REGULARITY RESULTS FOR LOCAL MINIMIZERS OF FUNCTIONALS WITH DISCONTINUOUS COEFFICIENTS
RISULTATI DI REGOLARITÀ DEI MINIMI LOCALI DI FUNZIONALI CON COEFFICIENTI DISCONTINUI

RAFFAELLA GIOVA - ANTONIA PASSARELLI DI NAPOLI

ABSTRACT. We give an overview on recent regularity results of local vectorial minimizers of under two main features: the energy density is uniformly convex with respect to the gradient variable only at infinity and it depends on the spatial variable through a possibly discontinuous coefficient. More precisely, the results that we present tell that a suitable weak differentiability property of the integrand as function of the spatial variable implies the higher differentiability and the higher integrability of the gradient of the local minimizers. We also discuss the regularity of the local solutions of nonlinear elliptic equations under a fractional Sobolev assumption.

SUNTO. Presentiamo alcuni recenti risultati di regolarità dei minimi locali vettoriali di funzionali integrali le cui caratteristiche principali sono che le densità di energia sono uniformemente convesse solo all' infinito e che, come funzioni della variabile spaziale possono essere discontinue. Tali risultati possono essere sintetizzati come segue: una opporuna differenziabilità debole dell' integrando rispetto alla variabile spaziale implica la maggiore differenziabilità e maggiore integrabilità del gradiente del minimo. Discutiamo anche la regolarità delle soluzioni locali di equazioni non lineari ellittiche sotto ipotesi di differenziabilità frazionaria.

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1. Introduction

Classical variational problems are related to the study of the existence and the regularity of local minimizers of integral functionals of the type

$$\mathcal{F}(u;\Omega) := \int_{\Omega} F(x, Du)\, dx, \tag{1.1}$$

where Ω is a bounded open set in $\mathbb{R}^n$, $n \geq 2$, $u : \Omega \to \mathbb{R}^N$, $N \geq 1$, and the integrand $F : \Omega \times \mathbb{R}^{nN} \to \mathbb{R}$ is a Carathéodory map such that $\xi \to F(x,\xi)$ is a convex function for almost every $x \in \Omega$. As far as the growth conditions are concerned, we assume that there exist exponents p, q, with $1 < p \leq q$, and a constant $L > 0$, such that

$$|\xi|^p \leq F(x,\xi) \leq L(1 + |\xi|^2)^{\frac{q}{2}}, \tag{1.2}$$

for almost every $x \in \Omega$ and for all $\xi \in \mathbb{R}^{nN}$.
We recall that $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is a *local minimizer* of the functional $\mathcal{F}$ in (1.1) if

$$F(x, Du) \in L^1_{\mathrm{loc}}(\Omega)$$

and

$$\mathcal{F}(u; \operatorname{supp}\varphi) \leq \mathcal{F}(u + \varphi; \operatorname{supp}\varphi),$$

for any $\varphi \in W^{1,1}(\Omega, \mathbb{R}^N)$ with $\operatorname{supp}\varphi \Subset \Omega$.
If $p = q$ we say that the functional $\mathcal{F}(u;\Omega)$ has p-growth or that satisfies standard growth conditions. If $p < q$, we say that the functional $\mathcal{F}(u;\Omega)$ has (p,q)-growth or that satisfies non standard growth conditions.
The regularity properties of minimizers of integral functionals of the type (1.1) under standard growth conditions has been widely investigated over the last 50 years in case the integrand $F(x,\xi)$ depends on the x-variable through a Hölder continuous function and is a strictly convex function with respect to the ξ- variable. Actually, the Hölder continuity of $F(x,\xi)$ with respect to the x variable leads to the C^1 partial regularity of the minimizers with a quantitative modulus of continuity that can be determined in dependence on the modulus of continuity of the coefficients (for an exhaustive treatment, we refer the interested reader to [45, 22] and the references therein).

It is worth pointing out that partial regularity results are a common feature when treating vectorial minimizers. Namely, in the vectorial setting everywhere regularity cannot be proven as it is shown by the counterexamples due to De Giorgi and by those due to Sverak and Yan ([25, 65]), unless some additional structure assumptions are imposed on the energy densities.

