ON THE FIRST BOUNDARY VALUE PROBLEM FOR HYPOELLIPTIC EVOLUTION EQUATIONS: PERRON-WIENER SOLUTIONS AND CONE-TYPE CRITERIA

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Abstract. For every bounded open set Ω in $\mathbb{R}^{N+1}$, we study the first boundary problem for a wide class of hypoelliptic evolution operators. The operators are assumed to be endowed with a well behaved global fundamental solution that allows us to construct a generalized solution in the sense of Perron-Wiener of the Dirichlet problem. Then, we give a criterion of regularity for boundary points in terms of the behavior, close to the point, of the fundamental solution of the involved operator. We deduce exterior cone-type criteria for operators of Kolmogorov-Fokker-Planck-type, for the heat operators and more general evolution invariant operators on Lie groups. Our criteria extend and generalize the classical parabolic-cone condition for the classical heat operator due to Effros and Kazdan. The results presented are contained in [K16].

Sunto. Per ogni aperto limitato Ω in $\mathbb{R}^{N+1}$, studiamo il primo problema al bordo per un' ampia classe di operatori di evoluzione ipoellittici. Assumiamo che gli operatori considerati abbiano una soluzione fondamentale globale con buone proprietà. Questo ci permetterà di costruire una soluzione generalizzata nel senso di Perron-Wiener del relativo problema di Dirichlet. Proviamo quindi un criterio di regolarità dei punti al bordo in termini di comportamento, vicino al punto stesso, della soluzione fondamentale dell' operatore. Ne deduciamo criteri di tipo cono per operatori di tipo Kolmogorov-Fokker-Planck, per l' operatore del calore e piú generali operatori di evoluzione invarianti sui gruppi di Lie. I nostri criteri estendono e generalizzano la condizione di cono-parabolico di Effros e Kazdan relativa all' operatore del calore classico. I risultati presentati in questa nota sono contenuti in [K16].

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1. Introduction

1.1. **The Dirichlet Problem for the Laplacian.** Let Ω be a bounded open set in $\mathbb{R}^N$ and let φ a continuos function on $\partial\Omega$. We say that u is a (classical) solution of the *Dirichlet problem*

$$\text{(DP)} \qquad \begin{cases} \Delta u = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = \varphi, \end{cases}$$

if

$$\Delta u = 0 \text{ in } \Omega \quad \text{and} \quad \lim_{x \to y} u(x) = \varphi(y) \quad \forall\, y \in \partial\Omega.$$

It was long believed that the Dirichlet problem related to the Laplacian always admits a solution, but in 1911 Zaremba [Zar11] gave the first counterexample. He considered a punctured open ball in $\mathbb{R}^2$ with prescribed boundary values 0 at the origin and 1 at the outer boundary. As the boundary at the origin consists of an isolated point, there is no way that a harmonic function attains both boundary data.

One year later, Lebesgue [L12] realized that not only do isolated boundary points cause problems, but so do other points such as the tip of a thorn, the so called *Lebesgue spine*, in $\mathbb{R}^3$.

Because of this, Perron [Per23] was led to formulation of the generalized Dirichlet problem. We recall soon the definition of *generalized solution in the sense of Perron-Wiener*, referring to [CC72, pp. 22–23] for a detailed historical note.

We need before the notion of *superharmonic* functions. We say that $u \in C(\Omega) \cap L^1_{\text{loc}}(\Omega)$ is *superharmonic* if and only if $\Delta u \leq 0$ in the weak sense of the distributions.

Then, we denote by $\overline{\mathcal{H}}(\Omega)$ the set of the superharmonic functions on Ω and, for every $\varphi \in C(\partial\Omega)$, we define

$$H_\varphi^\Omega := \inf\{u \in \overline{\mathcal{H}}(\Omega) \mid \liminf_{x \to y} u(x) \geq \varphi(y) \quad \forall\, y \in \partial\Omega\}.$$

H_φ^Ω is the *generalized solution in the sense of Perron-Wiener* of the Dirichlet Problem (DP).

H^{Ω}_{φ} is $C^{\infty}(\Omega)$ and satisfies $\Delta u = 0$ in Ω. When the Dirichlet problem (DP) has a solution u in the classical sense, it will turn out that $u = H^{\Omega}_{\varphi}$.

Viceversa, if

$$\lim_{x\to y} H^{\Omega}_{\varphi}(x) = \varphi(y) \quad \forall\, y \in \partial\Omega,$$

u is an harmonic function, continuous up to the boundary of Ω and solves the problem (DP) in classic sense.

