STEINER FORMULA AND GAUSSIAN CURVATURE IN THE HEISENBERG GROUP
FORMULA DI STEINER E CURVATURA DI GAUSS NEL GRUPPO DI HEISENBERG

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Abstract. The classical Steiner formula expresses the volume of the ϵ-neighborhood Ω_ϵ of a bounded and regular domain $\Omega \subset \mathbb{R}^n$ as a polynomial of degree n in ϵ. In particular, the coefficients of this polynomial are the integrals of functions of the curvatures of the boundary $\partial\Omega$. The aim of this note is to present the Heisenberg counterpart of this result. The original motivation for studying this kind of extension is to try to identify a suitable candidate for the notion of *horizontal Gaussian curvature.*

The results presented in this note are contained in the paper [4] written in collaboration with Zoltán Balogh, Fausto Ferrari, Bruno Franchi and Kevin Wildrick.

Sunto. La classica formula di Steiner afferma che il volume dell'ϵ-intorno Ω_ϵ di un dominio limitato e regolare $\Omega \subset \mathbb{R}^n$ si scriva come un polinomio di grado n in ϵ. In particolare, i coefficienti di questo polinomio sono dati da integrali di funzioni delle curvature del bordo $\partial\Omega$. In questa nota presenteremo l'analoga versione della formula di Steiner nel caso del primo gruppo di Heisenberg $\mathbb{H}$. La motivazione originale che ha portato allo studio della formula di Steiner in $\mathbb{H}$ consiste nella ricerca di un possibile candidato per la nozione di *curvatura di Gauss orizzontale.*

I risultati che presenteremo sono contenuti nel lavoro [4] scritto in collaborazione con Zoltán Balogh, Fausto Ferrari, Bruno Franchi and Kevin Wildrick.

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1. Introduction

The Steiner formula was first proved it in two and three dimensional Euclidean spaces for convex polytopes. Weyl extended it later (see [23]) to the setting of arbitrary smooth submanifolds of $\mathbb{R}^n$. In [12], Federer proved a localized version of the above formula for a large class of non-smooth submanifolds, introducing the concept of *sets of positive reach*.

Let us try to give a flavour of the content of the Steiner formula by stating it in $\mathbb{R}^n$. To this aim, let us denote by $\Omega \subset \mathbb{R}^n$ a bounded regular domain, and by Ω_ϵ its ϵ-neighborhood with respect to the standard Euclidean metric. The Steiner formula asserts that the volume $\mathcal{L}^n(\Omega_\epsilon)$ of Ω_ϵ can be expressed as a polynomial of degree n in ϵ:

$$\mathcal{L}^n(\Omega_\epsilon) = \sum_{k=0}^{n} a_k \epsilon^k, \tag{1}$$

where the coefficients a_k are integrals of suitable functions of the curvatures of of $\partial\Omega$. In particular, in the even simpler case of $\mathbb{R}^3$, the coefficients a_2 and a_3 are the integrals of scalar multiples of the mean curvature and of the Gaussian curvature of $\partial\Omega$, respectively. This explicit appearance of the curvatures of $\partial\Omega$ as integrands in the coefficients of the Steiner formula, has been our original motivation to look for a Heisenberg counterpart of such a formula. We must mention here that the existence of a Steiner formula in the first Heisenberg group has been already addressed by Ferrari in [13], where he proved the validity of a *global* Steiner formula for the case of Carnot-Carathéodory balls.

In order to state the first main result presented in this note, we need to introduce some notation. Let us denote by $\{X_1, X_2\}$ an orthonormal basis of the Lie algebra $\mathfrak{h}$ associated to the first Heisenberg group $\mathbb{H}$. Let $\Omega \subset \mathbb{H}$ be an open set and let $u : \Omega \to \mathbb{R}$ be a $\mathcal{C}^\infty$-smooth function, we will define the *iterated horizontal divergences* of u as follows:

$$\mathrm{div}_{\mathrm{H}}^{(i)} u := \begin{cases} 1, & i = 0, \\ \mathrm{div}_{\mathrm{H}}\left((\mathrm{div}_{\mathrm{H}}^{(i-1)} u) \cdot \nabla_{\mathrm{H}} u \right), & i \geq 1, \end{cases} \tag{2}$$

where $\nabla_{\mathrm{H}} u := (X_1 u) X_1 + (X_2 u) X_2$ is the *horizontal gradient* of u. Our first main result is then given by the *localized* Heisenberg counterpart of the Euclidean Steiner formula.

Theorem 1.1. *Let $\Omega \subset \mathbb{H}$ be a bounded smooth domain with $\mathcal{C}^\infty$-regular boundary and let $Q \subset \mathbb{H}$ be a localizing set with the property that $\partial\Omega \cap Q$ is free from characteristic* *points. We denote by δ_{cc} the signed Carnot-Carathédory distance function defined in a neighborhood of $\partial\Omega \cap Q$. For $\epsilon \geq 0$, let $\Omega_\epsilon \cap Q$ be a localized Heisenberg ϵ-neighborhood of Ω. Then there is a positive constant $\epsilon_0 > 0$ such that the function $\epsilon \mapsto \mathcal{L}^3(\Omega_\epsilon \cap Q)$ is analytic on the interval $[0, \epsilon_0)$ and has a power series expansion given by*

$$\mathcal{L}^3(\Omega_\epsilon \cap Q) = \mathcal{L}^3(\Omega \cap Q) + \sum_{i \geq 1} \frac{\epsilon^i}{i!} \int_{\partial\Omega\cap Q} (\mathrm{div}_{\mathrm{H}}^{(i-1)} \delta_{cc})\, d\mathcal{H}^3_{cc}. \tag{3}$$

We refer to Section 4 for the motivations and the precise description of the localization away from characteristic points given by the set Q.

