreasing $\tilde{g}$.

($g_{\rm reg}$) g is of class $C^1([0,\infty))$;

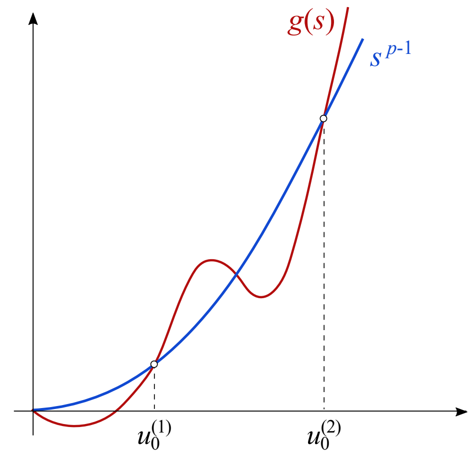
(g_0) $\lim_{s\to 0^+} \frac{g(s)}{s^{p-1}} \in (-\infty, 1)$;

(g_∞) $\liminf_{s\to\infty} \frac{g(s)}{s^{p-1}} \in (1, \infty]$;

(g_{u_0}) $\exists\, u_0 > 0$ such that $g(u_0) = u_0^{p-1}$ and

$$g'(u_0) > \begin{cases} (p-1)u_0^{p-2} & \text{if } p > 2, \\ \lambda_2^{\rm rad} + 1 & \text{if } p = 2, \end{cases}$$

where $\lambda_2^{\rm rad}$ denotes the second radial eigenvalue of Neumann Laplacian $-\Delta$.



In particular, the pure power s^{q-1} for $q > p$ satisfies ($g_{\rm reg}$)–(g_{u_0}).

As an immediate consequence of (g_0), $g(0) = 0$. Moreover, by (g_∞), the nonlinearity g can be taken Sobolev-supercritical. We further remark that, by the regularity of g and by (g_0) and (g_∞), we immediately have the existence of an intersection point $u_0 > 0$ between g and the power function s^{p-1}, with $g'(u_0) \geq (s^{p-1})'(u_0) = (p-1)u_0^{p-2}$. Hence, when $p > 2$, condition (g_{u_0}) is only needed to prevent the situation in which g is tangent to s^{p-1} at u_0. While for $p = 2$, the condition required at u_0 is stronger, being $\lambda_2^{\rm rad} > 0$. In both cases, $p > 2$ and $p = 2$, conditions (g_0) and (g_∞) are enough to prove the existence of a radial solution to (2) of minimax-type, while (g_{u_0}) is needed to prove that the solution found is non-constant. We finally observe that, due to the existence of $u_0 > 0$ for which $g(u_0) = u_0^{p-1}$, problem (2) admits at least the constant solution $u \equiv u_0$.

The main result in [6] reads as follows.

Theorem 2.1 ([2, Theorem 1.3] for $p = 2$, [6, Theorem 1.1] for $p > 2$)**.** *Let $p \geq 2$ and let g satisfy ($g_{\rm reg}$)–(g_{u_0}). There exists a non-constant, radial, radially non-decreasing solution of* (2)*. In addition, if $u_{0,1}, \dots, u_{0,n}$ are n different positive constants satisfying (g_{u_0}), then* (2) *admits n different non-constant, radial, radially non-decreasing solutions.*

We sketch below the proof of Theorem 2.1, see also [5]. We first observe that, since the equation in (2) is possibly supercritical, the energy associated to the problem might not be well-defined in the whole of $W^{1,p}(B_R)$. This prevents, a priori, the use of variational methods. Nevertheless, we take advantage of the idea in [12] and work in the cone of non-negative, radial, radially non-decreasing functions

$$\mathcal{C} := \Big\{ u \in W^{1,p}_{\mathrm{rad}}(B_R) \,:\, u \geq 0,\, u(r) \leq u(s) \text{ for all } 0 < r \leq s \leq R \Big\}, \tag{3}$$

where with abuse of notation we write $u(|x|) := u(x)$.

• **Sketch of the proof of Theorem 2.1.** We split the proof into five steps.

Step 1. *(A priori estimates)* The main reason for working in $\mathcal{C}$ is that

All functions in $\mathcal{C}$ are bounded.

Hence, in $\mathcal{C}$, the associated energy functional $I_{\mathcal{C}} : \mathcal{C} \to \mathbb{R}$ is well-defined. By the way, the cone $\mathcal{C}$ *has empty interior in the $W^{1,p}$-topology.* So, if a function u is such that

$$I'_{\mathcal{C}}(u)[\varphi] = 0 \quad \text{for all } \varphi \in \mathcal{C},$$

we cannot conclude in general that u is a weak solution of (2). The strategy used in [2, 6] to overcome this difficulty is based on a priori estimates in $\mathcal{C}$, namely:

All solutions of (2) *belonging to $\mathcal{C}$ are a priori bounded in $W^{1,p}(B_R)$ and in $L^\infty(B_R)$.*

Step 2. *(Truncation)* Thanks to the a priori estimates, we can *truncate the nonlinearity* g and redefine it at infinity, in order to deal with a *subcritical nonlinearity.* In this way, we end up with a new *truncated problem* with the property that all solutions of the truncated problem belonging to $\mathcal{C}$ solve also the original problem (2).