The study of the regularity properties of local minimizers of functionals satisfying (p, q)-growth conditions started with the pioneeristic papers by Marcellini ([56, 57, 58, 59]). It is important to remark that an example by Giaquinta ([40]) and Marcellini ([55]) implies that a bound on the gap between p and q is a necessary condition to the local regularity. It is now well known that, in general, to have the regularity of minimizers the gap between p and q must be not too large; in many cases this relation is expressed by an inequality of the type $q \leq c(n)p$, with $c(n) \to 1^+$ as n goes to infinity ([1, 4, 27, 28, 34, 35] and, for more details and references, [60]). It worth noting that, besides a condition on the distance between the growth and the ellipticity exponents, the dependence of the integrand on the x-variable can give substantial difficulties since the Lavrentiev phenomenon may appear ([28]).

The study of the regularity has been successfully carried out under weaker assumptions on the convexity of the integrand $F(x, \xi)$ with respect to the ξ-variable, i.e under a uniform convexity assumption only for large values of the modulus of the gradient (see for example [7, 29, 41, 54, 31, 32, 33, 64] for the case of standard growth conditions and [17, 21, 6] for the case of non standard growth conditions). So far, such regularity results have been proven under a smooth dependence of $F(x, \xi)$ from the x-variable.

On the other hand, in the last few years, in case of standard growth conditions the regularity of the local minimizers has been established under weaker assumptions on the function that measures the oscillations of the integrand with respect to the x-variable. Actually, when the partial map $x \to D_\xi F(x, \xi)$ belongs to a suitable Sobolev class the higher differentiability as well as the partial Hölder continuity of the gradient of the local minimizers have been obtained in [37, 42, 61, 62, 63, 43] (see also

[52, 53]). More recently, the regularity of the solutions of some parabolic systems with Sobolev coefficients has been faced in [39]. We also refer to [36, 38] for the case of functionals with variable exponents growth conditions.

In Section 2, we present some recent results concerning the regularity properties of the minimizers of integral functionals of the type (1.1), allowing a discontinuous dependence for the integrand $F(x,\xi)$ with respect to x- variable through a suitable weakly differentiable function in case the integrand is convex with respect to the gradient variable only at infinity. The case of non standard growth conditions is also considered.

More precisely, we shall illustrate that a suitable weak differentiability property of integer order of the partial map $x \to D_\xi F(x,\xi)$ implies the higher differentiability of the same order and the higher integrability of the gradient of the local minimizers.

Finally, in Section 3, we consider non linear elliptic equations of the form

$$\operatorname{div} A(x, Dv) = \operatorname{div}(|G|^{p-2}G) \qquad \text{in} \quad \Omega \subset \mathbb{R}^n,$$

where $A(x,\xi)$ is a p-harmonic type operator such that the partial map $x \to A(x,\xi)$ belongs to a fractional Sobolev space. Under this assumption, the fractional differentiability of the right hand side G transfers to the gradient of the solutions with no loss in the order of differentiability.

2. Functionals with Sobolev Coefficients

Let us consider the integral functional defined in (1.1) satisfying the growth assumption (1.2). We assume some convexity and regularity on $F(x,\xi)$ only for large values of the modulus of ξ. More precisely, we assume that there exists a radius $\bar{R} > 0$ such that:

• *Radial structure.* There exists $\tilde{F} : \Omega \times [\bar{R}, +\infty)$ s. t.

$$F(x,\xi) = \tilde{F}(x, |\xi|), \tag{2.1}$$

a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^{nN} \setminus B_{\bar{R}}(0)$.

• *C^2-asymptotic convexity or p-uniform convexity at infinity.* The function $\xi \mapsto F(x,\xi)$ is of a class $C^2(\mathbb{R}^{nN})$ and there exists $\nu > 0$ such that

$$\langle D_{\xi\xi}F(x,\xi)\lambda,\lambda\rangle \geq \nu(1+|\xi|^2)^{\frac{p-2}{2}}|\lambda|^2 \tag{2.2}$$

a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^{nN} \setminus B_{\bar{R}}(0)$.

• *Growth of $D_{\xi\xi}F$.* There exists a constant $L_1 > 0$ such that

$$|D_{\xi\xi}F(x,\xi)| \leq L_1(1+|\xi|^2)^{\frac{q-2}{2}} \tag{2.3}$$

a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^{nN} \setminus B_{\bar{R}}(0)$.

