CHARACTERIZING SPHERES IN $\mathbb{C}^2$ BY THEIR LEVI CURVATURE: A RESULT À LA JELLETT
UNA CARATTERIZZAZIONE DELLE SFERE DI $\mathbb{C}^2$ DI TIPO JELLETT TRAMITE LA CURVATURA DI LEVI

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Abstract. We investigate rigidity problems for a class of real hypersurfaces in $\mathbb{C}^2$ with constant Levi curvature. We present a recent result obtained in [18] in collaboration with V. Martino for the boundaries of starshaped circular domains.

Sunto. Si desiderano investigare problemi di rigidità e di caratterizzazione per una classe di ipersuperfici reali di $\mathbb{C}^2$ con curvatura di Levi costante. Viene qui presentato un recente risultato contenuto in [18], ottenuto in collaborazione con V. Martino, in cui si considerano le ipersuperfici che delimitano domini circolari e stellati.

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1. Introduction

The Levi curvatures of a real hypersurface in $\mathbb{C}^{n+1}$ can be defined as the elementary symmetric functions of the eigenvalues of the Levi form. They were introduced and studied in [2, 7]. Since the Levi form involves a kind of restriction of the second fundamental form to the holomorphic tangent space, such a restriction provides a lack of information and hence a lack of ellipticity in the operator describing the curvatures. Nevertheless, under suitable pseudo-convexity assumptions for the hypersurface, the direction of missing ellipticity is recovered through bracket commutations (we refer the reader to [21]). Therefore, such operators can be seen as non-linear degenerate-elliptic operators of sub-Riemannian type. This very special feature has been successfully exploited in the literature, e.g. by Citti-Lanconelli-Montanari in [6] where they were able to prove a regularity result for

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graphs in $\mathbb{C}^2$ with a prescribed smooth curvature.

Here we want to discuss symmetry/rigidity problems, and in particular an Aleksandrov *soap bubble*-type problem such as the characterization of the spheres as the only closed hypersurfaces with constant Levi curvatures. To this aim, the classical approaches by Aleksandrov in [1] and by Reilly in [23] seem not working so far. Let us briefly explain the main reasons. The approach by Aleksandrov is the celebrated moving-plane method and it is based, very roughly speaking, on comparisons between the given hypersurface M and suitable reflected copies of M. Interior strong comparison results hold true for the Levi curvatures by the results in [5, 21], whereas the boundary comparison principles are more delicate and may fail (see [9, 17, 25]). In any case, the reflections with respect to generic hyperplanes (which are the ones exploited in the classical moving-plane method) do not preserve the complex structure, and the Levi curvature is not invariant through them: this fact represents a strong impediment in running such method (however, in principle, other kinds of reflections might work). On the other side, the approach by Reilly is based on integral formulas. Concerning the Levi curvatures, integral representation formulas were proved in [15], whereas an analogue of the needed Minkowski formula does not hold in general (see for instance [20, 26, 19, 13]). There have bee recent developments in [19, 8] for different kind of Minkowski type formulas adapted to the complex structure, but it is not yet clear how to exploit them in a possible Reilly-type approach.

Despite of the various obstacles we have just described, in the literature some partial answers to the Aleksandrov-type problem for the Levi curvature have been given in [9, 10, 14, 24, 18] (see also [22, 16]). These results strongly rely on extra-assumptions about a priori symmetries for the hypersurface. Among them, we would like to highlight the result by Hounie and Lanconelli in [9]. They proved that the balls are the only bounded Reinhardt domains in $\mathbb{C}^2$ whose boundary has constant Levi curvature. An open subset Ω of $\mathbb{C}^2$ is called a Reinhardt domain (with center at the origin) if

$$(z_1, z_2) \in \Omega \qquad \Longrightarrow \qquad (e^{i\theta_1} z_1, e^{i\theta_2} z_2) \in \Omega \quad \forall\, \theta_1, \theta_2 \in \mathbb{R}.$$

