A MEASURE ZERO UDS IN THE HEISENBERG GROUP

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Abstract. We show that the Heisenberg group contains a measure zero set N such that every real-valued Lipschitz function is Pansu differentiable at a point of N.

Sunto. Proveremo che ogni gruppo di Heisenberg contiene un insieme di misura nulla tale che ogni funzione lipschitziana ammette almeno un punto di Pansu differenziabilità al suo interno.

Keywords. Universal differentiability sets, Heisenberg group, Pansu differentiability.

1. Introduction

In this note we describe the main ideas of a recent result obtained by the authors: in [19] we constructed a measure zero 'universal differentiability set' in the Heisenberg group (Theorem 2.9). This result was motivated by seemingly disjoint directions of research extending Rademacher's theorem on the differentiability of Lipschitz functions.

Rademacher's theorem states that every Lipschitz function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is differentiable almost everywhere with respect to Lebesgue measure. This result is classical but has many applications and has inspired much research. One direction of this research is the extension of Rademacher's theorem to more general spaces, while another involves finding points of differentiability in extremely small sets.

A Carnot group is a Lie group whose Lie algebra admits a stratification. This stratification decomposes the Lie algebra as a direct sum of finitely many vector spaces; one of these consists of privileged 'horizontal directions' which generate the other directions using Lie brackets. The Heisenberg group $\mathbb{H}^n$ (Definition 2.1) is the simplest non-Euclidean Carnot group.

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In the celebrated paper [16], Pansu introduced a notion of differentiability modeled on the geometry of Carnot groups, the so-called Pansu differentiability. Remarkably, Pansu proved that Lipschitz functions between Carnot groups are Pansu differentiable almost everywhere with respect to the Haar measure [16]. This extended Rademacher's theorem to Lipschitz maps between Carnot groups.

Differentiability and Rademacher-type results are also studied for functions between Banach spaces. There are versions of Rademacher's theorem for Gâteaux differentiability of Lipschitz functions, but the case of the stronger Fréchet differentiability is not fully understood [13]. Preiss [17] showed that any real-valued Lipschitz function on a Banach space with separable dual is Fréchet differentiable at a dense set of points. Here the main idea was that (almost local) maximality of directional derivatives implies differentiability.

Cheeger [4] gave a generalization of Rademacher's theorem for Lipschitz functions defined on metric spaces equipped with a doubling measure and satisfying a Poincaré inequality. This has inspired much research in the area of analysis on metric measure spaces. Bate [2] showed that Cheeger differentiability is strongly related to existence of many directional derivatives.

A rather different direction of research asks whether one can find points of differentiability in extremely small sets. In particular, we can ask if Rademacher's theorem is sharp: given a set $N \subset \mathbb{R}^n$ of Lebesgue measure zero, does there exist a Lipschitz function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ which is differentiable at no point of N?

If $n \leq m$ the answer is yes: for $n = 1$ this is rather easy [20], while the general case is very difficult and combines ongoing work of multiple authors [1, 6], see also the recent paper [7].

If $n > m$ the answer to our question is no: there are Lebesgue measure zero sets $N \subset \mathbb{R}^n$ such that every Lipschitz function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a point of N. The case $m = 1$ was a surprising corollary of the techniques of the previously mentioned result in Banach spaces by Preiss [17]. The case $m > 1$ were resolved by combining tools from the Banach space theory with a technique for avoiding porous sets [18]. In all cases, maximizing directional derivatives had a crucial role.

Sets $N \subset \mathbb{R}^n$ containing a point of differentiability for every real-valued Lipschitz function are now called universal differentiability sets. The argument in [17] was greatly refined to show that $\mathbb{R}^n$, $n > 1$, contains universal differentiability sets which are compact and of Hausdorff dimension one [8, 9]. This was improved to obtain a set which even has Minkowski dimension one [10].

In the present note we show the main ideas contained in [19] and in particular how to adapt [17] to the Heisenberg group. The interested reader can find the proofs of all the results mentioned in the present note in [19]. Our main result is Theorem 2.9 which asserts the following: there is a Lebesgue measure zero set $N \subset \mathbb{H}^n$ such that every Lipschitz function $f\colon \mathbb{H}^n \to \mathbb{R}$ is Pansu differentiable at a point of N.

