OPTIMAL CONCAVITY FOR NEWTONIAN POTENTIALS
CONCAVITÀ OTTIMALE PER POTENZIALI NEWTONIANI

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Abstract. In this note I give a short overview about convexity properties of solutions to elliptic equations in convex domains and convex rings and show a result about the optimal concavity of the Newtonian potential of a bounded convex domain in $\mathbb{R}^n$, $n \geq 3$, namely: if the Newtonian potential of a bounded domain is "sufficiently concave", then the domain is necessarily a ball. This result can be considered an unconventional overdetermined problem.

This paper is based on a talk given by the author in Bologna at the "Bruno Pini Mathematical Analysis Seminar", which in turn was based on the paper [26].

Sunto. In questa nota, darò un breve resoconto sulle proprietà di convessità di soluzioni di equazioni ellittiche in domini convessi o in anelli convessi e mostrerò un risultato di convessità ottimale per il potenziale Newtoniano di un dominio convesso in $\mathbb{R}^n$ ($n \geq 3$). In pratica: se il potenziale di un dominio convesso è "sufficientemente concavo", allora il dominio è necessariamente una palla. Questo risultato può essere considerato un problema sovradeterminato di tipo non convenzionale.

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1. Introduction

Convexity properties of solutions to partial differential equations have been an interesting issue of investigations since many years and to compile an exhaustive bibliography is almost impossible. Classical results are for instance the following.

1. The Torsion Problem

Let Ω be a convex domain in $\mathbb{R}^n$, $n \geq 2$, and u solve

$$
\begin{cases} \Delta u = -1 & \text{in } \Omega\,, \\ \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1}
$$

Then $\sqrt{u}$ is a concave function (this was first proved by L. Makar-Limanov [24] in the plane, for $n \geq 3$ see [2, 19]). The solution u of (1) is called *the torsion function* of Ω.

2. The Eigenvalue Problem

Let u_1 be a positive eigenfunction for the first positive Dirichlet eigenvalue of the Laplacian, i.e.

$$
\begin{cases} \Delta u_1 = -\lambda_1(\Omega) u_1 & \text{in } \Omega\,, \\ \\ u_1 = 0 \quad \text{on } \partial\Omega\,, & u_1 > 0 \quad \text{in } \Omega\,. \end{cases} \tag{2}
$$

If Ω is convex, then $\log u_1$ is a concave function (see [3]).

3. The Capacity problem

The *Newtonian capacity* of a bounded open set Ω in $\mathbb{R}^n$, $n \geq 3$, is defined as

$$
\mathrm{Cap}(\Omega) = \inf \left\{ \int_{\mathbb{R}^n} \frac{1}{2} |\nabla v|^2 dx \ : \ v \in C_0^\infty(\mathbb{R}^n), \ v \geq 1 \ \text{in } \Omega \right\} \tag{3}
$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. When Ω is sufficiently regular, and in particular if it is convex, (3) admits a unique minimizer u_Ω, which solves the following problem

$$\begin{cases} \Delta u_\Omega = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega}\,, \\ u_\Omega = 1 & \text{in } \overline{\Omega}\,, \\ u_\Omega \to 0 & \text{if } |x| \to \infty \end{cases} \tag{4}$$

and it is called *the Newtonian potential* of Ω.

Gabriel [14] and Lewis [20] proved that if Ω is convex, then all the superlevel sets $\{u_\Omega \geq t\}$ of u_Ω are convex.

The above results can be conveniently expressed by using the language of p-means and power concave functions, then let me introduce some notation.

Let $p \in [-\infty, +\infty]$ and $\mu \in (0,1)$. Given two real numbers $a > 0$ and $b > 0$, the quantity

$$M_p(a,b;\mu) = \begin{cases} \max\{a,b\} & p = +\infty \\ ((1-\mu)a^p + \mu b^p)^{\frac{1}{p}} & \text{for } p \neq -\infty, 0, +\infty \\ a^{1-\mu}b^\mu & p = 0 \\ \min\{a,b\} & p = -\infty. \end{cases}$$

is the *(μ-weighted)* p***-mean*** of a and b.
For $a, b \geq 0$, we set $M_p(a,b;\mu) = 0$ if $ab = 0$ (for any p).