The remarkable fact is that, *a priori*, the $(i-1)^{\mathrm{st}}$ iterated divergence $\mathrm{div}_{\mathrm{H}}^{(i-1)}\delta_{cc}$, $i > 1$, contains derivatives of order i, but this is actually not the case. This is the content of our second main result. In order to state it precisely we need to introduce some notation. Let us denote by X_3 the canonical left-invariant vertical vector field in $\mathbb{H}$. Let us also define the following quantities:

$$A := X_{11}\delta_{cc} + X_{22}\delta_{cc}, \quad B := -(4X_3\delta_{cc})^2, \quad C := -4\left((X_1\delta_{cc})(X_{32}\delta_{cc}) - (X_2\delta_{cc})(X_{31}\delta_{cc})\right),$$

$$D := 16X_{33}\delta_{cc}, \qquad E := 16\left((X_{31}\delta_{cc})^2 + (X_{32}\delta_{cc})^2\right),$$

where to simplify the notation we denoted $X_i(X_j\delta_{cc})$ by $X_{ij}\delta_{cc}$, for $i = 1, 2, 3$.

Theorem 1.2. *Under the conditions of Theorem 1.1, the following relations hold:*

$$\mathrm{div}_{\mathrm{H}}^{(1)}\delta_{cc} = A, \qquad \mathrm{div}_{\mathrm{H}}^{(2)}\delta_{cc} = B + 2C,$$

$$\mathrm{div}_{\mathrm{H}}^{(3)}\delta_{cc} = AB + 2D, \qquad \mathrm{div}_{\mathrm{H}}^{(4)}\delta_{cc} = B^2 + 2BC + 2AD - 2E,$$

and for all $j \geq 2$,

$$\mathrm{div}_{\mathrm{H}}^{(2j-1)}\delta_{cc} = B^{j-2}\left(AB + 2(j-1)D\right), \tag{4}$$

$$\mathrm{div}_{\mathrm{H}}^{(2j)}\delta_{cc} = B^{j-2}\left(B^2 + 2BC + 2(j-1)(AD - E)\right). \tag{5}$$

Let us make a small comment on the technique adopted to prove Theorem 1.1: the main ingredient in the proof is given by a systematic and iterated use of the divergence Theorem, conveniently adapted to our Heisenberg frame (see Proposition 4.2). This technique, that we will call *iterated divergence technique*, has been inspired by the works [21, 22] by Reilly. As far as we know, this approach is new even in the Euclidean case (see Section

2).

It is clear that the Heisenberg counterpart of the Steiner formula is pretty different from the Euclidean one. The first gap that we can notice comparing the Euclidean Steiner formula (1) to the Heisenberg one (3), is that the latter is not more a polynomial but a series in ϵ. Secondly, the recursive formula provided by Theorem 1.2 shows that the integrands, given by the *iterated divergences*, cannot be anymore related to the symmetric polynomials of a Hessian matrix (see Remark 4.1). On the other hand, this time in analogy with the Euclidean case, Theorem 1.2 proves that the coefficients of the series (3) are integrals of second order derivatives of the function defining $\partial\Omega$. Although the idea of looking for a suitable candidate for the notion of *horizontal Gaussian curvature* via the study of Steiner formula is definitely non-orthodox, the fact that $\mathrm{div}_{\mathrm{H}}^{(1)}\delta_{cc}$ is a scalar multiple of the so called *horizontal mean curvature*, could suggest that $\mathrm{div}_{\mathrm{H}}^{(2)}\delta_{cc}$ could be a reasonable candidate for the notion of *horizontal Gaussian curvature* away from characteristic points.

Let us now spend a few words on the existing literature concerning *horizontal curvatures* of smooth surfaces in $\mathbb{H}$. First, let us recall that the classical differential geometric approach to the study of the curvatures of a smooth embedded surface $\Sigma \subset \mathbb{R}^3$, is based on the study of the eigenvalues of the differential of the Gauss normal map. There were several attempts to propose a Heisenberg counterpart of the notion of Gauss normal map away from characteristic points (see e.g. [9, 10]). Restricting the attention to the case of graphs, there is a definition of *horizontal Gaussian curvature* modeled on the symmetrized horizontal Hessian (see e.g. [9, 8]), which has also been used in [17] to study the flow by horizontal Gaussian curvature by means of viscosity theory. A complete different definition of *horizontal Gaussian curvature*, based on an adapted covariant derivative, has been recently proposed by Diniz and Veloso in [11]. The story is much more clear when we deal with *horizontal mean curvature*. Indeed, there is already a well accepted notion of *horizontal mean curvature*, which plays a crucial role in the still under development theory of sub-Riemannian minimal surfaces. This concept was introduced in [20] by Pauls and it is obtained as limit of the Riemannian mean curvature in the Riemannian approximation scheme. The same approximation technique introduced by Pauls has been recently used in [5]. In this case the authors studied the limit of the Riemannian Gaussian (or sectional) curvature and suggested another possible candidate for the notion of *horizontal Gaussian curvature*. In this perspective, let us mention that the natural candidate suggested by the Steiner formula in $\mathbb{H}$ (see Theorem 1.2 and Remark 4.1) slightly differs from the one proposed in [5], and this indicates that there is still the need of a better understanding of these concepts.

All the proofs of the new results presented in this note are contained in [4].

2. A glance at the Steiner formula in $\mathbb{R}^3$

Let us start with a simple but enlightening example in $\mathbb{R}^3$.