2. The Heisenberg Group and Pansu Differentiability

More information on the topics in this section can be found in [3, 5, 12, 15]. Denote the Euclidean norm and inner product by $|\cdot|$ and $\langle \cdot, \cdot \rangle$ respectively. We represent points of $\mathbb{R}^{2n+1}$ as triples (a, b, c), where $a, b \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

Definition 2.1. The *Heisenberg group* $\mathbb{H}^n$ is $\mathbb{R}^{2n+1}$ equipped with the non-commutative group law:

$$(a, b, c)(a', b', c') = (a + a',\ b + b',\ c + c' - 2(\langle a, b' \rangle - \langle b, a' \rangle)).$$

The identity element in $\mathbb{H}^n$ is 0 and inverses are given by $x^{-1} = -x$.

Definition 2.2. For $r > 0$ define *dilations* $\delta_r\colon \mathbb{H}^n \to \mathbb{H}^n$ by:

$$\delta_r(a, b, c) = (ra, rb, r^2 c).$$

Dilations $\delta_r\colon \mathbb{H}^n \to \mathbb{H}^n$ and the projection $p\colon \mathbb{H}^n \to \mathbb{R}^{2n}$ onto the first $2n$ coordinates are group homomorphisms, where $\mathbb{R}^{2n}$ is considered as a group with the operation of addition.

As sets there is no difference between $\mathbb{H}^n$ and $\mathbb{R}^{2n+1}$. Nevertheless, we sometimes think of elements of $\mathbb{H}^n$ as points and elements of $\mathbb{R}^{2n+1}$ as vectors. Let e_i denote the standard basis vectors of $\mathbb{R}^{2n+1}$ for $1 \le i \le 2n+1$. That is, e_i has all coordinates equal to 0 except for a 1 in the i'th coordinate.

Definition 2.3. For $1 \leq i \leq n$ define vector fields on $\mathbb{H}^n$ by:

$$X_i(a,b,c) = e_i + 2b_i e_{2n+1}, \quad Y_i(a,b,c) = e_{i+n} - 2a_i e_{2n+1}.$$

Let $V = \mathrm{Span}\{X_i, Y_i : 1 \leq i \leq n\}$ and ω be the inner product norm on V making $\{X_i, Y_i : 1 \leq i \leq n\}$ an orthonormal basis. We say that the elements of V are *horizontal vector fields* or *horizontal directions*.

An easy calculation shows that if $E \in V$ then

$$x(tE(0)) = x + tE(x)$$

for any $x \in \mathbb{H}^n$ and $t \in \mathbb{R}$. That is, 'horizontal lines' are preserved by group translations. If $E \in V$ then $E(0)$ is a vector $v \in \mathbb{R}^{2n+1}$ with $v_{2n+1} = 0$. Conversely, for any such v there exists $E \in V$ such that $E(0) = v$. If $E \in V$ then $p(E(x))$ is independent of x, so we can unambiguously define $p(E) \in \mathbb{R}^{2n}$. The norm ω is then equivalently given by $\omega(E) = |p(E)|$.

We now use the horizontal directions to define horizontal curves and horizontal length in $\mathbb{H}^n$. Let I denote a subinterval of $\mathbb{R}$.

Definition 2.4. An absolutely continuous curve $\gamma \colon I \to \mathbb{H}^n$ is a *horizontal curve* if there is $h \colon I \to \mathbb{R}^{2n}$ such that for almost every $t \in I$:

$$\gamma'(t) = \sum_{i=1}^{n} (h_i(t) X_i(\gamma(t)) + h_{n+i}(t) Y_i(\gamma(t))).$$

Define the *horizontal length* of such a curve by:

$$L_{\mathbb{H}}(\gamma) = \int_I |h|.$$

Notice that in Definition 2.4 we have $|(p \circ \gamma)'(t)| = |h(t)|$ for almost every t, so $L_{\mathbb{H}}(\gamma)$ is computed by integrating $|(p \circ \gamma)'(t)|$. That is, $L_{\mathbb{H}}(\gamma) = L_{\mathbb{E}}(p \circ \gamma)$, where $L_{\mathbb{E}}$ is the Euclidean length of a curve in Euclidean space. It can be shown that left group translations preserve horizontal lengths of horizontal curves.