Notice that for $p = 1$ we have the usual *arithmetic mean*, for $p = 0$ we have the usual *geometric mean*. A simple consequence of Jensen's inequality is

$$M_p(a,b;\mu) \leq M_q(a,b;\mu) \quad \text{if } p \leq q\,. \tag{5}$$

Moreover, we have

$$\lim_{p\to\pm\infty} M_p(a,b;\mu) = M_{\pm\infty}(a,b;\mu) \quad \text{and} \quad \lim_{p\to 0} M_p(a,b;\mu) = M_0(a,b;\mu)\,.$$

Let Ω be an open convex set in $\mathbb{R}^n$ and $p \in [-\infty, \infty]$. A function $v : \Omega \to [0, +\infty)$ is said p ***-concave*** if

$$v((1-\mu)x + \mu y) \geq M_p(v(x), v(y); \mu)$$

for all x, $y \in \Omega$ and $\mu \in (0,1)$.

In the cases $p = 0$ and $p = -\infty$, v is also said ***log-concave*** and ***quasi-concave*** respectively. In other words, a non-negative function u, with convex support Ω, is p-concave if:

- it is a non-negative constant in Ω, for $p = +\infty$;
- u^p is concave in Ω, for $p > 0$ ($p = 1$ corresponds to usual concavity);
- $\log u$ is concave in Ω, for $p = 0$ (log-concave);
- u^p is convex in Ω, for $p < 0$;
- it is quasi-concave, i.e. all of its superlevel sets are convex, for $p = -\infty$.

For more details on power concave functions, see [19].

In force of the above introduced notation and terminology, we can rephrase the previous examples as follows:

1. the torsion function of a convex domain is (1/2)-concave;
2. the first positive eigenfunction of the Laplacian (with Dirichlet boundary condition) of a convex domain is log-concave;
3. the Newtonian potential of a convex domain is quasi-concave.

It follows from (5) that *if v is p-concave, then v is q-concave for any $q \leq p$.* Then, following [19], it makes sense to define the ***the concavity exponent*** of a quasi-concave function v as

$$a(v) = \sup\{\beta \in \mathbb{R} \,:\, v \text{ is } \beta\text{-concave}\} \tag{6}$$

and then to introduce ***the Newtonian concavity exponent*** of a convex domain Ω as

$$\alpha(\Omega) = a(u_\Omega)\,, \tag{7}$$

where u_Ω is the Newtonian potential of Ω, given by the solution of (4).

Thanks to the continuity of M_p with respect to p, it is easily seen that when $a(v) \in \mathbb{R}$ the supremum in (6) is in fact a maximum.

Since quasi-concavity is the *weakest* conceivable concavity property, the result above described about Newtonian potentials may look weak and one can expect $\alpha(\Omega) > -\infty$

if Ω is strictly convex and sufficiently regular. Indeed, when Ω is a ball, $\alpha(\Omega)$ is easily calculated: the potential of a ball B centered at x_0 of radius R is

$$u_B(x) = \frac{|x-x_0|^{2-n}}{R^{2-n}}\,,$$

then

$$\alpha(B) = \frac{1}{2-n}$$

Here we want to study the behavior of $\alpha(\Omega)$ and prove that

$$-\infty \leq \alpha(\Omega) \leq \frac{1}{2-n} \quad \text{for every convex set } \Omega \subset \mathbb{R}^n\,. \tag{8}$$

Moreover the two inequalities are sharp and the second one is also rigid, in the sense specified by the following two theorems from [26]

Theorem 1.1 (Proposition 5.1 in [26]). *For every $n \geq 3$, there exist (infinitely many) convex sets $\Omega \subset \mathbb{R}^n$ such that $\alpha(\Omega) = -\infty$.*

Theorem 1.2 (Theorem 1.1 in [26]). *For every bounded convex set $\Omega \subset \mathbb{R}^n$ it holds*

$$\alpha(\Omega) \leq \frac{1}{2-n}$$

and equality holds if and only if Ω is a ball.

The former theorem is proved just by showing an example. To prove the latter, we will use four main ingredients.

Ingredient 1. An easy relation existing between the Capacity of a generic level set of u_Ω and the Capacity of Ω.

Ingredient 2. An expression of Capacity through the behavior at infinity of the potential function.

Ingredient 3. A level sets characterization of the concavity of a function.

Ingredient 4. The Brunn-Minkowski inequality for Capacity and its equality condition.

Ingredients 1 and 2 are mainly needed to study another overdetermined problem which has its own interest.

Theorem 1.3 (Theorem 1.2 in [26]). *If the solution u_Ω of* (4) *has two homothetic convex level sets, then Ω is a ball.*

In particular: *if u_Ω has a level set that is homothetic to Ω (and Ω is convex), then Ω is a ball.* I recall here that two sets $A, B \subset \mathbb{R}^n$ are said homothetic if there exist $\rho > 0$ and $\xi \in \mathbb{R}^n$ such that $B = \rho A + \xi$, i.e. if they are dilate and translate of each other.