Example 2.1. *Let $B := B(0,r) = \{x \in \mathbb{R}^3 : \|x\|_{\mathbb{R}^3} < r\}$ be the open ball of center 0 and radius $r > 0$. Let $\epsilon > 0$ be small enough. Let us define the ϵ-neighborhood of B to be*

$$B_\epsilon = \big\{x \in \mathbb{R}^3 : \mathrm{dist}(x, B) < \epsilon\big\} = \{x \in \mathbb{R}^3 : \|x\|_{\mathbb{R}^3} < r + \epsilon\}.$$

Therefore

$$\mathcal{L}^3(B_\epsilon) = \frac{4}{3}\pi(r+\epsilon)^3 = \frac{4}{3}\pi r^3 + 4\pi r^2\epsilon + 4\pi\epsilon^2 + \frac{4}{3}\pi\epsilon^3.$$

Recalling that the mean curvature H of the sphere $\partial B(0,r)$ is $\frac{1}{r}$ and its Gaussian curvature K is $\frac{1}{r^2}$, we can write the Steiner formula for the Euclidean ball $B(0,r)$ as follows:

$$\mathcal{L}^3(B_\epsilon) = \mathcal{L}^3(B) + \epsilon\mathcal{H}^2(\partial B) + \epsilon^2 \int_{\partial B} H \, d\mathcal{H}^2 + \frac{\epsilon^3}{3}\int_{\partial B} K \, d\mathcal{H}^2,$$

where $\mathcal{L}^3$ denotes the Lebesgue measure and $\mathcal{H}^2$ the 2-Hausdorff measure.

This simple example seems to suggest that the Steiner formula could be a good tool to recover information concerning the curvature of suitably regular surfaces which are the boundary of given open and bounded sets. Since we are interested in extending the Steiner formula to the first Heisenberg group $\mathbb{H}$, we will briefly describe it in $\mathbb{R}^3$ but, as we mentioned in the Introduction, a similar statement holds true in $\mathbb{R}^n$. Let us introduce the notation and the standing assumptions that we will adopt throughout this Section.

- Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and regular set with Euclidean $\mathcal{C}^\infty$-smooth boundary $\partial\Omega$;
- let $\mathrm{dist}(\cdot, \Omega) : \mathbb{R}^3 \to \mathbb{R}_+$ be the distance function from Ω;

- let $\delta := \delta(x, \partial\Omega)$ be the signed distance function from $\partial\Omega$, defined as

$$\delta(x) := \begin{cases} \operatorname{dist}(x, \partial\Omega), & \text{if } x \in \mathbb{R}^3 \setminus \Omega, \\ -\operatorname{dist}(x, \partial\Omega), & \text{if } x \in \overline{\Omega}. \end{cases}$$

- let $\Omega_\epsilon := \Omega \cup \{x \in \mathbb{R}^3 : 0 \leq \delta(x) < \epsilon\}$ be the ϵ-neighborhood of Ω.

It is possible to prove that the signed distance function δ satisfies the eikonal equation,

$$\sum_{i=1}^{3} \left(\frac{\partial \delta}{\partial x_i} \right)^2 = 1. \tag{6}$$

In particular, if we think of $\partial\Omega$ as the 0-level set of the signed distance function δ,

$$\partial\Omega = \{x \in \mathbb{R}^3 : \delta(x) = 0\},$$

we have a globally defined unit outward-pointing normal ν to $\partial\Omega$, and it holds that $\nu = \nabla\delta$. It is very reasonable to expect the signed distance function δ to have one degree of regularity less than $\partial\Omega$ but deep results from [18], [14] and [16], show that actually there exists an open neighborhood $U \subset \mathbb{R}^3$ of $\partial\Omega$ such that $\delta|_U$ has the same regularity of $\partial\Omega$. This, coupled with the regularity assumptions made on $\partial\Omega$, shows that the normal ν is $\mathcal{C}^\infty$-smooth.

The approach we want to present here is based on the works [21] and [22] by Reilly: we will call it the *iterated divergences technique.* First, we need to define the *iterated divergences* of the signed distance function δ.

Definition 2.1. *Let Ω and δ be as before. We define the iterated divergences as follows*

$$\sigma_k := \begin{cases} 1, & \textit{for } k = 0, \\ \operatorname{div}(\sigma_{k-1} \cdot \nabla\delta), & \textit{for } k \geq 1. \end{cases}$$

We stress that, thank to the regularity assumed on $\partial\Omega$, the signed distance function δ is $\mathcal{C}^\infty$-smooth, and therefore the *iterated divergences* are certainly well defined.

We recall now the general definition of the symmetric polynomials.

Definition 2.2. *Let V be an n-dimensional vector space with an inner product, and let $A : V \to V$ be a linear symmetric transformation. Denote by $\lambda_1, \dots, \lambda_n$ its eigenvalues.*

For $0 \leq k \leq n$, we define the k^{th}-elementary symmetric function of the numbers λ_i's as

$$S_k(A) := \sum_{1\leq i_1<\ldots<i_k\leq n} \lambda_{i_1}\cdot \ldots \cdot \lambda_{i_k}.$$

The first remarkable result relates the *iterated divergences* to the elementary symmetric functions of the eigenvalues of Hess_δ of the signed distance δ.

Theorem 2.1. *Let Ω and δ be as before. Then*

$$\sigma_1 = S_1(\mathrm{Hess}_\delta), \quad \sigma_2 = 2\, S_2(\mathrm{Hess}_\delta), \quad \textit{and} \quad \sigma_k = 0, \quad \textit{for every } k \geq 3.$$

The already cited works [21] and [22] of Reilly play a fundamental role in the proof of Theorem 2.1.

Remark 2.1. *The main implication of Theorem 2.1 is that, despite their definition, the iterated divergences σ_k's can be described using only second order derivatives of the signed distance function δ. One of the key points to prove this result is that differentiating the eikonal equation* (6) *one can get several identities expressing higher order derivatives of the signed distance function δ in terms of only second order ones.*

The next step in the direction of proving the Steiner formula is to prove first the analyticity of volume function $\epsilon \mapsto \mathcal{L}^3(\Omega_\epsilon)$, and then to relate the derivatives of the volume function $\epsilon \mapsto \mathcal{L}^3(\Omega_\epsilon)$ to the integrals of the *iterated divergences*.