By Chow's Theorem, any two points of $\mathbb{H}^n$ can be joined by a horizontal curve of finite horizontal length. We use this fact to define the Carnot-Carathéodory distance.

Definition 2.5. Define the *Carnot-Carathéodory distance* d on $\mathbb{H}^n$ by:

$$d(x,y) = \inf\{L_{\mathbb{H}}(\gamma)\colon \gamma \text{ is a horizontal curve joining } x \text{ to } y\}.$$

Denote $d(x) = d(x,0)$ and $B_{\mathbb{H}}(x,r) := \{y \in \mathbb{H}^n\colon d(x,y) < r\}$.

It is known that geodesics exist in the Heisenberg group. That is, the infimum in Definition 2.5 is actually a minimum. The Carnot-Carathéodory distance respects the group law and dilations - for every $g,x,y \in \mathbb{H}^n$ and $r > 0$:

- $d(gx,gy) = d(x,y)$,
- $d(\delta_r(x), \delta_r(y)) = rd(x,y)$.

Notice $d(x,y) \geq |p(y) - p(x)|$, since the projection of a horizontal curve joining x to y is a curve in $\mathbb{R}^{2n}$ joining $p(x)$ to $p(y)$.

The Carnot-Carathéodory distance and the Euclidean distance are topologically equivalent but not Lipschitz equivalent. However, they are Hölder equivalent on compact sets.

If $f\colon \mathbb{H}^n \to \mathbb{R}$ or $\gamma\colon \mathbb{R} \to \mathbb{H}^n$ we denote the Lipschitz constant (not necessarily finite) of f or γ with respect to d (in the domain or target respectively) by $\mathrm{Lip}_{\mathbb{H}}(f)$ and $\mathrm{Lip}_{\mathbb{H}}(\gamma)$. If we use the Euclidean distance then we use the notation $\mathrm{Lip}_{\mathbb{E}}(f)$ and $\mathrm{Lip}_{\mathbb{E}}(\gamma)$. Throughout this note 'Lipschitz' means with respect to the Carnot-Carathéodory distance if the domain or target is $\mathbb{H}^n$, unless otherwise stated. For horizontal curves we have the following relation between Lipschitz constants.

Lemma 2.6. *Suppose $\gamma\colon I \to \mathbb{H}^n$ is a horizontal curve. Then:*

$$\mathrm{Lip}_{\mathbb{H}}(\gamma) = \mathrm{Lip}_{\mathbb{E}}(p \circ \gamma).$$

Lebesgue measure $\mathcal{L}^{2n+1}$ is the natural Haar measure on $\mathbb{H}^n$. It is compatible with group translations and dilations - for every $g \in \mathbb{H}^n$, $r > 0$ and $A \subset \mathbb{H}^n$:

- $\mathcal{L}^{2n+1}(\{gx\colon x \in A\}) = \mathcal{L}^{2n+1}(A)$,
- $\mathcal{L}^{2n+1}(\delta_r(A)) = r^{2n+2}\mathcal{L}^{2n+1}(A)$.

Definition 2.7. A function $L\colon \mathbb{H}^n \to \mathbb{R}$ is $\mathbb{H}$-linear if $L(xy) = L(x)+L(y)$ and $L(\delta_r(x)) = rL(x)$ for all $x,y \in \mathbb{H}^n$ and $r > 0$.

Let $f\colon \mathbb{H}^n \to \mathbb{R}$ and $x \in \mathbb{H}^n$. We say that f is *Pansu differentiable* at x if there is a $\mathbb{H}$-linear map $L\colon \mathbb{H}^n \to \mathbb{R}$ such that:

$$\lim_{y \to x} \frac{|f(y) - f(x) - L(x^{-1}y)|}{d(x,y)} = 0.$$

In this case we say that L is the *Pansu derivative* of f.

Clearly a $\mathbb{H}$-linear map is Pansu differentiable at every point. Pansu's theorem is the natural version of Rademacher's theorem in $\mathbb{H}^n$.

Theorem 2.8 (Pansu)**.** *A Lipschitz function $f\colon \mathbb{H}^n \to \mathbb{R}$ is Pansu differentiable Lebesgue almost everywhere.*

We can now state our main result.