Due to this symmetry, a Reinhardt domain $\Omega \subset \mathbb{C}^2$ can be naturally described with a domain in $\mathbb{R}^2$, and $\partial\Omega$ as a curve in $\mathbb{R}^2$. The fact of having constant Levi curvature gives rise to an ODE for the function parametrizing such curve. This is what Hounie and Lanconelli considered in [9]. To be more precise, by writing locally the defining function $f(z_1, z_2)$ as $F(|z_2|^2) - |z_1|^2$, they showed that F has to be solution of the ODE

$$sFF'' = sF'^2 - L(F + sF'^2)^{3/2} - FF'.$$

They proved a uniqueness result for the solutions to this degenerate second order ODE starting at $s = 0$, which lead to their Aleksandrov-type theorem.
The purpose of this note is to present a true extension of Hounie-Lanconelli's theorem which has been obtained together with V. Martino in [18] with completely different techniques. To this aim, we have to recall the following definition

Definition 1.1. *An open set $\Omega \subset \mathbb{C}^2$ is said to be circular (with center at the origin) if*

$$(z_1, z_2) \in \Omega \quad \Longrightarrow \quad (e^{i\theta} z_1, e^{i\theta} z_2) \in \Omega \quad \forall\, \theta \in \mathbb{R}.$$

Circular domains were studied by Carathéodory in [3] and by Cartan in [4] for the issues regarding the analytic representation (see also [12]). It is clear from the definition that Reinhardt domains, enjoying one more symmetry, form a special subclass of the class of circular domains. A defining function of a circular domain can be written in fact as a function of three real independent variables, and therefore ODE techniques are not allowed to treat the boundary of such domains. For our Aleksandrov-type result we are actually going to consider the class of domains which are circular and starshaped (with respect to the same point). It is interesting to notice that Cartan studied the role of the circular starshaped domains (*cerclé étoilé* in his terminology in [4]) and he proved, among other results, that holomorphic functions in a circular domain extend holomorphically to the smallest circular starshaped domain containing the initial domain.
The main consequence of our results in [18] is then the following

Theorem 1.1. *Let Ω be a circular starshaped bounded open subset of $\mathbb{C}^2$. Suppose $\partial\Omega$ is a smooth hypersurface with constant Levi curvature. Then Ω is a ball.*

We remark that a compact hypersurface with no boundary is forced to have positive Levi curvature at some point. Hence, if the Levi curvature is constant, it has to be a positive constant. In [18, Lemma 4.1] it is showed that a bounded Reinhardt domain of $\mathbb{C}^2$ whose boundary has positive Levi curvature must be starshaped with respect to the origin. That is why we can recover the result by Hounie and Lanconelli in [9] from Theorem 1.1.

The complete proof of Theorem 1.1 can be found in [18], where the circular assumption for Ω is seen as a particular case of a more general geometric condition. In this note we just present the main steps of the approach we adopted, which is a PDE-approach inspired by an old proof by Jellett in [11]. We are going to introduce a Hörmander operator $\mathcal{L}$ on the hypersurface which is a first order horizontal perturbation of 'the' sub-Laplacian, and we are going to exploit the strong maximum principle for $\mathcal{L}$ after the choice of a Jellet-type function. In Section 2 we recall the proof by Jellett and we see the difficulties in trying to transfer such a proof to our situation. The starshapedness assumption, which already appears in the classical Jellett's theorem, finds here the main justification. In Section 3 we stress the role of the circular assumption to get a crucial pointwise identity, and we conclude by obtaining a horizontal umbilicality condition for the hypersurface.

2. The Jellett approach

Let us say that a smooth orientable hypersurface in $\mathbb{R}^{n+1}$ is starshaped if it is a boundary of a bounded domain which is strictly starshaped. In 1853, i.e. one century before the complete characterization by Aleksandrov, Jellett proved in [11] that

any starshaped hypersurface in $\mathbb{R}^{n+1}$ with constant mean curvature is a sphere.

To be precise, Jellett proved his theorem for two-dimensional surfaces in $\mathbb{R}^3$, but the same arguments work in any dimension. Let us sketch here such a proof.