Step 3. *(Existence)* Due to the subcriticality of the truncated g, the energy functional I associated to the truncated problem is well-defined in the whole of $W^{1,p}(B_R)$. Hence, we can now apply variational methods. By Step 2., we need to find a critical point of I which belongs to $\mathcal{C}$. To this aim, we prove that a mountain pass-type theorem holds for I inside the cone $\mathcal{C}$. The main difficulty here is the construction of a descending flow that preserves $\mathcal{C}$, cf. [6, Lemmas 3.7–3.8]. When $p > 2$, this step presents the additional technical difficulty of proving the existence of a local Lipschitz vector field that preserves the cone $\mathcal{C}$, see [6, Lemmas 3.4–3.6].

Step 4. *(Non-constancy)* We want to prove that the solution found is nonconstant. To this aim, we further restrict our cone and work in a subset of $\mathcal{C}$ in which the only constant solution of (2) is $u \equiv u_0$. In this set, we build an admissible curve γ along which the energy is lower than the energy of the constant u_0. This gives immediately that the minimax solution found (whose energy is such that $I(u) = \min_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t))$, where Γ is the set of admissible curves) is not identically equal to u_0. More precisely, let ϕ_2 be the second eigenfunction of the Neumann p-Laplacian. Via second-order Taylor expansion of I, we prove that for every $s \in (-\varepsilon, \varepsilon) \setminus \{0\}$

$$I(t(s)(u_0 + s\phi_2)) - I(u_0) =$$
$$\begin{cases} \dfrac{s^2}{2} \displaystyle\int_{B_R} \left\{ \cancel{|\nabla u_0|^{p-2}|\nabla \phi_2|^2} + [(p-1)u_0^{p-2} - g'(u_0)]\phi_2^2 \right\} dx + o(s^2) < 0 & (p > 2), \\ \dfrac{s^2}{2} \displaystyle\int_{B_R} \left\{ |\nabla \phi_2|^2 + [1 - g'(u_0)]\phi_2^2 \right\} dx + o(s^2) < 0 & (p = 2), \end{cases}$$

where $t(s)$ is a suitable continuous function. We stress that the inequality signs in the above computation, both for $p > 2$ and for $p = 2$, are due to condition (g_{u_0}). This makes apparent the reason why we need to require different conditions for $p > 2$ and $p = 2$. Now, to get the curve $\gamma \in \Gamma$ along which the energy is lower than $I(u_0)$, it is enough to rescale suitably the curve $s \mapsto t(s)(u_0 + s\phi_2)$. Finally, we observe here that this part of the proof uses heavily the C^2-regularity of the energy functional I, thus it cannot be generalized to the case $1 < p < 2$.

<u>Step 5.</u> *(Multiplicity)* If there is more than one constant u_0 satisfying condition (g_{u_0}), we take advantage of the fact that, since we work in the restricted cone containing exactly one constant solution, we automatically localize each minimax solution. This allows us to prove the multiplicity result stated in Theorem 2.1, by simply repeating the same argument in each cone restricted about each $u_{0,i}$.

3. Some comments on the case $p \geq 2$

From Step 4. above, one could get the impression that condition (g_{u_0}) is only a technical *ad hoc* assumption imposed on g in order to let the machinery of the proof work fine. Actually, with reference to the bifurcation diagrams in Figures 1 and 2, one can see that the values $q = p$ for $p > 2$ and $q = 2 + \lambda_2^{\text{rad}}$ for $p = 2$, involved in condition (g_{u_0}) when $g(s) = s^{q-1}$, arise naturally from the problem. Despite this, one should be aware that it has been proved in [1] that, for $p = 2$ and $N \geq 3$, the value $2 + \lambda_2^{\text{rad}}$ is not sharp.

Let us first comment the case $p = 2$. We notice that, in the semilinear case, condition (g_{u_0}) involves the second radial eigenvalue of $-\Delta$ with Neumann boundary conditions. This is coherent with the result in [1], where the authors show that a bifurcation phenomenon underlies the existence result, at least in the case of the prototype nonlinearity $g(s) = s^{q-1}$. They prove that at $q = 2 + \lambda_{k+1}^{\text{rad}}$, $k \geq 1$, a new branch of solutions bifurcates from the constant branch $u \equiv u_0 = 1$.