The rest of the paper is organized as follows. Section 2 supplies the needed ingredients 1-4. In section Section 3 we will use the ingredients to cook the proofs of Theorem 1.3 and Theorem 1.2. Section 4 contains the proof ot Theorem 1.1, while in Section 5 there are some final remarks and comments about similar results for the torsional rigidity.

2. Preliminaries

2.1. **Ingredient 1: the capacity of a level set of the potential.** Let u be the Newtonian potential of a domain Ω and set

$$\Omega(t) = \{x \in \mathbb{R}^n \,:\, u(x) \geq t\}$$

for $t \in (0, 1]$. Then it is easily seen that the potential u_t of $\Omega(t)$ is given by $u_t(x) = t^{-1}u(x)$ and an integration by parts yields

$$\mathrm{Cap}(\Omega) = \int_{\partial\Omega} |\nabla u| \, d\sigma = \int_{\partial\Omega(t)} |\nabla u| \, d\sigma \quad \text{for every } t \leq 1\,,$$

whence

$$\mathrm{Cap}(\Omega(t)) = \int_{\partial\Omega(t)} |\nabla u_t| \, d\sigma = t^{-1} \int_{\partial\Omega(t)} |\nabla u| \, d\sigma = t^{-1}\mathrm{Cap}(\Omega)\,.$$

This is the first ingredient that we rewrite and label for better convenience:

$$\mathrm{Cap}(\Omega(t)) = t^{-1}\mathrm{Cap}(\Omega)\,. \tag{9}$$

2.2. **Ingredient 2: an expression of Newtonian capacity through the behavior at infinity of the potential.** The following relation between the Newton capacity of a convex domain and the behavior at infinity of the newtonian potential holds:

$$\mathrm{Cap}(\Omega) = (n-2)\,\omega_n \lim_{|x|\to\infty} u(x)|x|^{n-2}\,, \tag{10}$$

where ω_n denotes the surface area of the unit sphere in $\mathbb{R}^n$. An analogous relation holds also for p-Capacity, $p \in (1, n)$ (refer to [10] for instance).

2.3. **Ingredient 3: how to read concavity on the level sets.** A function u is concave if and only if the following relation between its level sets holds:

$$\{u \geq (1-\lambda)\ell_0 + \lambda\ell_1\} \supseteq (1-\lambda)\{u \geq \ell_0\} + \lambda\{u \geq \ell_1\} \tag{11}$$

for every $\ell_0, \ell_1 \in \mathbb{R}$ and every $\lambda \in [0, 1]$.

Here "+" stands for the Minkowski addition, which is defined as follows

$$A + B = \{x + y \mid x \in A,\ y \in B\},$$

while $\lambda A = \{\lambda x : x \in A\}$ for any $\lambda \in \mathbb{R}$, as usual.

To verify (11) is trivial.

Analogously we have the following characterization of power concavity:

The function $u \geq 0$ is p-concave if and only if

$$\{u \geq M_p(t_0, t_1; \lambda)\} \supseteq (1-\lambda)\{u \geq t_0\} + \lambda\,\{u \geq t_1\} \tag{12}$$

for every $t_0, t_1 > 0$ and for every $\lambda \in (0, 1)$. The proof is straightforward.

2.4. **Ingredient 4: the Brunn-Minkowski inequality for p-capacity.** We will use the following theorem.

Theorem 2.1. *Let K_1 and K_2 be n–dimensional convex bodies (i.e. compact convex subsets of $\mathbf{R}^n$ with non–empty interior), $n \geq 3$. Then*

$$[\mathrm{Cap}(\lambda K_1 + (1-\lambda)K_2)]^{\frac{1}{n-2}} \geq \lambda\ [\mathrm{Cap}(K_1)]^{\frac{1}{n-2}} + (1-\lambda)\ [\mathrm{Cap}(K_2)]^{\frac{1}{n-2}}\ , \tag{13}$$

for every $\lambda \in [0, 1]$. Moreover equality holds if and only if K_1 and K_2 are homothetic.

Inequality (13) was proved by C. Borell [2] and more recently in [5] L.A. Caffarelli, D. Jerison and E.H. Lieb treated the equality case. In [10] the treatments of the inequality and of its equality case are unified and the results are extended to the so called p-capacity.

Roughly speaking (13) says that $\mathrm{Cap}(\cdot)^{\frac{1}{n-2}}$ is a concave function in the class of convex bodies endowed with the Minkowsky addition. But here we are mainly interested in the equality condition: equality holds in (13) if and only if K_1 and K_2 are homothetic.