Theorem 2.9. *There is a Lebesgue measure zero set $N \subset \mathbb{H}^n$ such that every Lipschitz function $f\colon \mathbb{H}^n \to \mathbb{R}$ is Pansu differentiable at a point of N.*

The set N in Theorem 2.9 is called a *universal differentiability set*. Similar results are known for Lipschitz maps $f\colon \mathbb{R}^n \to \mathbb{R}^m$ with $n > m$, but Theorem 2.9 is the first such result outside the Euclidean setting.

3. Maximality of directional derivatives implies Pansu differentiability

We now define directional derivatives in horizontal directions (Definition 3.1) and compare them to the Lipschitz constant (Lemma 3.2). We discuss how the origin can be joined to other points by relatively simple curves (Lemma 3.3). These are the tools used to prove (Theorem 3.4), which states that existence of a maximal horizontal directional derivative implies Pansu differentiability.

Definition 3.1. Let $f\colon \mathbb{H}^n \to \mathbb{R}$ be a Lipschitz function and $E \in V$. Define

$$Ef(x) := \lim_{t \to 0} \frac{f(x + tE(x)) - f(x)}{t}.$$

whenever it exists.

If $E \in V$ then the horizontal line $\gamma(t) = x + tE(x)$ is a Lipschitz map from $\mathbb{R}$ into $\mathbb{H}^n$ and the composition $f \circ \gamma \colon \mathbb{R} \to \mathbb{R}$ is Lipschitz, so differentiable almost everywhere. Hence Lipschitz functions have many directional derivatives in horizontal directions.

Directional derivatives give precise information about what happens on small scales in some direction, while the Lipschitz condition gives information about arbitrary directions on large scales. It is useful to have some connection between the two. For this, we recall that the Lipschitz constant of a function between Euclidean spaces is given by the supremum of directional derivatives over directions of Euclidean length 1. We now give a similar statement for the Carnot-Carathéodory distance.

Lemma 3.2. *Suppose $f \colon \mathbb{H}^n \to \mathbb{R}$ is Lipschitz. Then:*

$$\mathrm{Lip}_{\mathbb{H}}(f) = \sup\{|Ef(x)| \colon x \in \mathbb{H}^n,\ E \in V,\ \omega(E) = 1,\ Ef(x) \text{ exists}\}.$$

Lemma 3.2 is easy to prove using the Fundamental Theorem of Calculus along curves.

Lemma 3.3. *Let $a, b \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Suppose $(a, b) \neq (0, 0)$ and let $L = |(a, b)|$. Then there is a Lipschitz horizontal curve γ joining $(0, 0, 0) \in \mathbb{H}^n$ to $(a, b, c) \in \mathbb{H}^n$, which is a concatenation of two line segments, such that:*

(1) $\mathrm{Lip}_{\mathbb{H}}(\gamma) \leq L\left(1 + \frac{c^2}{L^4} + \frac{4c^2}{L^2}\right)^{\frac{1}{2}}$,

(2) *$\gamma'(t)$ exists and $|\gamma'(t) - (a, b, 0)| \leq \frac{c}{L}(1 + 4L^2)^{\frac{1}{2}}$ for $t \in [0, 1] \setminus \{1/2\}$.*

To prove Lemma 3.3 one uses the characterization of horizontal curves in $\mathbb{H}^n$ as lifts of curves in $\mathbb{R}^{2n}$ [3]. The height of the final coordinate is determined by various areas described by the curve in $\mathbb{R}^{2n}$. For the curve in Lemma 3.3, using two line segments in $\mathbb{R}^{2n}$ before lifting to $\mathbb{H}^n$ allows one to specify these areas and hence reach the correct vertical coordinate.

The following Theorem is an adaptation to the Heisenberg group of [11, Theorem 2.4]. By Lemma 3.2, existence of a maximal horizontal directional derivative is equivalent to the agreement of a directional derivative with the Lipschitz constant.