Definition 2.3. *Let $\Omega \subset \mathbb{R}^3$ as before and let Ω_ϵ be its ϵ-neighborhood. For $k \geq 0$, let σ_k be as in Definition (2.1). Define*

$$I_k = \begin{cases} \mathcal{L}^3(\Omega), & \textit{for } k = 0, \\ \int_{\partial\Omega} \sigma_{k-1}\, d\mathcal{H}^2, & \textit{for } k = 1, 2, 3. \end{cases}$$

Similarly, for $k = 1, 2, 3$,

$$I_k(\epsilon) := \int_{\{\delta=\epsilon\}} \sigma_{k-1}\, d\mathcal{H}^2,$$

where $\{\delta = \epsilon\} := \{x \in \mathbb{R}^3 : \delta(x) = \epsilon\}$ denotes the level sets of the distance function δ from Ω.

One can then prove the following Theorem:

Theorem 2.2. *Let Ω, Ω_ϵ, δ, ν and σ_k be as before. Then*

$$\mathcal{L}^3(\Omega_\epsilon) = \sum_{k=0}^{3} I_k \, \frac{\epsilon^k}{k!}.$$

Sketch of the proof. Let $\epsilon > 0$. Let us denote by $a_0(\epsilon) := \mathcal{L}^3(\Omega_\epsilon)$ and then recursively,

$$a_k(\epsilon) := \lim_{s \to 0+} \frac{a_{k-1}(\epsilon + s) - a_{k-1}(\epsilon)}{s}, \quad s > 0.$$

One can now show that the above limits are all well defined and that $a_k(\epsilon) = I_k(\epsilon)$, for all $k \in \mathbb{N}$. Finally we let $\epsilon \to 0^+$. □

Remark 2.2. *We can now notice that we can weaken our regularity assumption on $\partial\Omega$ up to $\mathcal{C}^4$-smooth. Roughly speaking, the reason is the following: requiring $\mathcal{C}^4$-smoothness of $\partial\Omega$ implies $\mathcal{C}^3$ regularity of the normalized defining function δ of $\partial\Omega$. Theorem 1.2 now shows that the iterated divergences can be expressed only in terms of second order derivatives of the defining function δ, but in the explicit proof one needs to derive once these expressions. We want also to stress that this approach based on the properties of the signed distance function δ is possible only for submanifolds S of codimension 1, because we need to give a meaning to the notions of inside and outside of S.*

Let us also spend a few words on the geometric meaning of the coefficients appearing in the Steiner formula. We recalled that the coefficients of the polynomial given by the Steiner formula are integrals of the *iterated divergences*. By Theorem 2.1 these are precisely the symmetric polynomials in the eigenvalues of the Hessian matrix Hess_δ of the signed distance function δ. It is a classical result of differential geometry of Euclidean $\mathcal{C}^2$-smooth submanifolds, that those eigenvalues are nothing but the *principal curvatures* of $\partial\Omega$ (see for instance [16, Chapter 14]). In particular, we have that the *first iterated divergence* σ_1 and σ_2 are, respectively, scalar multiples of the *mean curvature* of $\partial\Omega$ and of the *Gaussian curvature* of $\partial\Omega$.

3. Preliminaries on the Heisenberg group

We will recall only the basic notation needed in the following Section. We refer to the monographs [6] and [7] for an introduction to the subject.

The first Heisenberg group $\mathbb{H}$ is the simplest model of a non commutative Carnot group. $\mathbb{H}$ is identified with $\mathbb{R}^3$ with a non-commutative group law $*$ defined as

$$(y_1, y_2, y_3) * (x_1, x_2, x_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + 2(x_1 y_2 - x_2 y_1)) \,,$$

whose inverse is given by $(x_1, x_2, x_3)^{-1} = (-x_1, -x_2, -x_3)$, and whose neutral element is the origin $0 = (0, 0, 0)$. In this way $(\mathbb{H}, *)$ is a Lie group. The first Heisenberg group $\mathbb{H}$ admits a 2-step stratification, $\mathfrak{h} = \mathbb{V}_1 \oplus \mathbb{V}_2$, where $\mathbb{V}_1 = \operatorname{span}\{X_1, X_2\}$ and $\mathbb{V}_2 = \operatorname{span}\{X_3\}$, for $X_1 = \partial_{x_1} + 2x_2\partial_{x_3}$, $X_2 = \partial_{x_2} - 2x_1\partial_{x_3}$ and $X_3 = -\frac{1}{4}[X, Y] = \partial_{x_3}$.
The horizontal vector fields X_1 and X_2 are of fundamental importance in the context of $\mathbb{H}$ because they define a 2-dimensional plane distribution $H\mathbb{H}$, known as the *horizontal distribution*:

$$H_g\mathbb{H} := \operatorname{span}\{X_1(g), X_2(g)\}, \quad g \in \mathbb{H}.$$

This smooth distribution of planes is a subbundle of the tangent bundle of $\mathbb{H}$, and due to the fact that $[X_1, X_2] = -4X_3 \notin H\mathbb{H}$, it is a non integrable distribution. The horizontal distribution $H\mathbb{H}$ makes the Heisenberg group $\mathbb{H}$ one of the easiest examples of a sub-Riemannian manifold. We define an inner product $\langle \cdot, \cdot \rangle_{g,\mathrm{H}}$ on $H\mathbb{H}$, so that for every $g \in \mathbb{H}$, $\{X_1(g), X_2(g)\}$ forms a orthonormal basis of $H_g\mathbb{H}$. We will then denote by $\| \cdot \|_{g,\mathrm{H}}$ the horizontal norm induced by the scalar product $\langle \cdot, \cdot \rangle_{g,\mathrm{H}}$. In both cases, we will omit the dependance on the base point $g \in \mathbb{H}$ when it is clear. With these notions, we are allowed to make measurements of all the horizontal objects. Among them, the horizontal curves are of fundamental importance.