Let us pick the occasion to recall that the original form of the Brunn–Minkowski inequality involves volumes of convex bodies and states that $\mathrm{Vol}_n(\cdot)^{1/n}$ is a concave function with respect to the Minkowski addition, i.e.

$$[\mathrm{Vol}_n(\lambda K_1 + (1-\lambda)K_2)]^{\frac{1}{n}} \geq \lambda\, [\mathrm{Vol}_n(K_1)]^{\frac{1}{n}} + (1-\lambda)\, [\mathrm{Vol}_n(K_2)]^{\frac{1}{n}} \tag{14}$$

for every convex bodies K_1 and K_2 and $\lambda \in [0,1]$. Here Vol_n is the n-dimensional Lebesgue measure. Inequality (14) is one of the fundamental results in the modern theory of convex bodies; it can be extended to measurable sets and it is intimately connected to several other important inequalities of analysis and geometry, e.g. the isoperimetric inequality.

Suitable versions of the Brunn-Minkowski inequality hold also for the other quermassintegrals (see for instance [31]) and recently Brunn-Minkowski type inequalities have been proved for several important functionals of calculus of variations (see for instance the beautiful survey paper [15] by R. Gardner and [1, 9, 27] for more recent references). Notice that in all the known cases, equality conditions are the same as in the classical Brunn-Minkowski inequality for the volume, i.e. equality holds if and only if the involved sets are (convex and) homothetic (i.e. translate and dilate of each other).

3. Proof of the main theorems

First we prove Theorem 1.3.

Proof of Theorem 1.3. Hereafter, for simplicity we write u instead of u_Ω. Assume u has two homothetic level sets $\Omega(r)$ and $\Omega(s)$, for some $0 < r < s \leq 1$. This means that there exist $\rho > 1$ and $\xi \in \mathbb{R}^n$ such that $\Omega(r) = \rho\,\Omega(s) + \xi$. Up to a translation, we can assume $\xi = 0$, i.e.

$$\Omega(r) = \rho\,\Omega(s)\,. \tag{15}$$

Then, if we denote as u_r and u_s the Newtonian potentials of $\Omega(r)$ and $\Omega(s)$ respectively, it must hold

$$u_r(x) = u_s\big(\frac{x}{\rho}\big)\,.$$

On the other hand,

$$u_r(x)=\frac{u(x)}{r}\quad\text{for } x\in\mathbb{R}^n\setminus\Omega(r)\,,$$

$$u_s(x)=\frac{u(x)}{s}\quad\text{for } x\in\mathbb{R}^n\setminus\Omega(s)\,,$$

as we have already observed when introducing Ingredient 1.

Coupling the latter and the former, we finally get

$$u(x)=\frac{r}{s}\,u\big(\frac{x}{\rho}\big)\,,\quad x\in\mathbb{R}^n\setminus\Omega(r)\,. \tag{16}$$

whence

$$\Omega(t)=\rho\,\Omega(\frac{s}{r}\,t)\quad\text{for } t<r\,.$$

Moreover Ingredient 2 yields

$$\left(\frac{\mathrm{Cap}(\Omega)}{(n-2)\omega_n}\right)^{\frac{1}{p-1}}=\lim_{|x|\to\infty}u(x)|x|^{\frac{n-p}{p-1}}=\frac{r}{s}\lim_{|x|\to\infty}\,u\big(\frac{x}{\rho}\big)|x|^{\frac{n-p}{p-1}}$$

$$=\frac{r}{s}\,\rho^{\frac{n-p}{p-1}}\,\lim_{|x|\to\infty}\,u\big(\frac{x}{\rho}\big)\left(\frac{|x|}{\rho}\right)^{\frac{n-p}{p-1}}$$

$$=\frac{r}{s}\,\rho^{\frac{n-p}{p-1}}\,\left(\frac{\mathrm{Cap}(\Omega)}{(n-2)\omega_n}\right)^{\frac{1}{p-1}}\,,$$

whence

$$\frac{r}{s}=\rho^{\frac{p-n}{p-1}}\,. \tag{17}$$

Hence, by setting

$$s_0=s\,,\quad ,s_1=r\,,\quad s_k=\left(\frac{r}{s}\right)^k s=\rho^{\frac{k(p-n)}{p-1}}s\,,\quad k=2,3,\dots,$$

it holds

$$\lim_{k\to\infty}s_k=0$$

and

$$\Omega(s_k)=\rho\,\Omega(s_{k-1})=\rho^2\Omega(s_{k-2})=\cdots=\rho^k\Omega(s_0)=\rho^k\Omega(s)\,. \tag{18}$$