Theorem 3.1 ([1])**.** *Let $p = 2$, $g(s) = s^{q-1}$ with $q > 2$, and λ_k^{rad} denote the k-th eigenvalue for the Neumann Laplacian for any integer $k \geq 1$.*

(i) *If $q > 2 + \lambda_{k+1}^{\text{rad}}$, there exist at least k non-constant radial solutions $u_1, \ldots, u_k$ of* (2)*. Furthermore, $u_j - 1$ has exactly j zeros for any $j = 1, \ldots, k$.*

(ii) *If $2^* > q > 2 + \lambda_{k+1}^{\text{rad}}$ (where 2^* is the Sobolev critical exponent), there exist at least $2k$ non-constant radial solutions $u_1^\pm, \ldots, u_k^\pm$ of* (2)*. Furthermore, $u_j^\pm - 1$ has exactly j zeros for any $j = 1, \ldots, k$.*

This theorem was proved by means of the Crandall-Rabinowitz bifurcation technique in the parameter q. Moreover, part (i) of the previous theorem was also recovered in [3, Corollary 1.5-(ii)] via shooting method.

We present now some numerical simulations performed with the software AUTO-07p for problem (2) in dimension $N = 1$, with $R = 1$ and $g(s) = s^{q-1}$.

In Figure 1, we represent the first three bifurcation branches for this problem with $p = 2$. The black line represents the constant solution $u \equiv 1$; the branches bifurcate at points $q = 2 + \lambda_k^{\text{rad}}$, $k = 2, 3, 4$. The solutions belonging to the lower part of the first branch are monotone increasing, the ones belonging to the upper part of the first branch are monotone decreasing, in both cases they all intersect once the constant solution $u \equiv 1$. Solutions of the lower part of the second branch present exactly one interior maximum point, solutions of the upper part of the second branch have exactly one interior minimum point, in both cases they have two intersections with $u \equiv 1$, and so on.

In [2], it was conjectured that a similar behavior should hold also for a general nonlinearity g, when $p = 2$. For g asymptotically linear (and hence Sobolev-subcritical), this conjecture was proved to be true in [8]. In [3, Corollary 1.3] (see Theorem 4.1-(i) below),

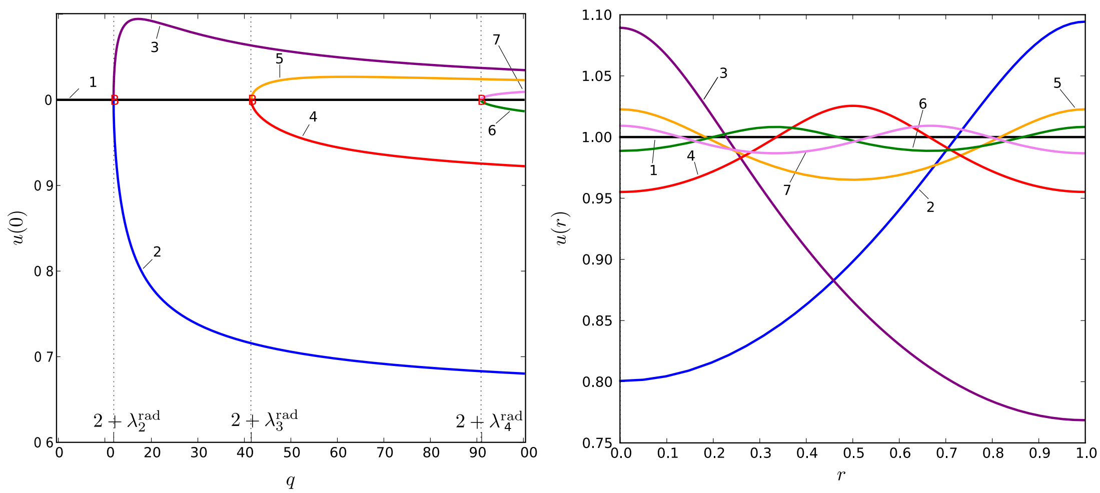


FIGURE 1. The first three bifurcation branches for problem (2) in the case $N = 1$, $R = 1$, $p = 2$, $g(s) = s^{q-1}$. *On the left*: bifurcation diagram $u(0)$ as a function of q. *On the right*: solutions belonging to the first three branches. The color of each solution in the right plot corresponds to the color of the branch it belongs to in the left plot. More precisely, the numbers along the branches in the left plot are located in correspondence with the initial condition $u(0)$ of the solution represented in the right plot.

we prove the conjecture, without assuming any growth conditions at infinity on g, via shooting method.

Concerning the case $p > 2$, from Theorem 2.1 we know that a non-constant solution of (2) arises as soon as the exponent $q > p$. Even more, Theorem 1.1-(ii) guarantees that when $g(s) = s^{q-1}$, (2) has infinitely many solutions for $q > p$. Here the eigenvalues of the operator are not involved. In Figure 2, we present some numerical simulations for the case $p = 2.1 > 2$. A bifurcation phenomenon from the constant solution seems to persists also when $p > 2$. In this figure only the two branches of monotone solutions are detected, we refer to [3, Section 3] for more simulations with $p > 2$. In [3], we conjecture that in this case infinite branches bifurcate from the same point $q = p$, giving rise to a very degenerate situation. This would be coherent with the result of Theorem 1.1-(ii). We further remark that the solution found in Theorem 2.1 is non-decreasing, so with reference to Figure 2, it belongs to the lower (blue) branch of solutions.