• *Sobolev regularity with respect to the x-variable.* The function $x \to D_\xi F(x,\xi)$ is weakly differentiable for all $\xi \in \mathbb{R}^{nN} \setminus B_{\bar{R}}(0)$ and there exists $k(x) \in L^r_{\mathrm{loc}}(\Omega)$, with $r \geq p+2$, such that

$$|D_x D_\xi F(x,\xi)| \leq |k(x)|(1+|\xi|^2)^{\frac{q-1}{2}} \tag{2.4}$$

a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^{nN} \setminus B_{\bar{R}}(0)$.

If we take into account that many properties are requested only *at infinity*, a model functional in our setting is

$$\mathcal{F}(u;\Omega) := \int_\Omega |Du|^p + a(x)|Du|^q\,dx, \qquad 1 < p \leq q, \tag{2.5}$$

with $a \in W^{1,r}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$, $a \geq 0$ and $a \equiv 0$ in a subset of positive measure.

A very recent paper by Eleuteri, Marcellini and Mascolo ([26]) deals with integral functionals that are uniformly convex only for large values of the modulus of the gradient and that depend on the x-variable through a Sobolev coefficient.
More precisely, besides the (p,q)-growth conditions, they deal with an integrand F satisfying (2.1) for all $\xi \in \mathbb{R}^{nN}$, and satisfying the assumption (2.4) with a function k belonging to L^r, with $r > n$. They proved the local Lipschitz continuity of the local minimizers of the functional $\mathcal{F}$ if

$$\frac{q}{p} < 1 + \frac{1}{n} - \frac{1}{r}. \tag{2.6}$$

Note that this assumption on the gap $\frac{q}{p}$ is sharp as shown in [28] and [30] (see also [10] and [11] for related results).

The aim in [15] is to show that, in the Sobolev dependence on the x-variable expressed by the assumption (2.4), it is sufficient to assume $r = n$ to prove that, under a p-convexity condition only at infinity, the local minimizers of the functional $\mathcal{F}$ have the gradient locally in L^t for every $t > p$ and, therefore, that they are locally Hölder continuous for every exponent $0 < \alpha < 1$.

More precisely, taking into account that, without loss of generality, we can assume $\tilde{R} = 1$, the result is the following.

Theorem 2.1. *Let $F : \Omega \times \mathbb{R}^{nN} \to [0, +\infty)$, $n \geq 3$, be a Carathéodory function such that $F = F(x, \xi)$ is convex and C^2 with respect to the last variable and satisfies the assumptions* (1.2) *with $1 < p = q$ and* (2.1)-(2.4) *with $r = n$. Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a local minimizer of* (1.1)*. Then u is locally α-Hölder continuous for all $\alpha \in (0, 1)$. Moreover, for all $t > p$ and for all $B_\rho(x_0) \subset B_R(x_0) \Subset \Omega$, we have that*

$$\left[\int_{B_\rho(x_0)} |Du|^t \, dx \right]^{\frac{1}{t}} \leq C \left[\int_{B_R(x_0)} (1 + F(x, Du)) \, dx \right]^{\frac{1}{p}},$$

where $C = C(n, N, p, t, L, L_1, \nu, \rho, R)$.

The assumption $p = q$ is not surprising: indeed, looking at the condition (2.6), which is sharp as suggested by the examples in [28] and [30], if r goes to n, then the condition on the gap reduces to $\frac{q}{p} \leq 1$.

We also remark that Hölder continuity results for any exponent α strictly less than 1 are not uncommon when the integrands depend on the x-variable. We refer to [14] and [62] for examples of not locally Lipschitz continuous minimizers, but Hölder continuous for every exponent $0 < \alpha < 1$.

The proof of Theorem 2.1 is achieved by establishing first an a priori estimate and then using an approximation argument.

The key tool in the proof of the a priori estimate is the construction of test functions that are proportional to a suitable power of the gradient of the minimizer and that vanish in the set where the ellipticity of the Euler Lagrange system associated to the functional is lost.

Moreover, a suitable iteration argument is needed to reabsorb terms with critical summability exponents. Next, we construct a sequence of integrands that are regular with respect to the x variable and p-uniformly convex with respect to the gradient variable in the whole $\mathbb{R}^{nN}$ and we show that the a priori estimate is preserved in passing to the limit.