Now let $x, y \in \partial\Omega(s)$ (with $x \neq 0$ and $y \neq 0$), i.e.

$$u(x) = u(y) = s\,,$$

and set

$$x_k = \rho^k x\,, \quad y_k = \rho^k y\,.$$

Then

$$\lim_{k\to\infty} |x_k| = \lim_{k\to\infty} |y_k| = \infty$$

and Ingredient 2 yields

$$\lim_{k\to\infty} u(x_k)|x_k|^{\frac{p-1}{n-p}} = \left(\frac{\mathrm{Cap}(\Omega)}{(n-2)\omega_n}\right)^{1/(p-1)} = \lim_{k\to\infty} u(y_k)|y_k|^{\frac{p-1}{n-p}}\,. \tag{19}$$

On the other hand

$$u(x_k) = u(y_k) = s_k\,,$$

hence (19) reads

$$\lim_{k\to\infty} s_k|x_k|^{\frac{p-1}{n-p}} = \lim_{k\to\infty} s_k|y_k|^{\frac{p-1}{n-p}}\,,$$

that is

$$\lim_{k\to\infty}\left(\rho^{\frac{k(p-n)}{p-1}} s \left|\rho^k x\right|^{\frac{p-1}{n-p}}\right) = \lim_{k\to\infty}\left(\rho^{\frac{k(p-n)}{p-1}} s \left|\rho^k y\right|^{\frac{p-1}{n-p}}\right)\,.$$

Since

$$\frac{k(p-n)}{p-1} + \frac{k(p-1)}{n-p} = 0\,,$$

we finally have

$$|x| = |y| =: R\,,$$

which means that $\Omega(s)$ is a ball or radius R centered at the origin. Then u is radial in $\mathbb{R}^n \setminus \overline{\Omega(s)}$ and, by analytic continuation, it is radial in $\mathbb{R}^n \setminus \overline{\Omega}$ and Ω is a ball.

$\square$

Now we can prove Theorem 1.2.

Proof of Theorem 1.2. Let $u \in C^2(\mathbb{R}^n \setminus \overline{\Omega}) \cap C(\mathbb{R}^n)$ be the Newtonian potential of the convex set Ω and

$$q = -\frac{1}{n-2}\,.$$

First we recall that if u is α-concave for some $\alpha \geq q$, then it is q-concave. Next we will proceed by proving that, if u is q-concave, then all its level sets are homothetic and the proof will be concluded thanks to Theorem 1.3.

Assume that u is q-concave, i.e.

$$v = u^q \qquad \text{is convex in } \mathbb{R}^n\,.$$

Now take $r, s \in (0,1]$, fix $\lambda \in (0,1)$ and set

$$t = M_q(r, s; \lambda) = [(1-\lambda) r^q + \lambda\, s^q]^{1/q}\,. \tag{20}$$

Ingredient 3 yields

$$\Omega(t) \supseteq (1-\lambda)\,\Omega(r) + \lambda\,\Omega(s)\,. \tag{21}$$

where

$$\Omega(r) = \{u \geq r\}\,,\ \ \Omega(s) = \{u \geq s\}\,,\ \ \Omega(t) = \{u \geq t\}\,.$$

Thanks to the monotonicity of capacity with respect to set inclusion, (21) implies

$$\mathrm{Cap}(\Omega(t)) \geq \mathrm{Cap}\big((1-\lambda)\,\Omega(r) + \lambda\,\Omega(s)\big)$$

and using the Brunn-Minkowski inequality for capacity (Ingredient 4) we get

$$\begin{aligned} \mathrm{Cap}(\Omega(t))^{1/(n-2)} \geq\ & \mathrm{Cap}\big((1-\lambda)\,\Omega(r) + \lambda\,\Omega(s)\big)^{1/(n-2)} \geq \\ \geq\ & (1-\lambda)\,\mathrm{Cap}(\Omega(r))^{1/(n-2)} + \lambda\,\mathrm{Cap}(\Omega(s))^{1/(n-2)}\,. \end{aligned} \tag{22}$$

