LINEAR PARABOLIC MIXED PROBLEMS IN SPACES OF HÖLDER CONTINUOUS FUNCTIONS: OLD AND NEW RESULTS
PROBLEMI MISTI PARABOLICI LINEARI IN SPAZI DI FUNZIONI HÖLDERIANE: RISULTATI VECCHI E NUOVI

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Abstract. We illustrate some old and new results, concerning linear parabolic mixed problems in spaces of Hölder continuous functions: we begin with the classical Dirichlet and oblique derivative problems and continue with dynamic and Wentzell boundary conditions.

Sunto Presentiamo alcuni risultati vecchi e nuovi riguardo a problemi misti lineari parabolici in spazi di funzioni hölderiane. Iniziamo con le classiche condizioni al contorno di Dirichlet e di derivata obliqua e continuiamo con condizioni dinamiche e di Wentzell.

The main aim of this seminar is to illustrate some old and new results concerning linear mixed parabolic problems in spaces of Hölder continuous functions. The common feature of them is that they are "maximal regularity results", in the sense that they establish linear and topological isomorphisms between spaces of solutions and spaces of data. For the sake of simplicity, we are going to consider only autonomous problems (in the sense that we are not considering the case that the coefficients depend on time). Moreover, we shall consider only second order linear problems. We observe that maximal regularity results usually allow a much simpler treatment of more complicated situations (nonautonomous, nonlinear), by linearization and perturbation arguments. So the parabolic equation will always be the following:

$$D_t u(t,\xi) - A(\xi, D_\xi) u(t,\xi) = f(t,\xi), \quad t \in (0,T), \xi \in \Omega, \tag{1}$$

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We shall always assume that:

(AA1) Ω *is an open bounded subset of* $\mathbb{R}^n$*, lying on one side of its boundary* $\partial\Omega$*, which is a submanifold of class* $C^{2+\beta}$ *of* $\mathbb{R}^n$*, for some* $\beta \in (0,1)$*;*

(AA2) $A(\xi, D_\xi) = \sum_{|\alpha|\le 2} a_\alpha(\xi) D_\xi^\alpha$, $a_\alpha \in C^\beta(\Omega)$ $\forall\alpha$ *with* $|\alpha| \le 2$*; if* $|\alpha| = 2$, a_α *is real valued and* $\sum_{|\alpha|=2} a_\alpha(\xi)\eta^\alpha \ge N|\eta|^2$ *for some* $N \in \mathbb{R}^+$, $\forall\xi \in \overline{\Omega}$, $\forall\eta \in \mathbb{R}^n$*;*

We shall consider several different boundary conditions, starting from the most classical (Dirichlet and oblique derivative), up to other boundary conditions (dynamic and Wentzell) which have been studied in detail more recently.

Of course, the main interest in this kind of results lies in the fact that spaces of Hölder continuous functions are the closest relatives of spaces of purely continuous functions where maximal regularity results hold.

We begin by recalling the definitions of the spaces we shall employ.

Let Ω be an open subset of $\mathbb{R}^n$. We shall indicate with $C(\Omega)$ the space of complex valued, uniformly continuous and bounded functions with domain Ω. If $f \in C(\Omega)$, it is continuously extensible to its topological closure $\overline{\Omega}$. We shall identify f with this extension. If $m \in \mathbb{N}$, we indicate with $C^m(\Omega)$ the class of functions f in $C(\Omega)$, whose derivatives $D^\alpha f$, with order $|\alpha| \le m$, belong to $C(\Omega)$. $C^m(\Omega)$ admits the natural norm

$$\|f\|_{C^m(\Omega)} := \max\{\|D^\alpha f\|_{C(\Omega)} : |\alpha| \le m\}, \tag{2}$$

with $\|f\|_{C(\Omega)} := \sup_{x\in\Omega} |f(x)|$. If

$$[f]_{C^\beta(\Omega)} := \sup_{x,y\in\Omega, x\ne y} |x-y|^{-\beta}|f(x) - f(y)|\}$$