Definition 3.1 (Horizontal curves). *An absolutely continuous curve $\gamma : [a, b] \subset \mathbb{R} \to \mathbb{H}$ is said to be horizontal if*

$$\dot{\gamma}(t) \in H_{\gamma(t)}\mathbb{H}, \quad \textit{for a.e.}\, t \in [a, b].$$

We can then define the *horizontal length* of the horizontal curves exploiting the scalar product previously defined on $H\mathbb{H}$.

Definition 3.2 (Horizontal length). *Let $\gamma : [a,b] \to \mathbb{H}$ a horizontal curve. The horizontal length $l_{\mathrm{H}}(\gamma)$ of γ is defined as*

$$l_{\mathrm{H}}(\gamma) := \int_a^b \|\dot{\gamma}\|_{\mathrm{H}}\, dt.$$

The importance of the notion of horizontal curve relies on the fact that, as in the Riemannian setting, we can use them to define a path-metric (and therefore the notion of a geodesic), better known as Carnot-Carathéodory metric (*cc*-metric in short).

Definition 3.3 (*cc*-metric). *Let $x, y \in \mathbb{H}$, with $x \neq y$. The cc-distance of x, y is defined as*

$$d_{cc}(x,y) := \inf\{l_{\mathrm{H}}(\gamma) | \gamma : [a,b] \to \mathbb{H}, \gamma(a) = x, \gamma(b) = y\}$$

We also recall the notions of *horizontal gradient* of a function and of *horizontal divergence* of a horizontal vector field.

Definition 3.4 (Horizontal gradient). *Let $u : \mathbb{H} \to \mathbb{R}$ be a Euclidean C^1-smooth function. The horizontal gradient $\nabla_{\mathrm{H}} u$ of u is the projection of the Euclidean gradient ∇u of u onto the horizontal distribution, namely*

$$\nabla_{\mathrm{H}} u = (X_1 u) X_1 + (X_2 u) X_2.$$

Definition 3.5 (Horizontal divergence). *Let $V = aX_1 + bX_2$ be a differentiable and horizontal vector field. The horizontal divergence $\mathrm{div}_{\mathrm{H}} V$ of V is defined as*

$$\mathrm{div}_{\mathrm{H}} V = X_1 a + X_2 b.$$

4. Steiner formula in the Heisenberg group

The aim of this Section is to describe the steps followed in [4] to prove Theorem 1.1, which is the localized counterpart of the Euclidean Steiner formula in the context of the first Heisenberg group $\mathbb{H}$. In order to do it properly, we need to introduce the notation, to identify the main ingredients used in the Euclidean case and to recall some basic results. Let $\Omega \subset \mathbb{H}$ be an open, bounded and regular domain with Euclidean $\mathcal{C}^\infty$-smooth boundary $\partial\Omega$. As in the Euclidean case, we need to work with the *cc-signed distance function* δ_{cc} from $\partial\Omega$.

Definition 4.1. *Let* $d_{cc}:\mathbb{H}\times\mathbb{H}\to\mathbb{R}_+$ *be the Carnot-Carathéodory distance on* $\mathbb{H}$. *The cc-distance from* $\partial\Omega$ *is defined as*

$$\mathrm{dist}_{cc}(g,\partial\Omega):=\inf_{h\in\partial\Omega} d_{cc}(g,h).$$

The signed distance $\delta_{cc}:\mathbb{H}\to\mathbb{R}$ *from* $\partial\Omega$ *is defined as*

$$\delta_{cc}(g)=\begin{cases}\mathrm{dist}_{cc}(g,\partial\Omega), & \textit{if } g\in\mathbb{H}\setminus\Omega,\\ -\mathrm{dist}_{cc}(g,\partial\Omega), & \textit{if } g\in\overline{\Omega}.\end{cases}$$

Let us also define the ϵ-neighborhood Ω_ϵ of Ω with respect to δ_{cc}:

$$\Omega_\epsilon:=\Omega\cup\{g\in\mathbb{H}:0\le\delta_{cc}(g)<\epsilon\}.$$

Due to the assumptions made on the regularity of $\partial\Omega$, we have a well defined Euclidean outward-pointing normal to $\partial\Omega$, whose components with respect to the standard basis of $\mathbb{R}^3$ are given by

$$\nu(g)=(\nu_1(g),\nu_2(g),\nu_3(g)),\qquad g\in\partial\Omega.$$

As usual in the study of the Heisenberg geometry of submanifolds, for every $g\in\partial\Omega$ we can also consider the *horizontal normal* $N(g):=\langle X_1(g),\nu(g)\rangle X_1(g)+\langle X_2(g),\nu(g)\rangle X_2(g)$, where $\langle\cdot,\cdot\rangle$ is the usual scalar product in $\mathbb{R}^3$. One of the typical geometrical obstructions arising in the Heisenberg group $\mathbb{H}$ is provided by the so called *characteristic set*

$$\mathrm{char}(\partial\Omega):=\{g\in\partial\Omega:T_g\Omega=H_g\mathbb{H}\}.$$

It is important to stress that, as in the Euclidean case, we can think of the set $\partial\Omega$ as the zero level set of δ_{cc},

$$\partial\Omega=\{g\in\mathbb{H}:\delta_{cc}(g,\partial\Omega)=0\}.$$

In particular, we will have that $N=\nabla_{\mathrm{H}}\delta_{cc}$. We want to follow the same ideas presented in Section 2. In particular, we want to define a Heisenberg counterpart of the *iterated divergences*. Formally, we can define them as follows:

$$\mathrm{div}_{\mathrm{H}}^{(i)}\delta_{cc}=\begin{cases}1, & \text{for } i=0,\\ \mathrm{div}_{\mathrm{H}}\left((\mathrm{div}_{\mathrm{H}}^{(i-1)}\delta_{cc})\cdot\nabla_{\mathrm{H}}\delta_{cc}\right), & \text{for } i\ge 1.\end{cases}\tag{7}$$

The main problem in the previous definition could come from the regularity of the *cc*-signed distance function δ_{cc}. In this perspective, a result of Arcozzi and Ferrari in [1], and recently generalized in [3], states that δ_{cc} is as regular as $\partial\Omega$, as in the Euclidean case, but only away from $\mathrm{char}(\partial\Omega)$. The precise statement is the following.