We remark that the assumption $n \geq 3$ is necessary in establishing the a priori estimate and, as far as the case $p = n = 2$ is concerned, the only available result for the vectorial setting is contained in [62]. More precisely, let us assume that there exist constants $\ell, L > 0$ such that

$$|\xi|^2 \leq F(x,\xi) \leq L(1+|\xi|^2), \tag{2.7}$$

$$\langle D_\xi F(x,\xi) - D_\xi F(x,\eta), \xi - \eta \rangle \geq \frac{\ell}{2}|\xi-\eta|^2 \tag{2.8}$$

and assume that there exists $k \in L^2_{\mathrm{loc}}(\Omega)$, such that

$$|D_x D_\xi F(x,\xi)| \leq |k(x)|(1+|\xi|) \tag{2.9}$$

a.e. $x, y \in \Omega \subset \mathbb{R}^2$ and for every $\xi, \eta \in \mathbb{R}^{2\times N}$.

Then the following result holds.

Theorem 2.2. *Let $F : \Omega \times \mathbb{R}^{2N} \to [0,+\infty)$, be a Carathéodory function such that $F = F(x,\xi)$ is C^1 with respect to the last variable and satisfies the assumptions* (2.7)-(2.9). *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ be a local minimizer of* (1.1). *Then $Du \in W^{1,t}_{loc}(\Omega)$, for every $t < 2$.*

Moreover there exists $R_0 = R_0(N,L,t)$ such that

$$\int_{B_R} |D^2u|^t \, dx \leq \frac{C}{R^2}\left(\int_{B_{2R}} |k|^2 \, dx\right)^{\frac{1}{2}} \int_{B_{2R}} |Du|^2 \, dx,$$

for every R such that $B_{2R} \subset B_{R_0} \subset \Omega$. In particular, $Du \in L^t_{\mathrm{loc}}(\Omega)$ for every $t > 1$.

As before, the proof is based on an a priori estimate and an approximation argument. In this case the test functions are constructed combining the difference quotient method with the following stability property of the Hodge decomposition due to Iwaniec and Sbordone ([48]).

Lemma 2.1. *Let* $w \in W^{1,p}(\mathbb{R}^n)$*, and let* $1 < p < \infty$*. Then there exist vector fields* $\mathcal{E} \in L^{p'}(\mathbb{R}^n)$ *with* $\mathrm{curl}(\mathcal{E}) = 0$ *and* $\mathcal{B} \in L^{p'}(\mathbb{R}^n)$ *with* $\mathrm{div}(\mathcal{B}) = 0$ *such that*

$$Dw|Dw|^{p-2} = \mathcal{E} + \mathcal{B}.$$

Moreover

$$\|\mathcal{E}\|_{L^{p'}(\mathbb{R}^n)} \le C\, \|Dw\|^{p-1}_{L^p(\mathbb{R}^n)}$$

and

$$\|\mathcal{B}\|_{L^{p'}(\mathbb{R}^n)} \le C\, \max\{p-2, p'-2\}\, \|Dw\|^{p-1}_{L^p(\mathbb{R}^n)},$$

where C *is a universal constant.*

The Hodge decomposition allows us to obtain an estimate for the second derivatives of the minimizer in a space of summability slightly larger than the natural one. Then a double iteration procedure is needed to handle integrals with critical integrability exponent.

We'd like to notice that Theorem 2.2 is sharp. More precisely, we can not have, in general, neither $D^2u \in L^2$ nor $Du \in L^\infty$.

Indeed, the function

$$u(x_1, x_2) = x_1(1 - \log|x|) \qquad x \in B\left(0, \frac{1}{e}\right) \subset \mathbb{R}^2$$

solves the equation $\mathrm{div}(A(x)Du) = 0$, where $A(x)$ is a matrix satisfying the assumptions of Theorem 2.2 with

$$K(x) = \frac{1}{|x|(1 - \log|x|)} \in L^2\left(B\left(0, \frac{1}{e}\right)\right).$$

We have

$$|Du| \sim (1 - \log|x|) \qquad \text{and} \qquad |D^2u| \sim \frac{1}{|x|}$$

and so $Du \in L^t$ for all $t > 1$ but $Du \notin L^\infty$. Moreover, $u \in W^{2,q}$ for all $1 < q < 2$, but $u \notin W^{2,2}$.