and $m \in \mathbb{N}_0$, we set

$$\|f\|_{C^{m+\beta}(\Omega)} := \max\{\|f\|_{C^m(\Omega)}, \max\{[D^\alpha f]_{C^\beta(\Omega)} : |\alpha| = m\}\}, \tag{3}$$

and, of course, $C^{m+\beta}(\Omega) = \{f \in C^m(\Omega) : \|f\|_{C^{m+\beta}(\Omega)} < \infty\}$.

Let X be a Banach space. If A is a set, we shall indicate with $B(A; X)$ the Banach space of bounded functions from A to X. If $m \in \mathbb{N}_0$, $\beta \ge 0$ and Ω is an open subset of $\mathbb{R}^n$, the definitions of $C^m(\Omega; X)$ and $C^\beta(\Omega; X)$ and of the norms $\|\cdot\|_{C^\beta(\Omega;X)}$ can be obtained by obvious modifications of the corresponding, in the case $X = \mathbb{C}$.

If I is an open interval in $\mathbb{R}$, Ω is an open subset of $\mathbb{R}^n$ and α, β are nonnegative, we set

$$C^{\alpha,\beta}(I \times \Omega) := C^{\alpha}(I; C(\Omega)) \cap C^{\beta}(\Omega; C(I)), \tag{4}$$

equipped with its natural norm

$$\|f\|_{C^{\alpha,\beta}(I\times\Omega)} := \max\{\|f\|_{C^{\alpha}(I;C(\Omega))}, \|f\|_{C^{\beta}(\Omega);C(I))}\}.$$

We shall be in particularly interested in spaces $C^{1+\beta/2,2+\beta}(I \times \Omega)$, with $\beta \in (0,1)$. The following properties of this space will be important for us:

Lemma 1. *Let $\beta \in (0,1)$ and suppose that there exists a common linear bounded extension operator, mapping $C^{\gamma}(\Omega)$ into $C^{\gamma}(\mathbb{R}^n)$, $\forall \gamma \in [0, 2+\beta]$. Then*
(I) $C^{1+\beta/2,2+\beta}(I \times \Omega) = C^{1+\beta/2}(I; C(\Omega)) \cap B(I; C^{2+\beta}(\Omega))$;
(II) if $f \in C^{1+\beta/2,2+\beta}(I \times \Omega)$, $D_t f \in B(I; C^{\beta}(\Omega))$;
(III) $C^{1+\beta/2,2+\beta}(I \times \Omega) \subseteq C^{\frac{1+\beta}{2}}(I; C^1(\Omega)) \cap C^{\beta/2}(I; C^2(\Omega))$.

Observe that (II) and (III) state that the membership of f to $C^{1+\beta/2,2+\beta}(I \times \Omega)$ implies some form of mixed regularity in the space-time variables.

In all the previous definitions Ω can be replaced with a suitably smooth, compact differentiable manifold.

We begin by considering Dirichlet boundary conditions: we are going to discuss the system

$$\begin{cases} D_t u(t,\xi) - A(\xi, D_\xi)u(t,\xi) = f(t,\xi), \quad t \in (0,T), \xi \in \Omega, \\ \\ u(t,\xi') = g(t,\xi'), \quad t \in (0,T), \xi' \in \partial\Omega, \\ \\ u(0,\xi) = u_0(\xi), \quad \xi \in \Omega. \end{cases} \tag{5}$$

The following result is classical:

Theorem 1. *Suppose that (AA1)-(AA2) hold. Then the following conditions are necessary and sufficient, in order that (5) have a unique solution u in $C^{1+\beta/2,2+\beta}((0,T) \times \Omega)$:*

(a) $f \in C^{\beta/2,\beta}((0,T) \times \Omega)$*;*

(b) $g \in C^{1+\beta/2,2+\beta}((0,T) \times \partial\Omega)$*;*

(c) $u_0 \in C^{2+\beta}(\Omega)$*;*

(d) $u_{0|\partial\Omega} = g(0,\cdot)$*;*

(e) if $\xi' \in \partial\Omega$*,* $A(\xi', D_\xi)u_0(\xi') + f(0,\xi') = D_t g(0,\xi')$*.*

For proofs, see [11], Theorem 5.1.15, [9], Chapter IV.