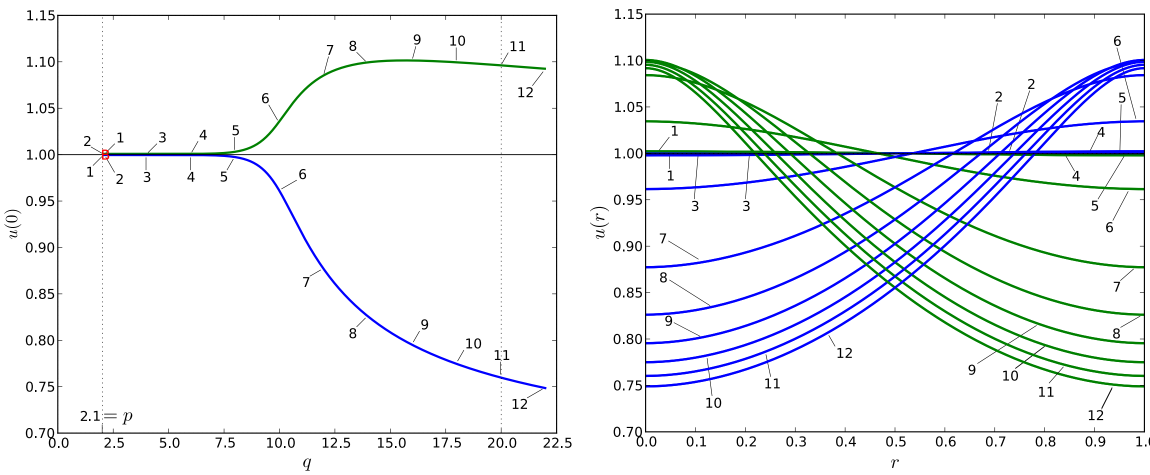


FIGURE 2. Partial bifurcation diagram for problem (2) in the case $N = 1$, $R = 1$, $p = 2.1$, and $g(s) = s^{q-1}$. *On the left*: bifurcation diagram $u(0)$ as a function of q: the first two branches of solutions bifurcating at $q = p > 2$. The green branch is the one of decreasing solutions, the blue branch is the one of increasing solutions. *On the right*: solutions belonging to the first branches. Blue (increasing) solutions belong to the blue branch in the left plot, green (decreasing) solutions belong to the green branch.

4. Existence and multiplicity via shooting method for $p > 1$

In [3], we consider problem (2) for every $p > 1$. We require slightly different (essentially weaker) conditions on g, with respect to the ones introduced in Section 2. Namely,

$(g_{\mathrm{reg}})'$ $\quad g \in C([0,\infty) \cap C^1((0,\infty))$;

$(g_0)'$ $\quad \lim_{s\to 0^+} \frac{g(s)}{s^{p-1}} \in (-\infty, 1]$;

$$(g_{\mathrm{eq}}) \quad g(s) - s^{p-1} \begin{cases} < 0 & \text{if } 0 < s < 1 \\ = 0 & \text{if } s = 1 \\ > 0 & \text{if } s > 1; \end{cases}$$

(g_1) $\quad$ there exists $C_1 \in [0,\infty]$ such that $\lim_{s\to 1} \frac{g(s)-s^{p-1}}{|s-1|^{p-2}(s-1)} = C_1$.

We note that (g_{eq}) means that g intersects only *once* the power s^{p-1} at a point u_0, which without loss of generality is taken equal to 1. Furthermore, we observe that while the assumption in zero (i.e., $(g_0)'$) is just slightly more general than before (i.e., (g_0)), we have replaced (g_{u_0}) with (g_1). Condition (g_1) is implied by the regularity of g when $1 < p \le 2$. Indeed, since g is of class C^1 at 1, hypothesis (g_1) holds automatically with $C_1 \in [0, \infty)$ for $p = 2$, and with $C_1 = 0$ for $p < 2$. The only case in which the existence of the limit in (g_1) is not implied by the regularity of g (and so (g_1) is really an additional assumption) is when $p > 2$ and $g'(1) = 0$. We further observe that for $p > 2$, condition (g_{u_0}) required in [6] is stronger than (g_1), since (g_{u_0}) (for $u_0 = 1$) implies (g_1) with $C_1 = \infty$.

With this set of hypotheses, the prototype nonlinearity can be taken also of the form

$$g(s) = s^{q-1} + s^{p-1} - s^{r-1} \quad \text{with } p \le r < q,$$

so that the model equation becomes a little more general than the one in (1), namely

$$-\Delta_p u + u^{r-1} = u^{q-1} \quad \text{in } B_R.$$

Concerning the shape of the domain, we point out that the main feature needed here is the radial symmetry. In [3] it is also treated the case set in an *annular domain*. Since the arguments are similar to the ones for the ball, for the sake of simplicity we present here only the case of the ball. The main result in [3] is the following.