But, in general H^{Ω}_{φ} does *not* assume the datum φ on Ω. A point $y \in \partial\Omega$ such that

$$\lim_{x\to y} H^{\Omega}_{\varphi}(x) = \varphi(y) \quad \forall\, \varphi \in C(\partial\Omega)$$

is called Δ-*regular* for Ω. So now a crucial question arises: which points of $\partial\Omega$ are regular?

In [Zar11], Zaremba provided us of a regularity criterion for the Laplace operator.

Let B be a compact set in $\mathbb{R}^N$, int $B \neq \emptyset$. We define *cone with vertex in* 0 the following set

$$K = \{rx \mid x \in B, 0 < r < 1\},$$

and *cone with vertex in* y the translated set

$$K + y.$$

Now, let Ω be a bounded open set, $\Omega \subseteq \mathbb{R}^N$, and let y be a point in $\partial\Omega$.

If there exists a cone with vertex in y contained in $\mathbb{R}^N \setminus \Omega$, then

$$y \text{ is } \ \mathit{\Delta}\textit{-regular.}$$

This is the *cone criterion* due to Zaremba. We have to wait almost sixty years to have an analogous criterion for the heat equation.

1.2. **The Dirichlet Problem for the Heat operator.** Let now Ω be a bounded open set in $\mathbb{R}^{N+1}$, $\varphi \in C(\partial\Omega)$ and consider the following Dirichlet problem

$$\text{(DP)} \qquad \begin{cases} \mathcal{L}u = \Delta u - \partial_t u = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = \varphi. \end{cases}$$

A function u is a (classical) solution of the Dirichlet problem (DP) if

$$\mathcal{L}u = 0 \text{ in } \Omega \quad \text{and} \quad \lim_{x\to y} u(x) = \varphi(y) \quad \forall\, y \in \partial\Omega.$$

We say that $u \in C(\Omega) \cap L^1_{\text{loc}}(\Omega)$ is *superharmonic* in Ω if and only if

$$\mathcal{L}u \le 0 \text{ in } \Omega \quad \text{in the weak sense of the distributions.}$$

Also in this case, we denote by $\overline{\mathcal{H}}(\Omega)$ the set of the superharmonic functions.

From the classical parabolic Potential Theory it follows that (DP) always admits a generalized solution in the sense of Perron-Wiener for every $\varphi \in C(\partial\Omega)$:

$$H^\Omega_\varphi := \inf\{u \in \overline{\mathcal{H}}(\Omega) \mid \liminf_{z\to\zeta} u(z) \ge \varphi(\zeta) \quad \forall\, \zeta \in \partial\Omega\}.$$

$$H^\Omega_\varphi \text{ is } C^\infty(\Omega) \text{ and satisfies } \Delta u - \partial_t u = 0 \text{ in } \Omega.$$

Again we have the problem of the boundary regularity: which $z_0 \in \partial\Omega$ are such that the generalized solution is continuous up to z_0 and there assume the boundary datum?

In 1970, Effros and Kazdan [EK70], see also [EK71], extend to the parabolic setting the Zaremba cone criterion.

Let B a compact set of $\mathbb{R}^N, \text{int}B \neq \emptyset$, we define *parabolic cone with vertex in* 0 the set

$$K = \{(rx, -Tr^2) \mid x \in B, 0 < r < 1\}$$

and *parabolic cone with vertex in* z_0

$$K + z_0.$$

Given Ω a bounded open set in $\mathbb{R}^{N+1}$ and z_0 in $\partial\Omega$, if there exists a cone with vertex in z_0 contained in $\mathbb{R}^{N+1}\backslash\Omega$, then

$$z_0 \text{ is } \mathcal{L}\text{-regular.}$$

We generalize this *parabolic cone* criterion of Effros and Kazdan to a wide class of evolution operators.

2. Our class of operators

The operators we are dealing with are hypoelliptic partial differential operators of the following type

$$\mathcal{L} = \sum_{i,j=1}^{N} a_{ij}(z)\partial_{x_i x_j} + \sum_{i=1}^{N} b_i(z)\partial_{x_i} - \partial_t.$$

The coefficients $a_{ij} = a_{ji}$ and b_i are of class C^∞ in the strip

$$S = \{z = (x,t) \in \mathbb{R}^{N+1} \mid x \in \mathbb{R}^N,\ T_1 < t < T_2\}$$

(with $-\infty \le T_1 < T_2 \le +\infty$). Moreover, the characteristic form of the operator is nonnegative definite and non totally degenerate, i.e.,

$$q_{\mathcal{L}}(z,\xi) := \sum_{i,j=1}^{N} a_{ij}(z)\xi_i\xi_j \ge 0, \qquad q_{\mathcal{L}}(z,\cdot) \not\equiv 0 \text{ for every } z \in S.$$