Theorem 3.4. *Let $f \colon \mathbb{H}^n \to \mathbb{R}$ be Lipschitz, $x \in \mathbb{H}^n$ and $E \in V$ with $\omega(E) = 1$. Suppose $Ef(x)$ exists and $Ef(x) = \mathrm{Lip}_{\mathbb{H}}(f)$. Then f is Pansu differentiable at x with derivative $L(x) := \mathrm{Lip}_{\mathbb{H}}(f)\langle x, E(0)\rangle = \mathrm{Lip}_{\mathbb{H}}(f)\langle p(x), p(E)\rangle$.*

While the proof of Theorem 3.4 in [19] is really direct, the rough idea is as follows. If f is not Pansu differentiable at x, then there is a nearby point y such that $f(y)-f(x)$ is too large. We then consider a curve joining a point $x-tE(x)$ to y which is a group translation of the curves constructed in Lemma 3.3. If t is relatively large, the length of this curve is very close to t. Hence the Lipschitz constant gives an upper bound on $f(y)-f(x-tE(x))$. However, $f(x)-f(x-tE(x))$ is large because $Ef(x)=\mathrm{Lip}_{\mathbb{H}}(f)$ is a maximal directional derivative and $f(y)-f(x)$ is large due to the failure of differentiability. This gives a contradiction.

4. The universal differentiability set and almost maximality implies Pansu differentiability

A general Lipschitz function may not have a maximal directional derivative in the sense of Theorem 3.4, especially inside a null set. In this section we identify our measure zero universal differentiability set (Lemma 4.1) and discuss almost maximal directional derivatives which are enough to prove Pansu differentiability (Theorem 4.3). The argument is based on that of [17, Theorem 4.1], but using horizontal curves and directional derivatives in horizontal directions.

If $y\in\mathbb{H}^n$ with $p(y)\neq 0$, let γ_y be the curve constructed in Lemma 3.3 joining 0 to y. Recall that a set in a topological space is G_δ if it is a countable intersection of open sets.

Lemma 4.1. *There is a Lebesgue measure zero G_δ set $N\subset\mathbb{H}^n$ containing all straight lines which are also horizontal curves and join pairs of points of $\mathbb{Q}^{2n+1}$. Any such set contains the image of:*

(1) *the line $x+tE(x)$ whenever $x\in\mathbb{Q}^{2n+1}$ and $E\in V$ is a linear combination of $\{X_i,Y_i:1\leq i\leq n\}$ with rational coefficients,*
(2) *all curves of the form $x\gamma_y$ for $x,y\in\mathbb{Q}^{2n+1}$ with $p(y)\neq 0$.*

Notation 4.2. Fix a Lebesgue null G_δ set $N\subset\mathbb{H}^n$ as in Lemma 4.1 for the remainder of the note. For any Lipschitz function $f:\mathbb{H}^n\to\mathbb{R}$ define:

$$D^f:=\{(x,E)\in N\times V:\omega(E)=1,\ Ef(x)\text{ exists}\}.$$

We now make precise the idea that almost maximality suffices for differentiability. For true maximality one should compare a fixed directional derivative with all directional derivatives using pairs in D^f. Instead, we use a subcollection of pairs with the property that changes in slopes are bounded by changes in directional derivatives.

Theorem 4.3. *Let* $f\colon \mathbb{H}^n \to \mathbb{R}$ *be a Lipschitz function with* $\mathrm{Lip}_{\mathbb{H}}(f) \leq 1/2$*. Suppose* $(x_*, E_*) \in D^f$*. Let* M *denote the set of pairs* $(x, E) \in D^f$ *such that* $Ef(x) \geq E_*f(x_*)$ *and*

$$|(f(x + tE_*(x)) - f(x)) - (f(x_* + tE_*(x_*)) - f(x_*))| \leq 6|t|((Ef(x) - E_*f(x_*))\mathrm{Lip}_{\mathbb{H}}(f))^{\frac{1}{4}}$$

for every $t \in (-1, 1)$*. If*

$$\lim_{\delta \downarrow 0} \sup\{Ef(x) \colon (x, E) \in M \text{ and } d(x, x_*) \leq \delta\} \leq E_*f(x_*)$$

then f *is Pansu differentiable at* x_* *with Pansu derivative*

$$L(x) = E_*f(x_*)\langle x, E_*(0)\rangle = E_*f(x_*)\langle p(x), p(E_*)\rangle.$$

The proof of Theorem 4.3 is by contradiction. The idea is to first modify the line $x_* + tE_*(x_*)$ to form a Lipschitz horizontal curve g in N which passes through a nearby point showing non-Pansu differentiability at x_*. Then, by applying a suitable mean value type theorem (see [17, Lemma 3.4]) it is possible to obtain a large directional derivative along g and estimates for difference quotients in the new direction. Finally, these estimates have to be improved to show that the new point and direction form a pair in M. This shows that there is a nearby pair in M giving a larger directional derivative than (x_*, E_*), a contradiction.