Remark 1. We limit ourselves to check that the conditions (a)-(e) are necessary: (b)-(c)-(d) are obvious. (a) is necessary, because, by Lemma 1, $D_t u$ and $A(\cdot, D_\xi)u$ belong to $C^{\beta/2,\beta}((0,T) \times \Omega)$. Finally, if $\xi' \in \partial\Omega$,

$$A(\xi', D_\xi)u_0(\xi') + f(0,\xi') = D_t u(0,\xi') = D_t g(0,\xi').$$

So even (e) is necessary.

Now we consider the second classical situation: the oblique derivative condition. The problem we are going to discuss is

$$(6) \qquad \begin{cases} D_t u(t,\xi) - A(\xi, D_\xi)u(t,\xi) = f(t,\xi), \quad t \in (0,T), \xi \in \Omega, \\ \\ B(\xi', D_\xi)u(t,\xi') = h(t,\xi'), \quad t \in (0,T), \xi' \in \partial\Omega, \\ \\ u(0,\xi) = u_0(\xi), \quad \xi \in \Omega. \end{cases}$$

Then we have:

Theorem 2. *Suppose that (AA1)-(AA2) hold. Suppose also that*

$$B(\xi', D_\xi) = \sum_{|\alpha| \leq 1} b_\alpha(\xi') D_\xi^\alpha,$$

with $b_\alpha \in C^{1+\beta}(\partial\Omega)$ $\forall \alpha$ *with* $|\alpha| \leq 1$*; if* $|\alpha| = 1$*,* b_α *is real valued and* $\sum_{|\alpha|=1} b_\alpha(\xi')\nu(\xi')^\alpha \neq 0$ $\forall \xi' \in \partial\Omega$*, where we have indicated with* $\nu(\xi')$ *the unit normal vector to* $\partial\Omega$ *in* ξ' *pointing outside* Ω*.*

Then the following conditions are necessary and sufficient, in order that (5) have a unique solution u in $C^{1+\beta/2,2+\beta}((0,T)\times\Omega)$:

(a) $f \in C^{\beta/2,\beta}((0,T)\times\Omega)$;

(b) $h \in C^{\frac{1+\beta}{2},1+\beta}((0,T)\times\partial\Omega)$;

(c) $u_0 \in C^{2+\beta}(\Omega)$;

(d) $[B(\xi', D_\xi)u_0](\xi') = h(0,\xi')\ \forall \xi' \in \partial\Omega$.

For a proof, see [11], Theorem 5.1.19, and [9], Chapter IV.

Remark 2. Exactly as in the case of Dirichlet boundary conditions, the necessity of assumptions (a)-(d) follows from Lemma 1.

We pass to consider problems with dynamic and Wentzell boundary conditions: we begin with a problem in the form

$$
\begin{cases}
D_t u(t,\xi) - A(\xi, D_\xi)u(t,\xi) = f(t,\xi), \quad t \in (0,T), \xi \in \Omega, \\
\\
D_t u(t,\xi') + B(\xi', D_\xi)u(t,\xi') = h(t,\xi'), \quad t \in (0,T), \xi' \in \partial\Omega, \\
\\
u(0,\xi) = u_0(\xi), \quad \xi \in \Omega.
\end{cases} \tag{7}
$$