Theorem 4.1 (Theorems 1.2 and 1.4 of [3])**.** *Let $p > 1$ and λ_k^{rad} denote the k-th radial eigenvalue of $-\Delta_p$ with Neumann boundary conditions for any integer $k \ge 1$. If g satisfies $(g_{\text{reg}})'$–(g_1), then the following implications hold.*

(i) *If $C_1 > \lambda_{k+1}^{\text{rad}}$, then (2) admits at least k different non-constant radial solutions $u_1, \dots, u_k$. Furthermore, $u_j - 1$ has exactly j zeros for any $j = 1, \dots, k$.*

(ii) *If $C_1 = \infty$, then problem (2) admits infinitely many non-constant radial solutions.*

(iii) *If $C_1 = 0$, then*[3] *for every integer $k \ge 1$ there exists $R_*(k) > 0$ such that if $R > R_*(k)$, (2) admits at least $2k$ different non-constant radial solutions $u_1^\pm, \dots, u_k^\pm$. Furthermore, $u_j^\pm - 1$ has exactly j zeros for any $j = 1, \dots, k$.*

[3]When the domain is an annulus $A(R_1, R_2)$, part (iii) of Theorem 4.1 reads as

If $C_1 = 0$, then for every integer $k \ge 1$ and any $\varepsilon > 0$ there exists $R_*(k, \varepsilon) > 0$ such that if $R_1 < \varepsilon R_2$ and $R_2 > R_*(k, \varepsilon)$, (2) admits at least $2k$ different non-constant radial solutions $u_1^\pm, \dots, u_k^\pm$.

The oscillating behavior is the same as for the solutions in the ball.

We observe that in part (i), condition $C_1 > \lambda_{k+1}^{\mathrm{rad}}$ can be also read in terms of the radius R of the ball. Since the eigenvalues $\lambda_k^{\mathrm{rad}} = \lambda_k^{\mathrm{rad}}(R)$ are decreasing in R, keeping C_1 fixed (i.e., g fixed), we can increase the radius R in order to have the condition satisfied. In this way, the assumption in (i) becomes much more akin to the one in (iii).

Furthermore, when $g(s) = s^{q-1}$ with $q > p$, the constant C_1 in condition (g_1) specializes in

$$C_1 = \begin{cases} +\infty & \text{if } p > 2, \\ q-2 & \text{if } p = 2, \\ 0 & \text{if } 1 < p < 2, \end{cases}$$

and consequently Theorem 4.1 specializes in Theorem 1.1.

We sketch below the proof of Theorem 4.1, we refer to [3] for more details.

Clearly, part (ii) of the theorem can be seen as an immediate consequence of part (i), being $C_1 = \infty$ greater than every eigenvalue λ_k^{rad}. Furthermore, the proof of part (iii) is rather technical. Hence, we present below only the main ideas behind the proof of part (i) and we highlight the main reasons why the results for $C_1 = 0$ differs so much from the ones for $C_1 \in (0, +\infty)$.

• **Sketch of the proof of Theorem 4.1-(i).** We split the proof into six steps.

Step 1. *(Equivalent 1-dimensional problem)* Since we are dealing with radial positive solutions of (2), we can extend g to the whole of $\mathbb{R}$ in such a way that

$$f(s) := \begin{cases} g(s) - s^{p-1} & \text{if } s \geq 0, \\ 0 & \text{if } s < 0 \end{cases}$$

and write the problem in radial coordinates

$$\begin{cases} -(r^{N-1}|u'|^{p-2}u')' = r^{N-1}f(u) & \text{in } (0,R) \\ u'(0) = u'(R) = 0. \end{cases} \tag{4}$$

We observe that while the condition $u'(R) = 0$ comes from Neumann boundary conditions in (2), $u'(0) = 0$ is implied by symmetry and regularity of the solution.

Then we prove (cf. [3, Lemma 2.1]) the following maximum principle-type result.

If u solves (4), then either $u > 0$ in $[0, R]$ or $u \equiv -C$ for some $C \geq 0$.

As a consequence, in order to get (positive) solutions of the original problem (2), it is enough to find non-constant solutions of (4).

Step 2. *(Shooting method)* Let $\varphi_p(s) := |s|^{p-2}s$ and $v := r^{N-1}\varphi_p(u')$.

We consider the ODE system

$$\begin{cases} u' = \varphi_p^{-1}\left(\frac{v}{r^{N-1}}\right) & \text{in } (0,R), \\ v' = -r^{N-1}f(u) & \text{in } (0,R), \\ u(0) = d \in [0,1], \\ v(0) = 0. \end{cases} \tag{5}$$

We prove in [3, Lemma 2.2] global existence, uniqueness and continuous dependence for (5). These results are not trivial because the system (5) is not regular for three different reasons: at $r = 0$ we have a singularity of order $r^{-\frac{N-1}{p-1}}$ which is not integrable when $N \geq p$; φ_p^{-1} is not Lipschitz continuous at 0 when $p > 2$; f is not Lipschitz continuous at 0 when $1 < p < 2$. Nevertheless, using [9, Theorem 4], we are able to prove the following:

- *For all* $d \in [0,1]$ *there exists a unique* (u_d, v_d) *global solution of* (5).
- *If* $d_n \to d$ *then* $(u_{d_n}, v_{d_n}) \to (u_d, v_d)$ *uniformly in* $[0,R]$.