On the other hand, by Ingredient 1 we have

$$\mathrm{Cap}(\Omega(r)) = r^{-1}\mathrm{Cap}(\Omega)\,,$$

$$\mathrm{Cap}(\Omega(s)) = s^{-1}\mathrm{Cap}(\Omega)\,,$$

$$\mathrm{Cap}(\Omega(t)) = t^{-1}\mathrm{Cap}(\Omega)\,.$$

Plugging these into (22) and taking into account the expression (20) of t in terms of r and s, we finally get

$$\begin{aligned}\mathrm{Cap}(\Omega(t))^{1/(n-2)} &= [(1-\lambda)r^{-1/(n-2)} + \lambda\, s^{-1/(n-2)}]\mathrm{Cap}(\Omega)^{1/(n-2)} = \\ &= (1-\lambda)[r^{-1}\,\mathrm{Cap}(\Omega)]^{1/(n-2)} + \lambda\,[s^{-1}\mathrm{Cap}(\Omega)]^{1/(n-2)} = \\ &= (1-\lambda)\,\mathrm{Cap}(\Omega(r))^{1/(n-2)} + \lambda\,\mathrm{Cap}(\Omega(s))^{1/(n-2)}\,,\end{aligned}$$

i.e. equality holds in (22).

Hence *equality must hold in the Brunn-Minkowski inequality for Capacity* for $\Omega(r)$ and $\Omega(s)$. Then $\Omega(r)$ *and* $\Omega(s)$ *must be homothetic* and the conclusion follows from Theorem 1.3.

□

Remark 3.1. Notice that the above proof provides in fact a stronger result than Theorem 1.2. Indeed in the proof we do not use the full strength of the $1/(2-n)$-concavity of the potential function, but just the existence of three convex super level sets

$$\Omega(r) = \{x \in \mathbb{R}^n \,:\, u(x) \geq r\}\,,\;\; \Omega(s) = \{x \in \mathbb{R}^n \,:\, u(x) \geq s\}\,,\;\; \Omega(t) = \{x \in \mathbb{R}^n \,:\, u(x) \geq t\}$$

(say $0 < s < r \leq 1$) such that

$$[(1-\lambda)r^q + \lambda\, s^q]^{1/q} \leq t < r$$

and

$$\Omega(t) \supseteq (1-\lambda)\,\Omega(r) + \lambda\,\Omega(s)$$

for some $\lambda \in (0,1)$.

Let me also point out that convexity is needed only in order to apply the BM Inequality for Capacity, that has been proved only for convex sets up to now.

4. Genuinely quasi-concave potentials

Now, let us prove Theorem 1.1.

Let $n \geq 3$. We want to prove that there exist infinitely many convex set such that $\alpha(\Omega) = -\infty$, i.e whose Newtonian potential is not p-concave for any $p \in \mathbb{R}$.

Consider an hypercube $Q = \{x = (x_1, \ldots, x_n) : |x_i| < 1 \; i = 1, \ldots, n\}$ and let u_Q be its Newtonian potential. Then it is easily seen by a barrier argument (see for instance [25, Section 4.2]) that ∇u_Q blows up on the vertices and edges of Q (in fact at every singular point of ∂Q). Then, for any $p < 0$, the gradient of $v = u_Q^p$ blows up at the same points too, hence v can not be convex.

The argument obviously works for every convex polytopes. $\square$

A natural question is now whether for every $p \in (-\infty, -1/(n-2))$ there exists a convex set Ω such that $\alpha(\Omega) = p$. Without pretending to give here an exhaustive answer, we notice that the level sets $Q_t = \{u_Q \geq t\}$ of u_Q can provide a solution to this question, since they smoothly change from an hypercube to a ball as $t \to 0$. Then $\alpha(Q_t)$ smoothly increases from $-\infty$ to $-1/(n-2)$ as t decreases from 1 to 0.

5. Final comments and remarks

I finally want to address a question similar to that of Theorem 1.2 for problem (1).

As said in the introduction, the torsion function of a convex set is (1/2)-concave, and we can not say anything more in the general case, that is for every $p > 1/2$ there exist a convex set whose torsion function is not p-concave (see [19]). On the other hand, the solution of problem (1) when Ω is a ball (say $\Omega = B(0, R)$) is easily calculated as

$$u_B(x) = \frac{R^2 - |x|^2}{2n}$$

and we can see that u_B is concave, which is much more than (1/2)-concave. More generally, the same happens for every ellipsoid $E = \{x \in \mathbb{R}^n : \sum_{i=1}^n a_i(x_i - \bar{x}_i)^2 < R^2\}$, with $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ a point in $\mathbb{R}^n$, $a_i > 0$ for $i = 1, \ldots, n$, $\sum_{i=1}^n a_i = n$, whose torsion function is

$$u_E(x) = \frac{R^2 - \sum_{i=1}^n a_i(x_i - \bar{x}_i)^2}{2n},$$

Then we can wonder whether the concavity of the torsion function characterizes ellipsoids or not. But this is easily seen to be false: there are many other domains whose torsion function is concave, for instance smooth perturbations of ellipsoids, since u_E is uniformly concave.