The operator $\mathcal{L}$ is assumed to be endowed with a *global fundamental solution* Γ,

$$(z,\zeta) \longmapsto \Gamma(z,\zeta),$$

of class C^∞ in $\{(z,\zeta) \in \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} \mid z \ne \zeta\}$, $\Gamma \ge 0$ and $\Gamma(x,t,\xi,\tau) > 0$ if and only if $t > \tau$. Γ is a fundamental solution in the following sense:

$$\Gamma(\cdot,\zeta) \in L^1_{\text{loc}}(S) \quad \text{and} \quad \mathcal{L}(\Gamma(\cdot,\zeta)) = -\delta_\zeta$$

for every $\zeta \in S$. Furthermore, Γ satisfies the following properties:

(i) For every $\varphi \in C_0^\infty(\mathbb{R}^{N+1})$ and for every $(x_0, \tau) \in S$, one has

$$\int_{\mathbb{R}^N} \Gamma(x,t,\xi,\tau)\varphi(\xi)\, d\xi \to \varphi(x_0), \qquad \text{as } (x,t) \to (x_0,\tau),\ t > \tau;$$

(ii) for every fixed $z \in S$, $\limsup_{\zeta \to z} \Gamma(z, \zeta) = +\infty$;

(iii) $\Gamma(z,\zeta) \to 0$ for $\zeta \to \infty$ uniformly for $z \in K$, compact set of S,

and, analogously,

$\Gamma(z,\zeta) \to 0$ for $z \to \infty$ uniformly for $\zeta \in K$, compact set of S;

(iv) there exists a positive constant $C > 0$ such that $\forall\ z = (x,t)$ we have

$$\int_{\mathbb{R}^N} \Gamma(z;\xi,\tau)\, d\xi \le C, \qquad \text{if } t > \tau.$$

3. Potential Analysis for $\mathcal{L}$

Methods and results from Abstract Potential Theory apply to the operator $\mathcal{L}$. For every Ω, open subset of S, we define the sheaf of functions

$$\mathcal{H}(\Omega) = \{u \in C^\infty(\Omega) \mid \mathcal{L}u = 0 \text{ in } \Omega\}.$$

Then, following the lines of [LU10], we may verify that

$(S, \mathcal{H}(S))$ is a β-Harmonic space satisfying the Doob convergence property.

Indeed the following axioms are satisfied.

(A1) *Positivity axiom:*

For every $z \in S$, there exists a open set $V \ni z$ and a function $u \in \mathcal{H}(V)$ such that $u(z) > 0$.

(A2) *Doob convergence axiom:*

The limit of any increasing sequence of $\mathcal{H}$-harmonic functions in a open set $V \subseteq S$ is $\mathcal{H}$-harmonic whenever it is finite in a dense subset of V.

(A3) *Regularity axiom:*

There is a basis of the euclidean topology of S formed by $\mathcal{H}$-regular sets.

(A4) *Separation axiom:*

For every y and z in S, $y \neq z$, there exist two nonnegative superharmonic functions u and v in S such that $u(y)v(z) \neq u(z)v(y)$.

We refer to [BLU07, chapter 6], [CC72] and [Bau66] for a detailed description of the general theory of *harmonic spaces*.

As the operator $\mathcal{L}$ endows the strip S with a structure of Doob β-harmonic space, from the Wiener resolutivity theorem (see [CC72, Theorem 2.4.2]) it follows the existence of a *generalized solution in the sense of Perron-Wiener*, that we may define exactly as in Subsection 1.2 for heat operator, for every $\varphi \in C(\partial\Omega)$ to the Dirichlet problem

$$\text{(DP)} \qquad \begin{cases} \mathcal{L}u = 0 \text{ in } \Omega, & \Omega \text{ bounded open set, } \overline{\Omega} \subseteq S, \\ u|_{\partial\Omega} = \varphi. \end{cases}$$

This solution assumes the boundary data at every regular points of $\partial\Omega$.

In the next section, we formulate a criterion of boundary regularity which bases only on the behavior of the integral of the fundamental solution on a particular subset of $\mathbb{R}^N$.