Formally replacing in the second equation $D_t u(t,\xi')$ with $A(\xi', D_\xi)u(t,\xi') + f(t,\xi')$, we obtain the Wentzell mixed boundary value problem

$$
\begin{cases}
D_t u(t,\xi) - A(\xi, D_\xi)u(t,\xi) = f(t,\xi), \quad t \in (0,T), \xi \in \Omega, \\
\\
A(\xi', D_\xi)u(t,\xi') + B(\xi', D_\xi)u(t,\xi') = k(t,\xi'), \quad t \in (0,T), \xi' \in \partial\Omega, \\
\\
u(0,\xi) = u_0(\xi), \quad \xi \in \Omega.
\end{cases} \tag{8}
$$

Problems in the form (7) or (8) appear in applications connected with heat transfer problems in a solid in contact with a fluid ([8], [10], [13]), in neurological problems and diffusion on networks ([12]). They have been extensively studied in L^p settings ([1], [2],

[3], [4], [5], [8]), [6]. In particular, the papers [5] and [8] contains maximal regularity results for systems in the form (7) in L^p settings. The main results in [6] are contained in the following theorem and corollary:

Theorem 3. *Consider problem (7), with the following assumptions: (AA1)-(AA2) are fulfilled. Suppose also that*

$$B(\xi', D_\xi) = \sum_{|\alpha| \le 1} b_\alpha(\xi') D_\xi^\alpha,$$

with $b_\alpha \in C^{1+\beta}(\partial\Omega)$ $\forall \alpha$ *with* $|\alpha| \le 1$*; if* $|\alpha| = 1$*,* b_α *is real valued and* $\sum_{|\alpha|=1} b_\alpha(\xi')\nu(\xi')^\alpha > 0$ $\forall \xi' \in \partial\Omega$*; then the following conditions are necessary and sufficient, in order that (7) have a unique solution* u *in* $C^{1+\beta/2, 2+\beta}((0,T)\times\Omega)$*, with* $D_t u_{|(0,T)\times\partial\Omega}$ *in* $B((0,T); C^{1+\beta}(\partial\Omega))$*:*

(a) $f \in C^{\beta/2,\beta}((0,T) \times \Omega)$*;*

(b) $h \in C^{\beta/2, 1+\beta}((0,T) \times \partial\Omega)$*;*

(c) $u_0 \in C^{2+\beta}(\Omega)$*;*

(d) $\forall \xi' \in \partial\Omega$

$$A(\xi', D_\xi)u_0(\xi') + f(0,\xi') = -B(\xi', D_\xi)u_0(\xi') + h(0,\xi').$$

The prototype of (7) is

$$\begin{cases} D_t u(t,\xi) - \Delta_\xi u(t,\xi) = f(t,\xi), & t \in (0,T), \xi \in \Omega, \\ \\ D_t u(t,\xi') + \frac{\partial u}{\partial \nu}(t,\xi') = h(t,\xi'), & t \in (0,T), \xi' \in \partial\Omega, \\ \\ u(0,\xi) = u_0(\xi), & \xi \in \Omega. \end{cases}$$

Some remarks concerning Theorem 3 are in order.

First of all, it states the existence and uniqueness of a solution which is not only in $C^{1+\beta/2,2+\beta}((0,T) \times \Omega)$, but it satisfies a supplementary condition of regularity in $(0,T) \times \partial\Omega$. In fact, by Lemma 1 , the membership of u to $C^{1+\beta/2,2+\beta}((0,T) \times \Omega)$ implies only that $D_t u_{|(0,T)\times\partial\Omega} \in B((0,T); C^\beta(\partial\Omega))$. On the other hand, $B(\cdot, D_\xi)u_{|(0,T)\times\partial\Omega} \in B((0,T); C^{1+\beta}(\partial\Omega)$. The necessity of the conditions (a)-(c) easily follows from Lemma 1, while (d) follows from

$$A(\xi', D_\xi)u_0(\xi') + f(0,\xi') = D_t u(0,\xi') = -B(\xi', D_\xi)u_0(\xi') + h(0,\xi').$$