However, a deep inspection of the concavity of u_E reveals that we can say more about it and the following property holds:

$$\text{the function} \quad v_E = \sqrt{\frac{R^2}{2n} - u_E} \quad \text{is convex}\,.$$

And the above property is sharp, since

$$v_E(x) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n} a_i(x_i - \bar{x}_i)^2}\,,$$

whose graph is a convex cone with vertex at $(\bar{x}, 0)$.

Then we introduce the following definition.

Definition 5.1. *Let Ω be a bounded convex domain in $\mathbb{R}^n$. We say that a function $u \in C(\overline{\Omega})$ satisfies the property (A) if*

$$(A) \qquad w(x) = \sqrt{M - u(x)} \quad \textit{is a convex function}\,,$$

where $M = \max_{\overline{\Omega}} u$.

It is easily seen that property (A) implies the concavity of u (which in turn implies its (1/2)-concavity).

Then it is natural to formulate the following conjecture.

Conjecture. *The torsion function of a convex domain Ω satisfies the property (A) if and only if Ω is an ellipsoid.*

The conjecture has been very recently proved to be true in [16]. The proof is based on a completely different technique from the one used here for the Newtonian potential.

Proposition 5.1. [16, Theorem 1.4] *Let Ω be a bounded open set and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be the solution to* (1)*. If u satisfies property (A), then Ω is an ellipsoid and $u = u_E$.*

A similar conjecture can be obviously formulated for problem (2), as Lindqvist first did in [21]. Precisely, once defined

$$\Lambda(\Omega) = \sup\{\beta \in \mathbb{R} \,:\, u_1 \text{ is } \beta\text{-concave}\}\,,$$

where u_1 is the first positive Dirichlet eigenfunction of the Laplacian in Ω, Lindqvist asks which convex domains maximize Λ and he conjectures the answer is the ball, also showing some evidences to support the conjecture (he in fact consider more in general the first eigenfunction of the p-laplacian). This question is still open, to my knowledge. But the situation in this case is probably more intricate and power concavity may be not enough to characterize eigenfunctions of a ball: there should be a suitable concavity property which plays in this case the same role that property (A) plays for the torsional rigidity.

Let me finally notice that Theorem 1.2, Theorem 1.3 and Proposition 5.1, as well as the conjecture of Lindqvist, can be regarded as (unconventional) overdetermined problems. In general, an overdetermined problem is a Dirichlet problem coupled with some extra condition. The archetypal one is the following Serrin problem:

$$\begin{cases} \Delta u = -1 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \\ |\nabla u| = \text{constant} & \text{on } \partial\Omega\,. \end{cases} \tag{23}$$

In a seminal paper [28], Serrin proved that a solution to (23) exists if and only if Ω is a ball. Since then, the literature about overdetermined problems has been continuously growing, but usually the extra condition imposed to the involved Dirichlet problem regards the normal derivative of the solution on the boundary of the domain, like in (23), and the solution is given by the ball. Recently different conditions have been considered, like for instance in [6, 7, 8, 11, 12, 22, 23, 29, 30]. Theorem 1.2, Theorem 1.3 and Proposition 5.1 can be included in this framework when the overdetermination is given by the convexity of $u^{2/(2-n)}$, by the existence of two homothetic level sets and by property (A) respectively. In connection with Proposition 5.1, let me also recall that overdetermined problems where the solution is affine invariant and it is given by ellipsoids are considered in [4, 13, 17].

References

[1] M. Akman, J. Gong, J. Hineman, J. Lewis, A. Vogel, *The Brunn-Minkowski inequality and a Minkowski problem for nonlinear capacity*, https://arxiv.org/abs/1709.00447v1, preprint 2017.

[2] Ch. Borell, *Greenian potentials and concavity*, Math. Anal. 272 (1985), 155-160.

[3] H. J. Brascamp, E. H. Lieb, *On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log-concave functions, and with an application to the diffusion equation*, J. Funct. Anal. **22** (1976), 366-389.

[4] B. Brandolini, N. Gavitone, C. Nitsch, C. Trombetti, *Characterization of ellipsoids through an overdetermined boundary value problem of Monge-Ampère type*, J. Math. Pures Appl. (9) 101 (2014), no. 6, 828-841.

[5] L. A. Caffarelli, D. Jerison, E. H. Lieb, *On the case of equality in the Brunn-Minkowski inequality for capacity*, Adv. Math. **117** (1996), 193-207.