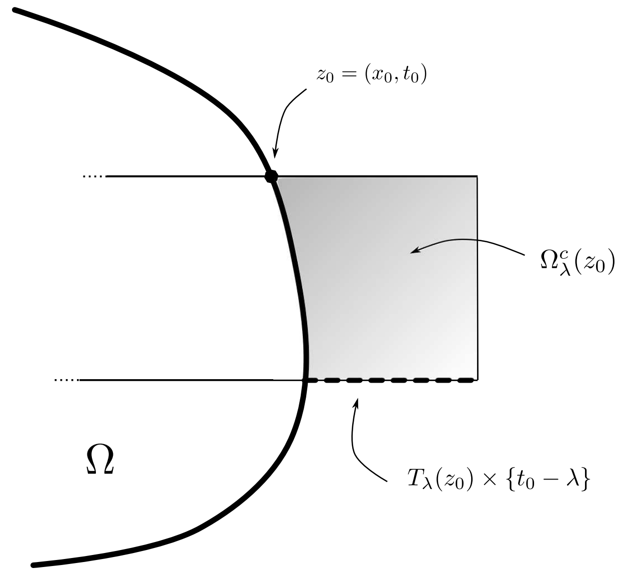
4. Our regularity criterion

Let Ω be a bounded open set with $\overline{\Omega} \subset S$ and z_0 a point of $\partial\Omega$.
Furthermore, let $(B_\lambda)_{0<\lambda<1}$ be a basis of closed neighborhood of x_0 (in $\mathbb{R}^N$) such that $B_\lambda \subseteq B_\mu$ if $0 < \lambda < \mu \leq 1$. We set

$$\Omega^c_\lambda(z_0) := (B_\lambda \times [t_0 - \lambda, t_0]) \backslash \Omega$$

and

$$T_\lambda(z_0) = \{x \in \mathbb{R}^N \ : \ (x, t_0 - \lambda) \in \Omega^c_\lambda(z_0)\}.$$



Theorem 4.1. *The point $z_0 \in \partial\Omega$ is $\mathcal{H}$-regular if*

$$\limsup_{\lambda \searrow 0} \int_{T_\lambda(z_0)} \Gamma(z_0; \xi, t_0 - \lambda)\, d\xi > 0.$$

This behavior of the integral of the fundamental solution together with a classical balayage-criterion will prove that the point $z_0 \in \partial\Omega$ is regular (see [K16, Proposition 4.1 and Theorem 5.1]).

Our criterion extends and generalizes the Effros and Kazdan *parabolic cone (or tusk) condition* for the heat operator. Furthermore, this criterion extends and generalizes also the cone-type condition proved in [LU10](see also [Ugu07]) for a class of hypoelliptic diffusion equations under the assumptions of doubling condition and segment property for an underlying distance and Gaussian bounds of the fundamental solution. We recall that for a class of elliptic-parabolic partial differential operators, a regularity criterion of de la Vallée-Poussin-type charachterazing boundary points was given in [NS84].

5. Applications: a cone-type condition

5.1. **Invariant operators on Lie groups.** In this section, we consider operators that are left translation invariant and homogeneous of degree two with respect to an homogeneous Lie group

$$\mathbb{G} = (\mathbb{R}^{N+1}, \circ, \delta_r).$$

The group of dilations $(\delta_r)_{r>0}$ is a group of *automorphism* on $(\mathbb{R}^{N+1}, \circ)$ of the following type

$$\delta_r : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}^{N+1}, \quad \delta_r(x_1, \dots, x_N, t) = (r^{\sigma_1} x_1, \dots, r^{\sigma_N} x_N, r^2 t),$$

where $\sigma_1, \dots, \sigma_p$ are positive integers such that $1 \le \sigma_1 \le ... \le \sigma_N$.

We set

$$D_r = \delta_r|_{\mathbb{R}^N}$$

the relevant group of dilations in $\mathbb{R}^N$.

We call δ_r*-cone with vertex in* $(0,0)$ every open set of the following type

$$\hat{C} := \{\delta_r(\xi, -T) \mid \xi \in B,\ 0 < r < 1\} = \{(D_r(\xi), -r^2 T) \mid \xi \in B,\ 0 < r < 1\},$$

where $T > 0$ and B is a bounded open set of $\mathbb{R}^N$, $\mathrm{int} B \neq \emptyset$. Moreover, we call δ_r*-cone with vertex in* z_0 the translated cone

$$z_0 \circ \hat{C},$$

where $\hat{C}$ is a δ_r-cone with vertex in 0.

From our criterion, we derive now the following cone-type criterion that extends the *parabolic cone condition* by Effros and Kazdan for the particular case of the classical heat operator $\Delta - \partial_t$.