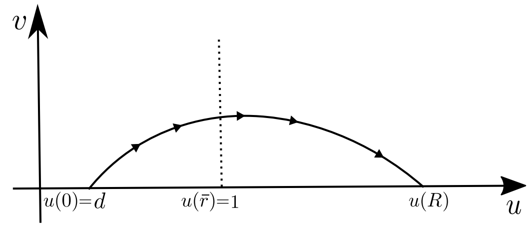
We observe that if (u,v) solves (5), then $u'(0) = 0$. This follows from the initial condition $v(0) = 0$, cf. [7]. Furthermore, by the definition of v, if $v(R) = 0$, also $u'(R) = 0$. Finally, for $d = 0$ and $d = 1$ we get the constant solutions $u \equiv 0$ and $u \equiv 1$, respectively. Hence, in order to get a non-constant solution of (2),

We look for $d \in (0,1)$ *such that the solution* (u_d, v_d) *of* (5) *satisfies* $v_d(R) = 0$.

This procedure is referred to as *shooting method.*

Step 3. *(Equivalent system in p-polar coordinates)*

If $v(0) = v(R) = 0$, by the regularity of v, there exists $\bar{r} \in (0,R)$ such that $v'(\bar{r}) = 0$. Thus, from the equation $v'(r) = -r^{N-1}f(u)$ and by (g_{eq}), $u(\bar{r}) = 1$.



Furthermore, by uniqueness, if $d \neq 1$, $(u_d(r), v_d(r)) \neq (1,0)$ for all $r \in [0,R]$. This means that *non-constant solutions of* (5) *having* $v(R) = 0$ *turn around the point* $(1,0)$ *in the phase plane* (u,v).

Hence, we can pass to p-polar coordinates[4] about $(1,0)$

$$\begin{cases} u - 1 = \rho^{\frac{2}{p}} \cos_p \theta \\ v = -\rho^{\frac{2}{p'}} \sin_p \theta \end{cases} \quad \Rightarrow \quad \text{if } \rho > 0: \quad \begin{array}{llll} u = 1 & \Leftrightarrow & \theta = (j + \frac{1}{2})\pi_p & (j \in \mathbb{Z}) \\ v = 0 & \Leftrightarrow & \theta = j\pi_p & (j \in \mathbb{Z}) \end{array}$$

to get the system

$$\begin{cases} \rho'(r) = \dfrac{p}{2\rho} u' \left[\varphi_p(u-1) - r^{(N-1)p'} f(u)\right] \\ \theta'(r) = r^{N-1} \left[\dfrac{p-1}{r^{(N-1)p'}} |\sin_p \theta|^{p'} + \dfrac{1}{\rho^2}(u-1) f(u)\right] \\ \theta(0) = \pi_p, \quad \rho(0) = (1-d)^{p/2}. \end{cases} \tag{6}$$

Thus, our goal becomes:

Find $d \in (0,1)$ *such that* $\theta_d(R) = j\pi_p$ *for some* $j \in \mathbb{Z}$.

We observe in passing that, by the equation for θ' in (6) and by (g_{eq}), we know that θ is monotone increasing.

Step 4. (Using the hypothesis $0 < C_1 < \lambda_{k+1}^{\mathrm{rad}}$*)* By (g_1) and by continuous dependence on d, we get for d close to 1

$$(u_d - 1) f(u_d) > (C_1 - \varepsilon)\, |u_d - 1|^p = (C_1 - \varepsilon)\, \rho_d^2 |\cos_p \theta_d|^p.$$

[4]See [3, Section 2 and Lemma 2.3] for the definition and properties of the functions p-cosine $\cos_p$ and p-sine $\sin_p$. Their name is due to the fact that these functions share many properties with the classical cosine and sine. For instance they are $2\pi_p$-periodic, where π_p is the number $\pi_p = \frac{2\pi(p-1)^{1/p}}{p \sin(\pi/p)}$. Furthermore, for $p = 2$, it holds $\cos_2 = \cos$, $\sin_2 = \sin$, and $\pi_2 = \pi$. The use of these functions is common in p-Laplacian problems, it allows to get the equation in ϑ of the associated eigenvalue system (9) *not coupled* with the equation in ϱ.