[6] G. Ciraolo, R. Magnanini, *A note on Serrin's overdetermined problem*, Kodai Math. Jour. 37 (2014), 728-736.

[7] G. Ciraolo, R. Magnanini, S. Sakaguchi, *Symmetry of minimizers with a level surface parallel to the boundary*, J. Eur. Math. Soc. (JEMS) 17 (2015), no. 11, 2789-2804.

[8] G. Ciraolo, M. Magnanini, V. Vespri, *Symmetry and linear stability in Serrin's overdetermined problem via the stability of the parallel surface problem*

[9] A. Colesanti, *Brunn-Minkowski inequalities for variational functionals and related problems*, Adv. Math. **194** (2005), 105-140.

[10] A. Colesanti and P. Salani, *The Brunn-Minkowski inequality for p-capacity of convex bodies*, Math. Ann. **327** (2003), 459-479.

[11] G. Cupini, E. Lanconelli *On an inverse problem in potential theory*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **27** (2016), 431-442.

[12] G. Cupini, E. Lanconelli, *Densities with the mean value property for sub-laplacians: an inverse problem*, in: Chanillo S., Franchi B., Lu G., Perez C., Sawyer E. (eds) Harmonic Analysis, Partial Differential Equations and Applications. Applied and Numerical Harmonic Analysis. Birkhuser, Cham.

[13] C. Enache, S. Sakaguchi, *Some fully nonlinear elliptic boundary value problems with ellipsoidal free boundaries*, Math. Nachr. **284** (2011), n. 14-15, 1872-1879.

[14] M. Gabriel, *A result concerning convex level–surfaces of three–dimensional harmonic functions*, London Math. Soc. J. **32** (1957), 286–294.

[15] R.J. Gardner, *The Brunn-Minkowski inequality*, Bull. Amer. Math. Soc. (N.S.) **39** (2002), 355-405.

[16] A, Henrot, C. Nitsch, P. Salani, C. Trombetti, *Optimal concavity of the torsion function*, https://arxiv.org/abs/1701.05821, preprint 2017.

[17] A. Henrot, G. A. Philippin, *Some overdetermined boundary value problems with elliptical free boundaries*, SIAM J. Math. Anal. **29** n. 2 (1998), 309-320.

[18] B. Kawohl, *Rearrangements and convexity of level sets in P.D.E.*, Lecture Notes in Mathematics, 1150, Springer, Berlin 1985.

[19] A. U. Kennington, *Power concavity and boundary value problems*, Indiana Univ. Math. J. **34** 3 (1985), 687-704.

[20] J. Lewis, *Capacitary functions in convex rings*, Arch. Rational Mech. Anal. **66** (1977), 201–224.

[21] P. Lindqvist, *A note on the nonlinear Rayleigh quotient* Analysis, algebra, and computers in mathematical research (Lule, 1992), 223231, Lecture Notes in Pure and Appl. Math., 156, Dekker, New York, 1994.

[22] R. Magnanini, S. Sakaguchi, *Matzoh ball soup: heat conductors with a stationary isothermic surface*, Annals of Mathematics 156 (2002), 941-956.

[23] R. Magnanini and S. Sakaguchi, Nonlinear diffusion with a bounded stationary level surface, Ann. Inst. H. Poincaré Anal. Non Linéaire, 27 (2010), pp. 937-952.

[24] L.G. Makar-Limanov, *The solution of the Dirichlet problem for the equation* $\Delta u = -1$ *in a convex region*, Mat. Zametki 9 (1971) 89-92 (Russian). English translation in Math. Notes 9 (1971), 52-53.

[25] O. Mendez, W. Reichel, *Electrostatic characterization of spheres*, Forum Math. 12 (2000), 223-245.

[26] P. Salani, *A characterization of balls through optimal concavity for potential functions*, Proc. AMS 143 (1) (2015), 173–183.

[27] P. Salani, *Convexity of solutions and Brunn-Minkowski inequalities for Hessian equations in* R^3, Adv. Math. 229 (2012), 1924-1948.

[28] J. Serrin, *A symmetry problem in potential theory*, Arch. Rational Mech. Anal., **43** (1971), 304-318.

[29] P. W. Schaefer, *On nonstandard overdetermined boundary value problems*, Proceedings of the Third World Congress of Nonlinear Analysis, Part 4 (Catania, 2000). Nonlinear Anal. **47** n. 4 (2001), 2203-2212.

[30] H. Shahgholian, *Diversifications of Serrin's and related problems*, Complex Var. Elliptic Equ. 57 (2012), no. 6, 653-665.

[31] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Cambridge University Press, Cambridge 1993.

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