We also notice that the assumption $k \in L^2$ in (2.9) can not be weakened (in the scale of Lebesgue spaces) and obtain the same regularity. In fact, the function

$$u(x_1, x_2) = x_1 |x|^{\alpha - 1} \qquad \alpha \in (0,1)$$

solves the equation $\operatorname{div}(A(x)Du) = 0$, where $A(x)$ is a matrix satisfying our assumptions with

$$k(x) \sim \frac{1}{|x|} \in L^p \qquad \forall p < 2$$

and

$$|Du| \sim |x|^{\alpha-1}, \qquad |D^2 u| \sim |x|^{\alpha-2}.$$

Therefore

$$u \in W^{1,t} \quad \forall t < \frac{2}{1-\alpha} \qquad \text{and} \qquad u \in W^{2,q} \quad \forall q < \frac{2}{2-\alpha}.$$

We now come to the case of functionals with non standard growth conditions, i.e. satisfying the assumptions (1.2) with $p < q$ and (2.1)–(2.4). First, we want to recall that the radial structure of the integrand expressed by (2.1) guarantees the local boundedness of the minimizers under the following condition

$$q < p^* = \frac{np}{n-p}$$

between the growth and the ellipticity exponents ([18, 19, 20, 23, 24]). Such local boundedness allow us to deal with functionals with non standard growth conditions that are uniformly convex only for large values of the modulus of the gradient of the minimizer and with a $W^{1,n}$ assumption on the partial map $x \to D_\xi F(x,\xi)$. More precisely, in the forthcoming paper [16], we establish the following

Theorem 2.3. *Let* $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ *be a local minimizer of the functional* $\mathcal{F}(u,\Omega)$*, satisfying the assumptions* (1.2)–(2.4) *with* $r \geq p+2$*. If*

$$1 < p \leq q \leq \min\left\{p + 1 - \frac{p+2}{r}, p^*\right\}$$

then, setting

$$\mathcal{G}(t) = \int_0^t (1+s)^{\frac{p-4}{2}} s\, ds$$

we have

$$\mathcal{G}((|Du|-1)_+) \in W^{1,2}_{\mathrm{loc}}(\Omega) \qquad \textit{and} \qquad Du \in L^{p+2}(\Omega).$$

Moreover

$$\int_{B_R} \left| \frac{(|Du|-1)_+}{1+(|Du|-1)_+} |Du|^{\frac{p-2}{2}} D^2 u \right|^2 \leq C\left(1 + \|k\|_{L^r(B_R)}\right)^{\frac{p+2}{q-p-1}} \left(\int_{B_{4R}(x_0)} (1+f(x,Du)) \right)^{\gamma}$$

with $\gamma = \gamma(p,q,n)$.

We remark that the assumption on k is weaker than $k \in L^n$ in the case $2 \leq p < n-2$.

The main tool in the proof of previous Theorem is the use of the local boundedness of the minimizers $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ combined with the following Gagliardo-Nirenberg type interpolation inequality (see [5]).

Lemma 2.2. *For every* $u \in C^2(\Omega, \mathbb{R}^N)$, $\eta \in C^1_c(\Omega)$, $\eta \geq 0$, *and every* $\kappa \in \mathbb{R}^N$, *we have*

$$\int_\Omega \eta^2 (1+|Du|^2)^{\frac{p}{2}} |Du|^2 \,\mathrm{d}x \leq c(p)||u-\kappa||^2_{L^\infty(\mathrm{supp}\eta)} \int_\Omega \eta^2 (1+|Du|^2)^{\frac{p-2}{2}} |D^2u|^2 \,\mathrm{d}x$$

$$+c||u-\kappa||^2_{L^\infty(\mathrm{supp}\eta)} \int_\Omega (|\eta|^2 + |\nabla\eta|^2)(1+|Du|^2)^{\frac{p}{2}} \,\mathrm{d}x,$$

for a positive constant $c = c(p)$.

Roughly speaking, the local boundedness of the minimizers together with the existence of the second derivatives implies the higher integrability of the gradient of the solutions with exponent $p+2$, which improves the one given by the Sobolev imbedding Theorem in case $p < n-2$. This is precisely the case in which we improve previous results, weakening the assumption on the summability of the weak derivative of the partial map $x \to D_\xi F(x,\xi)$.