Let Σ be a smooth orientable hypersurface in $\mathbb{R}^{n+1}$, with $n \geq 1$. Suppose that $\Sigma \subset \mathbb{R}^{n+1}$ is starshaped with respect to the origin. Let ν be the unit outward normal to Σ, $p \in \mathbb{R}^{n+1}$ be the position vector. Let us also denote respectively by Δ_Σ and h the Laplace-Beltrami operator on Σ and its second fundamental form. The mean curvature is then given by

$$H = \frac{1}{n}\mathrm{trace}(h).$$

If we assume H is constant, a straightforward computation and the use of Codazzi equations yield

$$\Delta_\Sigma\left(H\frac{|p|^2}{2}-\langle p,\nu\rangle\right)=\left(\|h\|^2-nH^2\right)\langle p,\nu\rangle\,.$$

Here we used the notation $\|\cdot\|^2$ for the squared norm of a matrix, namely the sum of all of its squared coefficients. The starshapedness assumption is saying that

$$\langle p,\nu\rangle>0\quad\forall\, p\in\Sigma.$$

On the other hand, for any symmetric $n\times n$ matrix A, we recall that

$$\|A\|^2\geq\frac{1}{n}(\mathrm{trace}(A))^2, \tag{1}$$

and the equality occurs if and only if the matrix A is a multiple of the identity. This is saying that also $\|h\|^2-nH^2\geq 0$. Therefore, the function $H\frac{|p|^2}{2}-\langle p,\nu\rangle$ is Δ_Σ-subharmonic, and thus constant being Σ compact without boundary. In particular, since $\langle p,\nu\rangle$ is strictly positive, $\|h\|^2=nH^2=\frac{1}{n}(\mathrm{trace}(h))^2$. The equality case in (1) says that Σ is umbilical. Hence it has to be a sphere by classical arguments.

Let us try to apply the same approach in our setting in $\mathbb{C}^2$ for the case of constant Levi curvature hypersurfaces. We first recall the needed notations. We identify $\mathbb{C}^2\simeq\mathbb{R}^4$, where the generic point $(z_1,z_2)=(x_1,y_1,x_2,y_2)$. We fix a smooth connected orientable hypersurface M, boundary of a bounded domain Ω. We put ν the unit outward normal to M and we denote by ∇ the Levi-Civita connection related to the standard inner product $\langle\cdot,\cdot\rangle$. The second fundamental form of M is the bilinear form on the tangent space TM given by

$$h(\cdot,\cdot):=\langle\nabla_{(\cdot)}\nu,\cdot\rangle,$$

and $H=\frac{\mathrm{trace}(h)}{3}$ is the mean curvature of M. In $\mathbb{C}^2$ we consider the standard complex structure J for which $J\partial_{x_j}=\partial_{y_j}$ and $J\partial_{y_j}=-\partial_{x_j}$. It is compatible with $\langle\cdot,\cdot\rangle$ and ∇ in the following sense:

$$\langle\cdot,\cdot\rangle=\langle J\cdot,J\cdot\rangle,\qquad J\nabla=\nabla J. \tag{2}$$

Thanks to J we can define the unit characteristic vector field $X_0\in TM$ by $X_0:=-J\nu$. The horizontal distribution or Levi distribution HM is the 2-dimensional subspace in TM

which is invariant under the action of J:

$$HM = TM \cap JTM,$$

that is a vector field $X \in TM$ belongs to HM if and only if also $JX \in HM$. Then TM splits in the orthogonal direct sum:

$$TM = HM \oplus \mathbb{R}X_0.$$

We can thus consider an orthonormal frame for TM of the form $E := \{X_0, X_1, X_2\}$, where $X_1 \in HM$ is a unit vector field and $X_2 = JX_1$. We denote by

$$h_{jk} = h(X_j, X_k), \qquad j,k = 0,1,2,$$

the coefficients of the second fundamental form with respect to E. In such a basis we have

$$h = \begin{pmatrix} h_{00} & h_{01} & h_{02} \\ h_{01} & h_{11} & h_{12} \\ h_{02} & h_{12} & h_{22} \end{pmatrix}.$$