5. Construction of an almost maximal directional derivative and Proof of Theorem 2.9

We now state Theorem 5.1 and we prove Theorem 2.9. Theorem 5.1 shows that given a Lipschitz function $f_0\colon \mathbb{H}^n \to \mathbb{R}$, there is a Lipschitz function $f\colon \mathbb{H}^n \to \mathbb{R}$ such that $f - f_0$ is $\mathbb{H}$-linear and f has an almost locally maximal horizontal directional derivative in the sense of Theorem 4.3. We will conclude that any Lipschitz function f_0 is Pansu differentiable at a point of N, proving Theorem 2.9.

Recall the measure zero G_δ set N and the notation D^f fixed in Notation 4.2. In particular, the statement $(x, E) \in D^f$ implies that $x \in N$. Note that if $f - f_0$ is $\mathbb{H}$-linear then $D^f = D^{f_0}$ and also the functions f and f_0 have the same points of Pansu differentiability.

Theorem 5.1. *Suppose $f_0 : \mathbb{H}^n \to \mathbb{R}$ is a Lipschitz function, $(x_0, E_0) \in D^{f_0}$ and $\delta_0, \mu, K > 0$. Then there is a Lipschitz function $f : \mathbb{H}^n \to \mathbb{R}$ such that $f - f_0$ is $\mathbb{H}$-linear with $\mathrm{Lip}_{\mathbb{H}}(f - f_0) \leq \mu$, and a pair $(x_*, E_*) \in D^f$ with $d(x_*, x_0) \leq \delta_0$ such that $E_*f(x_*) > 0$ is almost locally maximal in the following sense.*

For any $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ such that whenever $(x, E) \in D^f$ satisfies both:

(1) $d(x, x_*) \leq \delta_\varepsilon$, $Ef(x) \geq E_*f(x_*)$,
(2) *for any $t \in (-1, 1)$:*

$$\begin{aligned} &|(f(x + tE_*(x)) - f(x)) - (f(x_* + tE_*(x_*)) - f(x_*))| \\ &\qquad \leq K|t|(Ef(x) - E_*f(x_*))^{\frac{1}{4}}, \end{aligned}$$

then:

$$Ef(x) < E_*f(x_*) + \varepsilon.$$

Proof of Theorem 2.9. Let $f_0 \colon \mathbb{H}^n \to \mathbb{R}$ be a Lipschitz function. Multiplying f_0 by a non-zero constant does not change the set of points where it is Pansu differentiable. Hence we can assume $\mathrm{Lip}_{\mathbb{H}}(f_0) \leq 1/4$. Fix an arbitrary pair $(x_0, E_0) \in D^{f_0}$.

Apply Theorem 5.1 with $\delta_0 = 1$, $\mu = 1/4$ and $K = 8$. This gives a Lipschitz function $f \colon \mathbb{H}^n \to \mathbb{R}$ such that $f - f_0$ is $\mathbb{H}$-linear with $\mathrm{Lip}_{\mathbb{H}}(f - f_0) \leq 1/4$ and a pair $(x_*, E_*) \in D^f$, in particular $x_* \in N$, such that $E_*f(x_*) > 0$ is almost locally maximal in the following sense.