Theorem 4.1. *(*[1, Theorem 1.1]*) Let $\Omega \subset \mathbb{H}$ be an open regular domain with Euclidean $\mathcal{C}^k$-smooth boundary $\partial\Omega$, $k \geq 2$. Then $\nabla_{\mathrm{H}}\delta_{cc}$ and δ_{cc} are Euclidean $\mathcal{C}^{k-1}$-smooth in an open neighborhood of $\partial\Omega - \mathrm{char}(\partial\Omega)$ in $\mathbb{H}$.*

The previous result implies in particular that the *iterated divergences* introduced in (7) are well-defined objects.

One of the key points in the Euclidean case was Theorem 2.1, which provided the identification of the *iterated divergences* with the symmetric polynomials of the Hessian matrix Hess_δ of the signed distance function δ. A similar statement is unfortunately not available in the case of the Heisenberg group $\mathbb{H}$. A first question is whether we are able to really compute the *iterated divergences* defined in (7), and the answer is provided by Theorem 1.2. In order to make the few next pages more readable, let us fix the notation we will use in the following. We will denote the action of X_iX_j on any smooth function by X_{ij} and similarly $X_iX_jX_k$ by X_{ijk}, for $i,j,k \in \{1,2,3\}$. With this at hand, we can briefly recall the content of Theorem 1.2. Define the iterated divergences as in (7) and set

$$A := \Delta_{\mathrm{H}}\delta_{cc}, \qquad B := -(4X_3\delta_{cc})^2, \qquad C := -4\left((X_1\delta_{cc})(X_{32}\delta_{cc}) - (X_2\delta_{cc})(X_{31}\delta_{cc})\right),$$

$$D := 16X_{33}\delta_{cc}, \qquad E := 16\left((X_{31}\delta_{cc})^2 + (X_{32}\delta_{cc})^2\right).$$

Then Theorem 1.2 states that

$$\mathrm{div}_{\mathrm{H}}^{(1)}\delta_{cc} = A, \qquad \mathrm{div}_{\mathrm{H}}^{(2)}\delta_{cc} = B + 2C,$$

$$\mathrm{div}_{\mathrm{H}}^{(3)}\delta_{cc} = AB + 2D, \qquad \mathrm{div}_{\mathrm{H}}^{(4)}\delta_{cc} = B^2 + 2BC + 2AD - 2E,$$

and, as a recursive formula, we also have that for all $j \geq 2$,

$$\mathrm{div}_{\mathrm{H}}^{(2j-1)}\delta_{cc} = B^{j-2}\left(AB + 2(j-1)D\right), \tag{8}$$

$$\mathrm{div}_{\mathrm{H}}^{(2j)}\delta_{cc} = B^{j-2}\left(B^2 + 2BC + 2(j-1)(AD - E)\right). \tag{9}$$

Remark 4.1. *It is clear that there is already a huge difference with the Euclidean case (see Theorem 2.1). Indeed, we cannot anymore relate the iterated divergences to the symmetric polynomials of a Hessian matrix. On the other hand, we also have a similarity, namely the fact that, despite their definition, the iterated divergences can still be expressed only by means of second order derivatives of* δ_{cc} *with respect to the vector fields* X_1, X_2 *and* X_3.
We also want to stress that the natural candidate for a notion of horizontal Gaussian curvature would now be the term $\mathrm{div}^{(2)}_{\mathrm{H}}\delta_{cc}$. *We want to point out that this term is very close to the one found in* [5] *as a limit of the Riemannian sectional curvature: in the notation of Theorem 1.2, the object found in* [5] *is given by* $B + C$.

Remark 4.2. *We know by Theorem 4.1 that the cc-signed distance function* δ_{cc} *is smooth only away from the characteristic set. Therefore Theorem 1.2 must be read as a formal result, which will hold where* δ_{cc} *is smooth enough to allow the computations there involved, see Remark 4.4.*

The proof of Theorem 1.2 is quite technical and, as in the Euclidean case, relies on several identities that can be deduced directly from the eikonal equation. In order to streamline the exposition here, we will not write them explicitly but we refer to [4, Section 4] for all the details.

The validity of the eikonal equation is another delicate and crucial issue. A deep and far more general result contained in [19] states that the *cc*-signed distance function δ_{cc} satisfies the natural Heisenberg analog of the eikonal equation.

Theorem 4.2. *(*[19]*)The Carnot-Carathéodory signed distance function* δ_{cc} *satisfies the eikonal equation almost everywhere, namely*

$$\|\nabla_{\mathrm{H}}\delta_{cc}\|_{\mathrm{H}} = 1, \qquad a.e.\ in\ \mathbb{H}.$$

It is then quite clear that all the tools necessary in $\mathbb{R}^3$ are available in $\mathbb{H}$ as well. The main differences come from the *local regularity* of the *cc*-signed distance function δ_{cc} and from the fact that the *iterated divergences* do not identically vanish after some iterations. In particular, the regularity of the *cc*-signed distance function δ_{cc} forces us to look for a localized Steiner formula. To be more precise, we will construct a very specific localizing set Q where the cc-signed distance function δ_{cc} is smooth. For sake of completeness, we will briefly sketch the construction of the set Q, but we refer to [4, Section 3.1] for all the details. Let U_0 be an open subset of $\partial\Omega$ such that dist $(U_0, \mathrm{char}(\partial\Omega)) > 0$. Then, by Theorem 4.1, there exists an open neighborhood U of U_0 in $\mathbb{H}$, with $\overline{U}_0 \subset U$, such that δ_{cc} is smooth in U. It is clear that $d_0 := \mathrm{dist}(U_0, \partial U) > 0$.