Let us also define the horizontal part of h

$$h_H = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix}.$$

A possible way to define the Levi curvature of M is as the normalized trace of h_H, that is

$$L = \frac{1}{2}\mathrm{trace}(h_H) = \frac{h_{11} + h_{22}}{2}.$$

In other words,

$$3H = 2L + h_{00}.$$

Having in mind the Jellett's approach we need to consider a second order operator which plays the role of the Laplace-Beltrami operator. For any smooth function $u : M \to \mathbb{R}$, we define the Hessian of u as follows

$$Hess(u)(X,Y) = XYu - (\nabla^M_X Y)u, \qquad \forall\, X,Y \in TM,$$

where

$$\nabla^M_X Y = \nabla_X Y + h(X,Y)\nu, \qquad \forall\, X,Y \in TM.$$

Let us then consider the horizontal Hessian of u

$$Hess_H(u)(X,Y) = XYu - (\nabla^M_X Y)u, \qquad \forall\, X,Y \in HM.$$

The Laplace-Beltrami operator Δ_M acting on u can be seen either as the divergence of the gradient of u or as the trace of the Hessian of u. In order to define a model subelliptic operator such as 'the subLaplacian', it is worth to notice that the divergence of the horizontal gradient of u and the trace of the horizontal Hessian of u do not coincide in general. For this reason we might define two different operators. First, let us consider the divergence form operator

$$\Delta_H^{div} := \Delta_M - X_0 X_0.$$

This is in divergence form since the characteristic vector field X_0 is always divergence free: in fact, by (2), we have

$$\begin{aligned}\mathrm{div} X_0 &= \langle \nabla^M_{X_0} X_0, X_0\rangle + \langle \nabla^M_{X_1} X_0, X_1\rangle + \langle \nabla^M_{X_2} X_0, X_2\rangle \\ &= \langle \nabla_{X_1}\nu, X_2\rangle - \langle \nabla_{X_2}\nu, X_1\rangle = 0.\end{aligned}$$

On the other hand, we can also consider the trace of $Hess_H$, which can be written in our orthonormal frame E as

$$\Delta_H u := \sum_{j=1}^{2} \left(X_j X_j u - (\nabla^M_{X_j} X_j) u \right).$$

The difference of the two operators is given by the following first order horizontal vector field

$$V := \nabla^M_{X_0} X_0 = h_{02} X_1 - h_{01} X_2 \;\in HM. \tag{3}$$

As a matter of fact, we have

$$\Delta_H^{div} = \Delta_H - V.$$

Let us explicitly remark that V (which is the null vector if M is the sphere) is not divergence free in general. We also stress that both the operators are purely horizontal, in the sense that first order derivatives along X_0 do not appear. In fact, by the properties of the complex structure J and the symmetries of the second fundamental form, we get

$$\langle \nabla^M_{X_1} X_1, X_0\rangle + \langle \nabla^M_{X_2} X_2, X_0\rangle = \langle \nabla_{X_1} X_2, \nu\rangle - \langle \nabla_{X_2} X_1, \nu\rangle = 0.$$

From now on, let us assume $M = \partial\Omega$ has constant Levi curvature L. We recall that this easily implies that $L > 0$. It is important to notice that, since $L > 0$, both Δ_H^{div} and Δ_H are Hörmander type operators since we have the following step-two Hörmander condition

$$\langle [X_1, X_2], X_0\rangle = \langle \nabla_{X_1} X_2 - \nabla_{X_2} X_1, X_0\rangle = 2L > 0. \tag{4}$$

Inspired by the Jellett's proof, we suppose that M is starshaped with respect to 0 and we consider the function $u : M \longrightarrow \mathbb{R}$ defined as

$$u = L\frac{|p|^2}{2} - \langle p, \nu\rangle .$$

We can compute $\Delta_H u$ and $\Delta_H^{div} u$. To this aim, we introduce the notations

$$a_1 := \langle p, X_1\rangle , \qquad a_2 = \langle p, X_2\rangle , \qquad A_1 := h_{01}h_{12} - h_{11}h_{02}, \qquad A_2 := h_{22}h_{01} - h_{12}h_{02}.$$

It is proved in [18, Lemma 3.1 and Lemma 3.2], by using the fact that L is constant and the Codazzi equations, that

$$\Delta_H^{div} u = (\|h_H\|^2 - 2L^2)\langle p, \nu\rangle + ((h_{01})^2 + (h_{02})^2))\langle p, \nu\rangle - 3(a_1A_1 + a_2A_2) - L(a_1h_{02} - a_2h_{01}),$$

$$\Delta_H u = (\|h_H\|^2 - 2L^2)\langle p, \nu\rangle + ((h_{01})^2 + (h_{02})^2))\langle p, \nu\rangle - 2(a_1A_1 + a_2A_2).$$