It is worth to point out that the bound on q depends on the assumption on $k(x)$ in (2.4) and obviously improves if $r \to \infty$, where r is the summability exponent of $k(x)$.

Theorem 2.2, in case $p = q$ and with $F(x, \xi)$ strictly convex in the whole $\mathbb{R}^{nN}$ has been proven in [44].

3. Elliptic equations with Fractional Sobolev Coefficients

In this section we confine ourselves to the scalar case ($N = 1$) and we want to discuss some regularity results for weak solutions to

$$\operatorname{div} A(x, Dv) = 0 \qquad \text{in} \quad \Omega \subset \mathbb{R}^n, \tag{3.1}$$

where $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory map . We assume that there exist $2 \le p \le n$ and constants $\alpha, \beta > 0$ such that

$$\alpha(1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} |\xi - \eta|^2 \le \langle A(x, \xi) - A(x, \eta), \xi - \eta \rangle, \tag{3.2}$$

$$|A(x, \xi) - A(x, \eta)| \le \beta(1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} |\xi - \eta|. \tag{3.3}$$

Concerning the dependence on the x-variable, we assume that there exist $k(x) \in L^r(\Omega)$, $r > 1$, and $0 < \alpha \le 1$, such that

$$|A(x, \xi) - A(y, \xi)| \le (|k(x)| + |k(y)|)|x - y|^\alpha (1 + |\xi|^2)^{\frac{p-1}{2}}, \tag{3.4}$$

for a.e. $x, y \in \Omega$ and for all $\xi, \eta \in \mathbb{R}^n$.

It is known that the p-growth condition expressed by (3.3), the p-ellipticity expressed by (3.2) and the $W^{1,n}$-Sobolev assumption on the coefficients, i.e. the assumption (3.4) with $\alpha = 1$ and $r = n$ (see [46]), allow to expect higher differentiability results for the solutions with integer order.

In particular, the equations satisfying (3.2), (3.3) and (3.4) with $\alpha = 1$ and $\sigma = n$, (including the so called Beltrami equations in the case $p = n = 2$), were recently studied in [8, 61, 62, 42].

We assume, in this scalar case, that the inequality (3.4) holds, with $0 < \alpha < 1$. Roughly speaking, we deal with case in which the map $x \to A(x, \xi)$ enjoys a fractional differentiability property (for related results in this setting see [51]). To state the result properly, we recall that the following definition of the *Besov spaces* via difference quotients (see [66, Section 2.5.12]).

Definition 3.1. *Let* $1 \leq p < \infty$, $1 \leq q \leq \infty$ *and* $0 < \alpha < 1$. *Let* $v \in L^p(\mathbb{R}^n)$. *We say that*

- $v \in B^{\alpha}_{p,q}(\mathbb{R}^n)$ *if*

$$\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|v(x+h)-v(x)|^p}{|h|^{\alpha p}}dx\right)^{\frac{q}{p}}\frac{dh}{|h|^n}\right)^{\frac{1}{q}} < +\infty,$$

- $v \in B^{\alpha}_{p,\infty}(\mathbb{R}^n)$ *if*

$$\sup_{h\in\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|v(x+h)-v(x)|^p}{|h|^{\alpha p}}dx\right)^{\frac{1}{p}} < +\infty.$$

For a fixed ball B_R the Besov spaces $B^{\alpha}_{p,q}(B_R)$ consists of the function $v \in L^p(B_R)$, such that $\varphi v \in B^{\alpha}_{p,q}(\mathbb{R}^n)$, for some $\varphi \in C^{\infty}_0(B_R)$.

There is a link between Besov spaces and Lebesgue ones, given by the following embedding results (a proof can be found at [47, Prop. 7.12]).