Theorem 5.1. *Let $\mathcal{L}$ be an invariant evolution operator on $\mathbb{G} = (\mathbb{R}^{N+1}, \circ, \delta_r)$. Let Ω be a bounded open set of $\mathbb{R}^{N+1}$ and let $z_0 \in \partial\Omega$. If there exists a δ_r-cone with vertex in z_0 contained in $\mathbb{R}^{N+1} \setminus \Omega$, then*

$$z_0 \text{ is } \mathcal{L}\text{-regular for } \Omega.$$

5.2. **Kolmogorov-Fokker-Planck-type operators.** Let us consider the operator

$$\mathcal{L} = \mathrm{div}(A\nabla) + \langle Bx, \nabla \rangle - \partial_t, \text{ in } \mathbb{R}^{N+1} = \mathbb{R}^N_x \times \mathbb{R}_t, \tag{1}$$

where A and B are constant $N \times N$ real matrices of the following type

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} * & * & * & \dots & * \\ B_1 & * & * & \dots & * \\ 0 & B_2 & \dots & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & B_k & * \end{pmatrix}$$

where A_0 is a $p_0 \times p_0$ symmetric and positive semidefinite matrix, the B_j's are $p_j \times p_{j-1}$ matrices with rank p_j, $1 \leq j \leq k$, $p_0 \geq p_1 \geq \ldots \geq p_k \geq 1$ and $p_0 + p_1 + \ldots + p_k = N$.

These operators, introduced and studied in [LP94], have been subsequently studied by many authors as a basic model for general Kolmogorov-Fokker-Planck operators.

We collect before some properties from [LP94] and then we state our cone-type criterion.

Denote and introduce in $\mathbb{R}^{N+1}$ the following composition law

$$(x,t) \circ (y,\tau) := \; (y + E(\tau)x, t+\tau), \quad E(t) = \exp(-tB).$$

Then the operator $\mathcal{L}$ is left-translation invariant with respect to the Lie group

$$\mathbb{K} = (\mathbb{R}^{N+1}, \circ) \qquad\qquad (x,t) \circ (x',t') = (x' + E(t')x, t+t'),$$

where $E(t) := \exp(-tB), \quad t \in \mathbb{R}$.

Moreover, for every $r > 0$, we introduce the following group of dilations

$$\begin{aligned} \delta_r : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}^{N+1}, \quad \delta_r(x,t) &= \delta_r(x^{(p)}, x^{(p_1)}, \ldots, x^{(p_k)}, t) \\ &= (rx^{(p_0)}, r^3 x^{(p_1)}, \ldots, p^{2k+1} x^{(p_k)}, r^2 t), \end{aligned}$$

where $x^{(p_j)} \in \mathbb{R}^{p_i}$, for every $j = 0, \ldots, k$, and $r > 0$.

We observe that δ_r is an authomorphism of $\mathbb{G}$ if and only if the all the blocks $*$ in B are identically zero.

As in the previous subsection we call *δ_r-cone with vertex in* $(0,0)$ any open set of the kind

$$\hat{C} := \{\delta_r(\xi, -T) \mid \xi \in B,\ 0 < r < 1\},$$

where $T > 0$ and B is a bounded open set of $\mathbb{R}^N$.

We name *δ_r-cone with vertex in z_0* every set

$$z_0 \circ \hat{C},$$

where $\hat{C}$ is a δ_r-cone with vertex in 0.

Although, in general, the operator $\mathcal{L}$, is not δ_r-homogeneous, nevertheless the following regularity criterion holds.

Theorem 5.2. *Let $\mathcal{L}$ be a Kolmogorov-Fokker-Planck-type operator as in* (1).

Let Ω a bounded open set of $\mathbb{R}^{N+1}$ and let $z_0 \in \partial\Omega$. If there exists a δ_r-cone with vertex in z_0 contained in $\mathbb{R}^{N+1}\backslash\Omega$, then z_0 is $\mathcal{L}$-regular for Ω.

Cone-type criteria for homogeneous Kolmogorov-Fokker-Planck-type operators were obtained in [Sco81, Man97].

For parabolic operators with variable coefficients, cone-type conditions could be deduced from the Wiener criteria proved in [GL88, FEBL89].

We would like to emphasize that, in our general framework, i.e., for evolution equations with underline sub-Riemannian structures, the problem of characterizing the regularity of the boundary points in terms of Wiener-type series is still widely open. Nowadays, there are only few results in literature: the one related to the Kolmogorov equation in $\mathbb{R}^3$ due to Scornazzani [Sco81] and the Wiener criterion related to the heat operator on the Heisenberg group due to Garofalo and Segala [GS90]. Very recently, for the operators studied in [LU10], Lanconelli, Tralli and Uguzzoni in [LTU16] have given necessary and sufficient regularity conditions in terms of Wiener-type series; however, these criteria do not exactly characterize the boundary points.

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