Hence, by (6), since $C_1 > \lambda_{k+1}^{\mathrm{rad}}$, for $\varepsilon > 0$ sufficiently small and d close to 1

$$\begin{aligned}\theta_d'(r) &= r^{N-1}\left[\frac{p-1}{r^{(N-1)p'}}|\sin_p\theta_d|^{p'} + \frac{1}{\rho_d^2}(u_d-1)f(u_d)\right]\\ &> r^{N-1}\left[\frac{p-1}{r^{(N-1)p'}}|\sin_p\theta_d|^{p'} + (C_1-\varepsilon)\,|\cos_p\theta_d|^{p}\right]\\ &> r^{N-1}\left[\frac{p-1}{r^{(N-1)p'}}|\sin_p\theta_d|^{p'} + \lambda_{k+1}^{\mathrm{rad}}|\cos_p\theta_d|^{p}\right].\end{aligned}$$

Step 5. (The associated eigenvalue problem) We will estimate the number of times that the solutions of the problem turn around $(1,0)$ by the number of times that the radial eigenfunctions of the Neumann p-Laplacian turn around $(0,0)$ in the phase plane. To this aim, we introduce the associated eigenvalue problem

$$\begin{cases} -\Delta_p\phi = \lambda^{\mathrm{rad}}|\phi|^{p-2}\phi & \text{in } B_R,\\ \partial_\nu\phi = 0 & \text{on } \partial B_R. \end{cases} \tag{7}$$

In [10, Theorem 1] it has been proved what follows.

The eigenvalue problem (7) *has a countable number of eigenvalues* $0 = \lambda_1^{\mathrm{rad}} < \lambda_2^{\mathrm{rad}} < \dots$ *which go to infinity as* $k \to \infty$*. Furthermore, the* k*-th eigenfunction* ϕ_k *has exactly* $k-1$ *zeros in* $(0,R)$*.*

Since we are interested only in radial eigenvalues, we can write (7) as

$$\begin{cases} -(r^{N-1}\varphi_p(\phi'))' = \lambda r^{N-1}\varphi_p(\phi) & \text{in } (0,R),\\ \phi'(0) = \phi'(R) = 0. \end{cases} \tag{8}$$

We now pass to p-polar coordinates around $(0,0)$, that is to say

$$\begin{cases} \phi = \varrho^{\frac{2}{p}}\cos_p\vartheta\\ \psi := r^{N-1}|\phi'|^{p-2}\phi' = -\varrho^{\frac{2}{p'}}\sin_p(\vartheta) \end{cases} \Rightarrow \text{ if } \varrho > 0: \quad \begin{array}{lll} \phi = 0 \;\Leftrightarrow\; \vartheta = (j+\frac{1}{2})\pi_p & & (j\in\mathbb{Z})\\ \psi = 0 \;\Leftrightarrow\; \vartheta = j\pi_p & & (j\in\mathbb{Z}). \end{array}$$

Hence, system (8) becomes

$$\begin{cases} \varrho'(r) = \dfrac{p}{2\varrho}\left(1-\lambda r^{(N-1)p'}\right)\varphi_p(\phi)\phi'\\ \vartheta'(r) = r^{N-1}\left[\dfrac{p-1}{r^{(N-1)p'}}|\sin_p\vartheta|^{p'} + \lambda|\cos_p\vartheta|^{p}\right]\\ \vartheta(0) = \pi_p, \quad \vartheta(R) = j\pi_p \quad (\exists\, j\in\mathbb{Z}). \end{cases} \tag{9}$$

From the second equation of (9), we get $\vartheta'(r) > 0$. Therefore, the fact that ϕ_{k+1} has exactly k zeros reads as $\vartheta_{\lambda_{k+1}}(R) = (k+1)\pi_p$.

Step 6. *(Comparing solutions with eigenfunctions)* We now know that

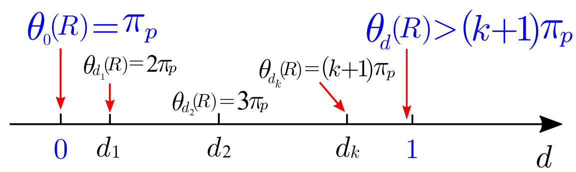
- $\theta_d'(r) > r^{N-1}\left[\frac{p-1}{r^{(N-1)p'}}|\sin_p \theta_d|^{p'} + \lambda_{k+1}^{\mathrm{rad}}|\cos_p \theta_d|^p\right]$ for d close to 1, by Step 4.;
- $\theta_d(0) = \vartheta_{\lambda_{k+1}}(0) = \pi_p$;
- $\vartheta'_{\lambda_{k+1}}(r) = r^{N-1}\left[\frac{p-1}{r^{(N-1)p'}}|\sin_p \vartheta_{\lambda_{k+1}}|^{p'} + \lambda_{k+1}^{\mathrm{rad}}|\cos_p \vartheta_{\lambda_{k+1}}|^p\right]$, by Step 5.

Therefore, by Comparison Theorem

$$\theta_d(R) > \vartheta_{\lambda_{k+1}}(R) = (k+1)\pi_p \quad \text{as } d \sim 1,$$

that is to say, the solution performs more than k half-turns around $(1,0)$ in the phase plane. Then, by continuous dependence of (u_d, v_d) and hence of (ρ_d, θ_d) on d, and by the fact that $\theta_0(R) = \pi_p$ (i.e., 0 turns), we obtain that there exist $d_1, \dots, d_k \in (0,1)$ such that

$$\theta_{d_j}(R) = (j+1)\pi_p \text{ for any } j = 1, \dots, k,$$



which correspond to the k non-constant radial solutions $u_1, \dots, u_k$. Furthermore, since $\theta_{d_j}(0) = \pi_p$, $\theta_{d_j}(R) = (j+1)\pi_p$, and θ_{d_j} is monotone increasing, we immediately get that $u_j - 1$ has exactly j zeros for any $j = 1, \dots, k$.