The term $(\|h_H\|^2 - 2L^2)\langle p, \nu\rangle$ is the exact analogous of the Euclidean case, where the term $(\|h\|^2 - nH^2)\langle p, \nu\rangle$ magically appears. The inequality (1) applied to the matrix h_H and the starshapedness assumption imply in fact that

$$(\|h_H\|^2 - 2L^2)\langle p, \nu\rangle \geq 0.$$

If we want to conclude that u is either Δ_H^{div} or Δ_H-subharmonic, the problem relies on the remaining terms. For both of them, we have no clue about their sign. Even more importantly, we don't have a significant class of examples for which we can guarantee a nonnegative sign. This is the reason why we introduce a first order perturbation of the two 'subLaplacians'.

3. Integral and pointwise identities

The subelliptic operator we want to deal with is the following

$$\mathcal{L} = \Delta_H + 4V = \Delta_H^{div} + 5V.$$

Such operator is (in general) neither the trace of the horizontal Hessian nor the divergence of the horizontal gradient, and it is not (in general) in divergence form. If we now compute $\mathcal{L}u$ (see [18, Proof of Theorem 1.1]), we get

$$\mathcal{L}u = (\|h_H\|^2 - 2L^2)\langle p,\nu\rangle + 3((h_{01})^2 + (h_{02})^2))\langle p,\nu\rangle + 2R^{\mathcal{L}},$$

where $R^{\mathcal{L}}$ is defined by the following quantity

$$R^{\mathcal{L}} := 2L(a_1 h_{02} - a_2 h_{01}) + (a_1 A_1 + a_2 A_2) - \left((h_{01})^2 + (h_{02})^2\right)\langle p,\nu\rangle .$$

At a first sight, it seems to be just another remainder term. The reason for the particular choice of the operator $\mathcal{L}$ relies in the following lemma, which is proved in [18, Lemma 3.3].

Lemma 3.1. *We have*

$$R^{\mathcal{L}} = W(\langle p, X_0\rangle),$$

where $W = JV = h_{01}X_1 + h_{02}X_2$.
Moreover, when L *is constant,* W *is divergence free.*

The divergence free condition implies that

$$\int_M R^{\mathcal{L}}\, d\sigma = 0,$$

which says that $\mathcal{L}u$ has nonnegative average. We still don't know if $\mathcal{L}u$ is pointwise nonnegative for a general starshaped hypersurface M with constant Levi curvature. What we have proved in [18, Proof of Corollary 1.1] is that

$$\text{if } \Omega \text{ is circular} \quad \Longrightarrow \quad \langle p, X_0\rangle = 0 \quad \forall\, p \in \partial\Omega,$$

which implies by Lemma 3.1 that $R^{\mathcal{L}} \equiv 0$. This is the crucial point where the assumption of being circular comes into play. Recalling now the starhapedness of M and the algebraic inequality (1) applied to the matrix h_H, we have

$$\mathcal{L}u \geq 0 \quad \text{in } M,$$

i.e. u is a smooth subsolution for $\mathcal{L}$. On the other hand, since M is compact, u must have a maximum point. Such a maximum point has to be interior since M has no boundary. We can then exploit the fact that $\mathcal{L}$ is an Hörmander-type operator by (4). This says that the strong maximum principle holds true for $\mathcal{L}$, and therefore u is forced to be constant and $\mathcal{L}u \equiv 0$. Since $0 = \mathcal{L}u = (\|h_H\|^2 - 2L^2)\langle p, \nu\rangle + 3((h_{01})^2 + (h_{02})^2))\langle p, \nu\rangle$ and having $\langle p, \nu\rangle > 0$, we thus get

$$\|h_H\|^2 = 2L^2 \qquad \text{and} \qquad h_{01} = h_{02} = 0.$$

This gives at one time that h_H is a multiple of the identity matrix (from the equality case in (1)) and the fact that the full second fundamental form h is diagonal. It is now not difficult to conclude that M is a sphere. For more details we refer to [18, Proof of Theorem 1.1], where the same desired conclusion is deduced from a weaker information.

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