Theorem 3.1. *Let* $0 < \alpha < 1$ *and* $1 \leq p < \frac{n}{\alpha}$ *and* $1 \leq q \leq \frac{np}{n-\alpha p} =: p^*_\alpha$. *Then the following inclusions hold:*

$$B^{\alpha}_{p,q} \subset L^{p^*_\alpha}$$

Moreover, if $p = \frac{n}{\alpha}$ *and* $1 \leq q \leq \infty$, *then*

$$B^{\alpha}_{\frac{n}{\alpha},q} \subset VMO.$$

We recall that VMO denotes the space of functions with vanishing mean oscillations; i.e.

$$f \in VMO \quad \Longleftrightarrow \quad \lim_{r\to 0}\sup_{x}\sup_{B_r(x)} \fint_{B_r(x)} |f - f_B|\, dx = 0$$

where $f_B := \fint_B f(x)\, dx$.

Note that the assumption (3.4) with $k(x) \in L^{\frac{n}{\alpha}}$ and $0 < \alpha < 1$ implies that the partial map $x \mapsto A(x,\xi)$ belongs to the Besov space $B^{\alpha}_{\frac{n}{\alpha},\infty}$.

In this framework the higher differentiability of solutions of Beltrami equations ($p = n = 2$) is obtained by Cruz, Mateu and Orobitg [12] and Baison, Clop and Orobitg [2].

Here we state the following recent result proved by Baison, Clop, Giova, Orobitg and Passarelli di Napoli in [3] in case $p = 2$ and by Clop, Giova and Passarelli di Napoli in [9] in case $p > 2$.

Theorem 3.2. *Let $0 < \alpha < 1$ and $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory map satisfying* (3.2) *and* (3.3)*. Assume that there exists a non negative function $k \in L^{\frac{n}{\alpha}}$, such that* (3.4) *holds. If $u \in W^{1,p}_{\mathrm{loc}}$ is a weak solution of* (3.1)*, then*

$$(1 + |Du|^2)^{\frac{p-2}{4}} Du \in B^{\alpha}_{2,\infty}(B_R),$$

for every ball $B_R \Subset \Omega$.

The proof relies on the basic fact that assumption (3.4) with $k \in L^{\frac{n}{\alpha}}$ implies that $x \mapsto A(x, \cdot)$ is locally uniformly VMO (see [3]). At this point, the theorem is achieved by using the following regularity result for solutions of equations with VMO coefficients, also proved in [3].

Theorem 3.3. *If A satisfies* (3.2) *and* (3.3) *and $x \mapsto A(x, \cdot)$ is VMO regular, then for every solution $u \in W^{1,p}$ of*

$$\operatorname{div}(A(x, Du)) = \operatorname{div}(|G|^{p-2}G) \tag{3.5}$$

with $G \in L^q_{\mathrm{loc}}(\Omega)$, $q > p$, we get $Du \in L^q_{\mathrm{loc}}(\Omega)$.

We'd like to mention that Theorem 3.3 extends to general linear elliptic equations previous results by Iwaniec and Sbordone [49] and Kinnunen and Zhou [50].

Improving the regularity of the coefficients, i.e. assuming that the partial map $x \mapsto A(x, \xi)$ belongs to the Besov space$B^{\alpha}_{\frac{n}{\alpha},q}$, with $q < \infty$ yields the following

Theorem 3.4. *Let $\mathcal{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory map satisfying* (3.2) *and* (3.3) *. Assume in addition that $x \mapsto A(x, \xi) \in B^{\alpha}_{\frac{n}{\alpha},q}$. Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a weak solution of* (3.5)*. If $|G|^{p-2}G \in B^{\alpha}_{2,q}(B_{2R})$ then*

$$(1 + |Du|^2)^{\frac{p-2}{4}} Du \in B^{\alpha}_{2,q}(B_R),$$

for every $1 \leq q \leq \frac{2n}{n-2\alpha} =: 2^*_\alpha$ *and for every ball* $B_R \subset B_{2R} \subset \Omega$*.*

We refer to [3] for the case $p = 2$ and to [9] for the case $p > 2$.

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DIPARTIMENTO DI STUDI ECONOMICI E GIURIDICI, UNIVERSITÀ DEGLI STUDI DI NAPOLI "PARTHENOPE", PALAZZO PACANOWSKY- VIA GENERALE PARISI, 13, 80132 NAPOLI, ITALY., DIPARTIMENTO DI MATEMATICA E APPL. "R. CACCIOPPOLI", UNIVERSITÀ DI NAPOLI "FEDERICO II", VIA CINTIA, 80126 NAPOLI, ITALY.

E-mail address: raffaella.giova@uniparthenope.it, antpassa@unina.it