For any $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ such that whenever $(x, E) \in D^f$ satisfies both:

(1) $d(x, x_*) \leq \delta_\varepsilon$, $Ef(x) \geq E_*f(x_*)$,

(2) for any $t \in (-1, 1)$:

$$|(f(x + tE_*(x)) - f(x)) - (f(x_* + tE_*(x_*)) - f(x_*))|$$
$$\leq 8|t|(Ef(x) - E_*f(x_*))^{\frac{1}{4}},$$

then:

$$Ef(x) < E_*f(x_*) + \varepsilon.$$

Combining $\mathrm{Lip}_{\mathbb{H}}(f_0) \leq 1/4$ and $\mathrm{Lip}_{\mathbb{H}}(f - f_0) \leq 1/4$ gives $\mathrm{Lip}_{\mathbb{H}}(f) \leq 1/2$. Notice that (x_*, E_*) is also almost locally maximal in the sense of Theorem 4.3, since the restriction on pairs above is weaker than that in Theorem 4.3. Hence Theorem 4.3 implies that f is Pansu differentiable at $x_* \in N$. A $\mathbb{H}$-linear function is Pansu differentiable everywhere. Consequently f_0 is Pansu differentiable at x_*, proving Theorem 2.9. □

References

[1] Alberti, G., Csornyei, M., Preiss, D.: *Differentiability of Lipschitz functions, structure of null sets, and other problems*, Proc. Int. Congress Math. III (2010), 1379–1394.

[2] Bate, D.: *Structure of measures in Lipschitz differentiability spaces*, J. Amer. Math. Soc. 28 (2015), 421–482.

[3] Bonfiglioli, A., Lanconelli, E., Uguzzoni, F.: *Stratified Lie Groups and Potential Theory for Their Sub-Laplacians*, Springer Monographs in Mathematics (2007).

[4] Cheeger, J.: *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Funct. Anal. 9(3) (1999), 428–517.

[5] Capogna, L., Danielli, D., Pauls, S., Tyson, J.: *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*, Birkhauser Progress in Mathematics 259 (2007).

[6] Csornyei, M., Jones, P.: *Product formulas for measures and applications to analysis and geometry*, announced result: http://www.math.sunysb.edu/Videos/dfest/PDFs/38-Jones.pdf.

[7] De Philippis, G., Rindler, F.: *On the structure of $\mathcal{A}$-free measures and applications*, Annals of Math. ArXiv: 1601.06543.

[8] Doré, M., Maleva, O.: *A compact null set containing a differentiability point of every Lipschitz function*, Math. Ann. 351(3) (2011), 633–663.

[9] Doré, M., Maleva, O.: *A compact universal differentiability set with Hausdorff dimension one*, Israel J. Math. 191(2) (2012), 889–900.

[10] Dymond, M., Maleva, O.: *Differentiability inside sets with upper Minkowski dimension one*, preprint available at arXiv:1305.3154.

[11] Fitzpatrick, S.: *Differentiation of real-valued functions and continuity of metric projections*, Proc. Amer. Math. Soc. 91(4) (1984), 544–548.

[12] Gromov, M.: *Carnot-Carathéodory spaces seen from within*, Progress in Mathematics 144 (1996), 79–323.

[13] Lindenstrauss, J., Preiss, D., Tiser, J.: *Fréchet differentiability of Lipschitz functions and porous sets in Banach spaces*, Annals of Mathematics Studies 179, Princeton University Press (2012).

[14] Magnani, V.: *Towards differential calculus in stratified groups*, J. Aust. Math. Soc. 95(1) (2013), 76–128.

[15] Montgomery, R.: *A tour of subriemannian geometries, their geodesics and applications*, American Mathematical Society, Mathematical Surveys and Monographs 91 (2006).

[16] Pansu, P.: *Metriques de Carnot-Carathéodory et quasiisometries des espaces symetriques de rang un*, Annals of Mathematics 129(1) (1989), 1–60.

[17] Preiss, D.: *Differentiability of Lipschitz functions on Banach spaces*, J. Funct. Anal. 91(2) (1990), 312–345.

[18] Preiss, D., Speight, G.: *Differentiability of Lipschitz functions in Lebesgue null sets*, Inven. Math. 199(2) (2015), 517–559.

[19] Pinamonti, A., Speight, G.: *A measure zero universal differentiability set in the Heisenberg group*, to appear in Mathematische Annalen, DOI: 10.1007/s00208-016-1434-x, preprint available at arXiv:1505.07986.

[20] Zahorski, Z.: *Sur l'ensemble des points de non-derivabilite d'une fonction continue*, Bull. Soc. Math. France 74 (1946), 147–178.

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