Theorem 3 admits the following corollary, concerning (8):

Corollary 1. *Consider problem (8), with the following assumptions: (AA1)-(AA2) are fulfilled and B is as in the statement of Theorem 3. Then the following conditions are necessary and sufficient, in order that (8) have a unique solution u in $C^{1+\beta/2,2+\beta}((0,T)\times\Omega)$, with $D_t u_{|(0,T)\times\partial\Omega}$ in $B((0,T);C^{1+\beta}(\partial\Omega))$:*

(a) $f \in C^{\beta/2,\beta}((0,T)\times\Omega)$;

(b) $f_{|(0,T)\times\partial\Omega} + k \in C^{\beta/2,1+\beta}((0,T)\times\partial\Omega)$;

(c) $u_0 \in C^{2+\beta}(\Omega)$;

(d) $\forall \xi' \in \partial\Omega$

$$A(\xi', D_\xi)u_0(\xi') + f(0,\xi') + B(\xi', D_\xi)u_0(\xi') = k(0,\xi').$$

Proof. The necessity of (a), (c), (d) is clear. Moreover, if u satisfies (8), it satisfies also (7), with

$$h(t,\xi') = f(t,\xi') + k(t,\xi').$$

So (b) is necessary also.

On the other hand, if (a)-(d) are satisfied, the assumptions of Theorem 3 are satisfied, with $h(t,\xi') = f(t,\xi') + k(t,\xi')$. We deduce that (7) has a unique solution u with the desired regularity and it is easy to check that u solves also (8).

$\square$

We conclude with another mixed problem with a dynamic boundary condition: we consider the system

$$(9)\quad \begin{cases} D_t u(t,\xi) - A(\xi, D_\xi)u(t,\xi) = f(t,\xi), \quad t\in(0,T), \xi\in\Omega, \\ D_t u(t,\xi') + B(\xi', D_\xi)u(t,\xi') - L[u(t,\cdot)_{|\partial\Omega}](\xi') = h(t,\xi'), \quad t\in(0,T), \xi'\in\partial\Omega, \\ u(0,\xi) = u_0(\xi), \quad \xi\in\Omega. \end{cases}$$

in the case that L is a second order strongly elliptic operator in $\partial\Omega$. The prototype is in this case

$$\begin{cases} D_t u(t,\xi) - \Delta_\xi u(t,\xi) = f(t,\xi), \quad t \in (0,T), \xi \in \Omega, \\ \\ D_t u(t,\xi') + \frac{\partial u}{\partial \nu}(t,\xi') - \Delta_{LB}[u(t,\cdot)_{|\partial\Omega}](\xi') = h(t,\xi'), \quad t \in (0,T), \xi' \in \partial\Omega, \\ \\ u(0,\xi) = u_0(\xi), \quad \xi \in \Omega. \end{cases}$$

where we have indicated with Δ_{LB} the Laplace-Beltrami operator.

In our knowledge, a problem in the form (9) was introduced for the first time in [13], in the particular case that $A(x, D_x) = \alpha(x)\Delta_x$, $B(x', D_x) = b(x')D_\nu$, $L = a(x)\Delta_{LB}u$, with α, a, b positive functions. [13] contains a physical interpretation of the problem: briefly, a heat equation with a heat source on the boundary, that depends on the heat flow along the boundary, the heat flux across the boundary and the temperature at the boundary.

In the paper [14] the authors studied the system (9) in the particular case $A(x, D_x) = \Delta_x$, $f \equiv 0$, $h \equiv 0$, $L(t) = l\Delta_{LB}$ with $l > 0$ and $B(t, x', D_x) = kD_\nu$, where k may be negative, showing that, in order to obtain a well-posed problem, it is not necessary that the condition $B(\xi', \nu(\xi')) > 0$ $\forall \xi' \in \partial\Omega$ is satisfied.