Remark 4.3. *The set $\{g \in \mathbb{H} : \delta_{cc}(g) = 0\} \cap U$ is really a manifold. Indeed, by Theorem 4.1 combined with the regularity assumptions on $\partial\Omega$, δ_{cc} is $\mathcal{C}^\infty$-smooth on $\partial\Omega \cap U$, and therefore the eikonal equation $\|\nabla_{\mathrm{H}}\delta_{cc}\|_{\mathrm{H}} = 1$, holds everywhere on $\partial\Omega \cap U$. Therefore*

$$\nabla\delta_{cc} \neq 0, \qquad \text{on } \partial\Omega \cap U,$$

otherwise we would have $\nabla_{\mathrm{H}}\delta_{cc} = 0$ on $\partial\Omega \cap U$.

Now, for sake of simplicity, let $\tilde{g} \in U_0 \subset \partial\Omega$ and let $r > 0$ such that

$$B_0 := \overline{B}_{\mathbb{R}^3}(\tilde{g}, r) \cap \partial\Omega,$$

lies in a connected component of U_0. The boundary ∂B_0 of B_0 can be then parametrized as follows: $\gamma : [-\tau, \tau] \longrightarrow \partial B_0$ for some $\tau > 0$. It is clear that one can easily consider as B_0 a general connected subset of U_0, whose boundary components admit a Lipschitz parametrization. The idea now is to follow the evolution of the set B_0 in the direction of the horizontal normal. Recalling the definition of the horizontal normal N, and the fact that it vanishes at characteristic points of $\partial\Omega$, we do expect that the time of existence depends on the distance d_0 from $\mathrm{char}(\partial\Omega)$.

Proposition 4.1. *There exists $s_0 > 0$ dependent on d_0, such that for any $g_0 \in U_0 \subseteq \partial\Omega$, the Cauchy problem*

$$\begin{cases} \dot{\varphi}(s) = N(\varphi(s)), \\ \varphi(0) = g_0 \in U_0, \end{cases}$$

has a local solution $\varphi_{g_0} : [-s_0, s_0] \to U$ satisfying

$$d_{cc}(g_0, \varphi_{g_0}(\sigma)) = |\sigma| \quad \text{and} \quad \delta_{cc}(\varphi_{g_0}(\sigma)) = \sigma,$$

for each $\sigma \in [-s_0, s_0]$.

Using Proposition 4.1 we can define the localizing set

$$Q := \{\varphi_g(s) : g \in B_0, |s| \leq s_0\},$$

We may think at this Q as a cylinder-type set which is going inside and outside Ω for a height equal to s_0. For technical reasons we also define

$$V_\epsilon := \{p \in V : 0 < \delta_{cc}(p, B_0) < \epsilon\} \quad \text{and} \quad Q_\epsilon := \{p \in Q : 0 < \delta_{cc}(p, B_0) < \epsilon\},$$

where we are obviously assuming that $\epsilon \leq s_0$.

Remark 4.4. *The cc-signed distance function is smooth in the set Q_ϵ, and it is precisely in Q_ϵ where we will use Theorem 1.2. We want also to stress that Proposition 4.1 implies that the cylinder-type set Q_ϵ is foliated by level sets of the cc-signed distance function δ_{cc}. Moreover, on each of these level sets $\nabla_{\mathrm{H}}\delta_{cc}$ is precisely the horizontal normal, and we have $\|\nabla_{\mathrm{H}}\delta_{cc}\|_{\mathrm{H}} = 1$.*

We now focus ourselves only on the set Q_ϵ, and we seek for a power series expansion of its volume. This expression is given by the series (3) stated in the Introduction, and can be considered as our localized Steiner formula in the first Heisenberg group $\mathbb{H}$. To simplify the readablilty of the manuscirpt, let us briefly recall the content of Theorem 1.1. Let $\Omega \subset \mathbb{H}$, $\partial\Omega$, Ω_ϵ and δ_{cc} and be as before. Let Q be the localizing set previously defined. Then,

$$\mathcal{L}^3(\Omega_\epsilon \cap Q) = \mathcal{L}^3(\Omega \cap Q) + \sum_{i \geq 1} \frac{\epsilon^i}{i!} \int_{\partial\Omega \cap Q} (\mathrm{div}_{\mathrm{H}}^{(i-1)} \delta_{cc}) \, d\mathcal{H}^3_{cc}.$$

Let us now spend the remaining pages to briefly describe the strategy of the proof of Theorem 1.1. The first step is to show that the volume function $\epsilon \mapsto \mathcal{L}^3(\Omega_\epsilon \cap Q)$ is actually analytic.

Theorem 4.3. *(*[4, Section 4.2]*) The volume function $\epsilon \mapsto \mathcal{L}^3(\Omega_\epsilon \cap Q)$ is real-analytic on the interval* $[0, s_0]$.

The proof of Theorem 4.3 is just an application of the recursive formula for the *iterated divergences* found in Theorem 1.2.

The second step is to determine the relation between the derivatives of the volume function

$$\epsilon \mapsto \mathcal{L}^3\left(\Omega_\epsilon \cap Q\right)$$

on the interval $[0, s_0)$ and the *iterated divergences* of δ_{cc}, which have been defined in (7). To this aim, let us define the sequence of derivatives $a^{(i)}\colon [0, s_0) \to \mathbb{R}$ by induction:

$$a^{(0)}(\epsilon) := \mathcal{L}^3\left(\Omega_\epsilon \cap Q\right), \quad \text{and} \quad a^{(i+1)}(\epsilon) := \begin{cases} \lim_{s\searrow 0} \frac{a^{(i)}(s)-a^{(i)}(0)}{s} & \epsilon = 0, \\ \lim_{s\to 0} \frac{a^{(i)}(\epsilon+s)-a^{(i)}(\epsilon)}{s} & \epsilon > 0. \end{cases}$$

The content of the next Theorem is that the above sequence is well-defined and can be expressed in terms of the *iterated divergences* of δ_{cc}.