Remark 4.1. *With reference to Figure 1, we observe that in the pure power case $g(s) = s^{q-1}$, from Step 2. we can see that the solutions detected in Theorem 4.1 belong to the lower parts of the branches, since they all satisfy $u(0) = d < 1$.*

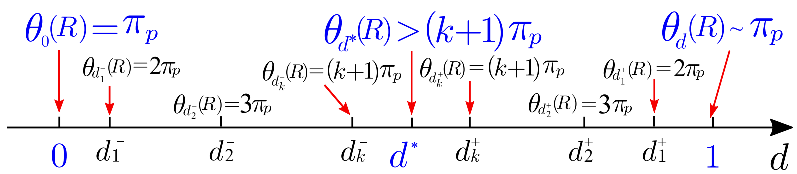
• **Comments on Theorem 4.1-(iii).** From a technical point of view, in the proof of part (i) it is crucial to have an estimate of the number of times that the solution of (5), shot from a point d close enough to 1 of the u-axis, turns around the point $(1,0)$ in the phase plane. Indeed, in Step 6. of part (i), we end up with the following estimate from below $\theta_d(R) > (k+1)\pi_p$ for $d \sim 1$. Instead, in the case $C_1 = 0$, thanks to an adaptation of [4, Corollary 5.1] (see [3, Lemma 2.8]), we get the following result:

If $R > R_*(k)$, there exists $d_* \in (0,1)$ such that $\theta_{d_*}(R) > (k+1)\pi_p$.

This means that we know that the number of half-turns is greater than $k+1$ for solutions shot at the *finite* distance $1-d^*$ from the point $(1,0)$, and not in the limit as $d \to 1$. Furthermore, by (g_1) and Gronwall's inequality, we prove for $\lambda = C_1 = 0$

$$\theta_d(R) \to \vartheta_0(R) = \pi_p \quad as\ d \to 1.$$

This allow us to make the continuous-dependence procedure effective both for solutions shot from $u(0) = d \in (0, d^*)$ and for solutions shot from $u(0) = d \in (d^*, 1)$. In this way, we obtain the double of the solutions found for $C_1 \in (0, \infty)$, as represented in the following picture.



More precisely, from one side, by continuous dependence on d and since $\theta_0(R) = \pi_p$, we have that

$$\textit{there exist } d_1^-, \dots, d_k^- \in (0, d_*) \ \textit{s.t. } \theta_{d_j^-}(R) = (j+1)\pi_p \ \textit{for all } j = 1, \dots, k.$$

On the other side, again by continuous dependence on d, being $\theta_{d_*}(R) > (k+1)\pi_p$, we obtain

$$\textit{there exist } d_1^+, \dots, d_k^+ \in (d_*, 1) \ \textit{s.t. } \theta_{d_j^+}(R) = (j+1)\pi_p \ \textit{for all } j = 1, \dots, k.$$

From a numerical point of view, some simulations performed in [3, Section 3] for $N = 1$, $R = 1$, $p = 1.97 < 2$, and $g(s) = s^{q-1}$ show that for values of $p < 2$ sufficiently close to 2 the branches of solutions persist. Differently from what we found for $p = 2$, now each branch splits into two and both the upper and the lower part of the branches *fold*, as represented in Figure 3. This heuristically explains why for $p < 2$ we find the double of solutions with respect to the case $p = 2$. Indeed, the shape of the branches is coherent with the result found in Theorem 1.1-(iii), since for every value of $q > p$, each *folded* branch contains now *two* different solutions having the same oscillatory behavior. Furthermore, none of the branches seem to bifurcate from the constant solution $u \equiv 1$, but each of

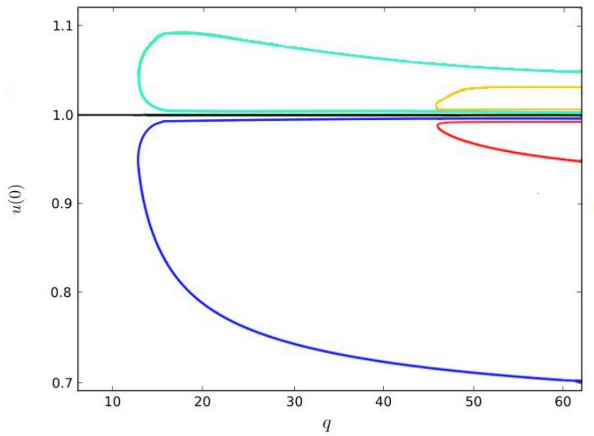


FIGURE 3. Qualitative representation of the first four branches of non-constant solutions for problem (2) in the case $N = 1$, $1 \ll p < 2$, $R = 1$, $g(s) = s^{q-1}$.

them seem to converge to the constant solution as $q \to \infty$. It looks like as the bifurcation point has escaped to infinity.

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