The following result is contained in [7]:

Theorem 4. *Consider problem (9), with the following assumptions:*

(a) (AA1)-(AA2) are fulfilled;

(b) $B(x', D_x) = \sum_{|\alpha| \le 1} b_\alpha(t, x')D_x^\alpha$, *with* $b_\alpha \in C^\beta(\partial\Omega)$*;*

(c) L *is a second order strongly elliptic partial differential operator in* $\partial\Omega$*. More precisely: for every local chart* (U, Φ)*, with* U *open in* $\partial\Omega$ *and* Φ $C^{2+\beta}-$ *diffeomorphism between* U *and* $\Phi(U)$*, with* $\Phi(U)$ *open in* $\mathbb{R}^{n-1}$*,* $\forall v \in C^{2+\beta}(\partial\Omega)$ *with compact support in* U*,*

$$Lv(x') = \sum_{|\alpha| \le 2} l_{\alpha,\Phi}(x') D_y^\alpha (v \circ \Phi^{-1})(\Phi(x'));$$

moreover, if $|\alpha| = 2$*,* $l_{\alpha,\Phi}$ *is real valued, for every open subset* V *of* U*, with* $\overline{V} \subset\subset U$*,* $l_{\alpha,\Phi|V} \in C^\beta(V)$*, and there exists* $\nu(V)$ *positive such that,* $\forall x' \in V$*,* $\forall \eta \in \mathbb{R}^{n-1}$*,* $\sum_{|\alpha|=2} l_{\alpha,\Phi}(x')\eta^\alpha \ge \nu(V)|\eta|^2$*.*

Then the following conditions are necessary and sufficient in order that (9) have a unique solution u belonging to $C^{1+\beta/2,2+\beta}((0,T)\times\Omega)$*:*

(a) $f \in C^{\beta/2,\beta}((0,T)\times\Omega)$*;*

(b) $h \in C^{\beta/2,\beta}((0,T)\times\partial\Omega)$*;*

(c) $u_0 \in C^{2+\beta}(\Omega)$*;*

(d) $A(\xi', D_\xi)u_0(\xi') + f(0,\xi') = -B(\xi', D_\xi)u_0(\xi') + L(u_{0|\partial\Omega})(\xi') + h(0,\xi'), \quad \forall \xi' \in \partial\Omega.$

Remark 3. Observe that in the statement of Theorem 4, apart some regularity of the coefficients, B may be an arbitrary first order (not necessarily tangential) operator in $\partial\Omega$.

As usual, Theorem 4 admits a corollary, concerning Wentzell boundary conditions:

Corollary 2. *Suppose that the assumptions (a)-(d) in Theorem 4 are fulfilled. Then the following conditions are necessary and sufficient in order that the system*

(10)
$$\begin{cases} D_t u(t,\xi) - A(\xi, D_\xi)u(t,\xi) = f(t,\xi), \quad t \in (0,T), \xi \in \Omega, \\ \\ A(\xi', D_\xi)u(t,\xi) + B(\xi', D_\xi)u(t,\xi') - L[u(t,\cdot)_{|\partial\Omega}](\xi') = k(t,\xi'), \quad t \in (0,T), \xi' \in \partial\Omega, \\ \\ u(0,\xi) = u_0(\xi), \quad \xi \in \Omega. \end{cases}$$

have a unique solution in $C^{1+\beta/2,2+\beta}((0,T)\times\Omega)$*:*

(a) $f \in C^{\beta/2,\beta}((0,T)\times\Omega)$*;*

(b) $h \in C^{\beta/2,\beta}((0,T)\times\partial\Omega)$*;*

(c) $u_0 \in C^{2+\beta}(\Omega)$*;*

(d) $A(\xi', D_\xi)u_0(x') + B(\xi', D_\xi) - L(u_{0|\partial\Omega})(\xi') = k(0,\xi'), \quad \forall \xi' \in \partial\Omega.$

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