Theorem 4.4. *(*[4, Theorem 3.4]*)For each integer $i \geq 1$ and $\epsilon \in [0, s_0)$, the limit $a^{(i)}(\epsilon)$ exists and is given by*

$$a^{(i)}(\epsilon) = \int_{\delta_{cc}^{-1}(\epsilon)\cap Q} (\mathrm{div}_{\mathrm{H}}^{(i-1)} \delta_{cc})\, d\mathcal{H}^3_{cc}.$$

Finally, we want to spend a few words on another technical issue related to the proof of Theorem 1.1. Without any aim of completeness, for which we refer to [4], we point out that in the proof we also need a slight modification of the following divergence theorem:

Theorem 4.5. *(*[15]*) Let Ω, $\partial\Omega$, ν and ν_H be as before. Let $a, b : \mathbb{H} \to \mathbb{R}$ be smooth real valued functions. Let $aX_1 + bX_2$ be a horizontal vector field. Then*

$$\int_\Omega \mathrm{div}_{\mathrm{H}}(aX_1 + bX_2)\, d\mathcal{L}^3 = \int_{\partial\Omega} \langle aX_1 + bX_2, \nu_H\rangle_{\mathrm{H}}\, d\mathcal{H}^3_{cc}. \tag{10}$$

where $d\mathcal{L}^3$ is the 3-dimensional Lebesgue measure.

To be more precise, we need to describe the boundary of certain sets, that we will call $Q_{s,t}$, related to Q_ϵ. Following [4], for $-s_0 < s < t < s_0$, we define

$$Q_{s,t} := \{g \in Q : s < \delta_{cc}(g) < t\} = \delta_{cc}^{-1}((s,t)) \cap Q,$$

so that $Q_\epsilon = Q_{0,\epsilon}$. We define the *initial boundary*, the *lateral boundary*, and the *final boundary* of $Q_{s,t}$ by

$$\partial_i Q_{s,t} := \{\varphi_g(s), g \in B_0\} = \delta_{cc}^{-1}(s) \cap Q,$$

$$\partial_l Q_{s,t} := \{\varphi_g(\epsilon) : g \in \partial B_0, s < \epsilon < t\},$$

$$\partial_f Q_{s,t} := \{\varphi_g(t), g \in B_0\} = \delta_{cc}^{-1}(t) \cap Q.$$

respectively. As one might expect, it holds that

$$\partial(Q_{s,t}) = \partial_i Q_{s,t} \cup \partial_l Q_{s,t} \cup \partial_f Q_{s,t}. \tag{11}$$

In order to apply the divergence theorem to the sets $Q_{s,t}$, we need to identify the horizontal normal to $\partial(Q_{s,t})$. First define the vector field $\mu\colon \partial(Q_{s,t}) \to \mathbb{R}^3$ by

$$\mu(p) := \begin{cases} -\frac{\nabla\delta_{cc}(p)}{||\nabla\delta_{cc}(p)||_{\mathbb{R}^3}} & p \in \partial_i Q_{s,t}, \\ w(p) & p \in \partial_l Q_{s,t}, \\ \frac{\nabla\delta_{cc}(p)}{||\nabla\delta_{cc}(p)||_{\mathbb{R}^3}} & p \in \partial_f Q_{s,t}, \end{cases}$$

where $w\colon \partial_l Q_{s,t} \to \mathbb{R}^3$ is the Euclidean outward unit normal vector to $\partial(Q_{s,t})$. Then μ is the Euclidean unit outward-pointing normal vector field to $\partial(Q_{s,t})$. Denote its projection onto the horizontal distribution by μ_{H}, so that

$$\mu_{\mathrm{H}}(p) = \begin{cases} -\frac{N(p)}{||\nabla\delta_{cc}(p)||_{\mathbb{R}^3}} & p \in \partial_i Q_{s,t}, \\ w_H(p) & p \in \partial_l Q_{s,t}, \\ \frac{N(p)}{||\nabla\delta_{cc}(p)||_{\mathbb{R}^3}} & p \in \partial_f Q_{s,t}, \end{cases}$$

where w_{H} is the projection of w onto the horizontal distribution.

We are now able to point out the main technical reason that led to the choice of such a precise localizing set Q. By construction we have that on the lateral boundary, the vector w_{H} is perpendicular to the horizontal normal N with respect to the scalar product $\langle\cdot,\cdot\rangle_{\mathrm{H}}$.

Lemma 4.1. *(*[4, Lemma 3.2]*)Let $p \in \partial_l Q_{s,t}$. Then*

$$\langle N(p), w_{\mathrm{H}}(p)\rangle_{\mathrm{H}} = 0.$$

As a consequence, we can adopt an *ad hoc* version of the divergence Theorem previously recalled.

Proposition 4.2. *(*[4, Proposition 3.3]*) Let $c\colon U \to \mathbb{R}$ be a $\mathcal{C}^\infty$-function and let $-s_0 < s < t < s_0$. Then the vector field $cN : U \to \mathbb{R}^3$ satisfies*

$$\int_{Q_{s,t}} \mathrm{div}_{\mathrm{H}}(cN)\ d\mathcal{L}^3 = \int_{\delta_{cc}^{-1}(t)\cap Q} c\, d\mathcal{H}^3_{d_{cc}} - \int_{\delta_{cc}^{-1}(s)\cap Q} c\, d\mathcal{H}^3_{d_{cc}}.$$

We want to stress that the construction of the localizing set Q is deeply connected with the notion of *metric normal* introduced and deeply studied in [1, 2], and turns out to be quite efficient to perform explicit computations (e.g [4, Section 5], [13]).

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