AN EIGENVALUE PROBLEM FOR NONLOCAL EQUATIONS

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Abstract. In this paper we study the existence of a positive weak solution for a class of nonlocal equations under Dirichlet boundary conditions and involving the regional fractional Laplacian operator, given by

$$(-\Delta)^s u(x) := -\int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy\,, \quad x\in\mathbb{R}^n\,,$$

where $s\in(0,1)$ is fixed and $n>2s$. More precisely, exploiting direct variational methods, we prove a characterization theorem on the existence of one weak solution for the nonlocal elliptic problem

$$\begin{cases} (-\Delta)^s u = \lambda f(u) & \text{in } \Omega \\ u>0 & \text{in } \Omega \\ u=0 & \text{in } \mathbb{R}^n\setminus\Omega, \end{cases}$$

where the nonlinear term f is a suitable continuous function and $\Omega\subset\mathbb{R}^n$ is open, bounded and with smooth boundary $\partial\Omega$. Our result extends to the fractional setting some theorems obtained recently for ordinary and classical elliptic equations, as well as some characterization properties proved for differential problems involving different elliptic operators. With respect to these cases studied in literature, the nonlocal one considered here presents some additional difficulties, so that a careful analysis of the fractional spaces involved is necessary, as well as some nonlocal L^q-estimates, recently proved in the nonlocal framework.

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1. Introduction

One of the most celebrated applications of critical point theory (see [1, 23, 24, 34, 35]) consists in the construction of nontrivial solutions of semilinear equations: in this context, the solutions are constructed with a variational method by a minimax procedure on the associated energy functional.

There is a huge literature on these classical topics, and, in recent years, also a lot of papers related to the study of fractional and nonlocal operators of elliptic type, through critical point theory, appeared. Indeed, a natural question is whether or not these techniques may be adapted in order to investigate the fractional analogue of the classical elliptic case. The answer is yes, even if the variational approach has to be adapted to the nonlocal setting. For this we refer to the recent book [15], which is dedicated to the analysis of fractional elliptic problems, via classical variational methods and other novel approaches.

The interest shown in the literature for nonlocal operators and problems is due both for the pure mathematical research and to their applications in a wide range of contexts, such as the thin obstacle problem, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes,

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flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science, water waves, just to name a few.

In this paper we focus our attention on fractional nonlocal problems studied through variational and topological methods. To be more precise, we consider the following nonlocal eigenvalue problem

$$\begin{cases} (-\Delta)^s u = \lambda f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.1}$$

where $s \in (0,1)$ is a fixed parameter, $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $n > 2s$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying suitable regularity and growth conditions, and λ denotes a positive real parameter. Here $(-\Delta)^s$ is the fractional Laplace operator defined, up to a normalization factor, by the Riesz potential as

$$(-\Delta)^s u(x) := -\int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\, dy\,, \quad x \in \mathbb{R}^n\,, \tag{1.2}$$

see [10, 15] for further details. Note that in (1.1) the homogeneous Dirichlet datum is given in $\mathbb{R}^n \setminus \Omega$ and not simply on $\partial\Omega$, as it happens in the classical case of the Laplacian, consistently with the nonlocal character of the operator $(-\Delta)^s$.

Alternatively, following the work of Caffarelli and Silvestre [9], the fractional Laplacian operator in the whole space $\mathbb{R}^n$ can be defined as a Dirichlet to a Neumann map:

$$(-\Delta)^s u(x) := -\kappa_s \lim_{y \to 0^+} y^{1-2s} \frac{\partial w}{\partial y}(x,y),$$

where κ_s is a suitable constant and w is the s-harmonic extension of a smooth function u. In other words, w is the function defined on the upper half-space $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,+\infty)$ which is solution to the local elliptic problem

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla w) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ w(x,0) = u(x) & \text{in } \mathbb{R}^n. \end{cases}$$

In order to define the fractional Laplacian operator in bounded domains, the above procedure has been adapted in [6, 8]. We point out that two notions of fractional operators on bounded domains were considered in the literature, namely the one considered in [6, 8] (called also *spectral* Laplacian operator) and the *integral* one given in (1.2). In [30, Theorem 1] the authors compare these two operators by studying their spectral properties obtaining, as consequence of this careful analysis, that these two operators are different. We refer also to [12] for an exhaustive study of this comparison.

In the framework of the the spectral Laplacian, the problem considered here has been treated in [14] by using the extension method and the Dirichlet to a Neumann map. With respect to this case, problem (1.1) presents some additional technical difficulties and to make the nonlinear methods work, some careful analysis of the fractional spaces involved is necessary.

Problem (1.1) has a variational structure and the natural space where finding solutions for it is a closed linear subspace of the classical fractional Sobolev space $H^s(\mathbb{R}^n)$. Indeed, in order to give the weak formulation of problem (1.1), we need to work in a special functional space, which allows us to encode the Dirichlet boundary condition in the variational formulation. We would note that, with this respect, the standard fractional Sobolev spaces are not enough in order to study the problem, and so, we overcome this difficulty working in a new functional space, whose definition will be recalled in Section 2.

1.1. **Main results of the paper.** Throughout this paper we suppose that $f : [0,+\infty) \to \mathbb{R}$ is a continuous function satisfying the following sign condition

$$f(0) = 0 \textit{ and } f(\xi) \geq 0 \textit{ for any } \xi \in (0,+\infty). \tag{1.3}$$

Furthermore, let $h : (0, +\infty) \to [0, +\infty)$ be the map defined by

$$h(\xi) := \frac{F(\xi)}{\xi^2},$$

where

$$F(\xi) := \int_0^\xi f(t)dt,$$

for each $\xi \in [0, +\infty)$. We will suppose that

(1.4) *there exists $a > 0$ such that h is non-increasing in the interval $(0, a]$.*

Note that assumptions (1.3) and (1.4) are the natural ones, when dealing with elliptic differential equations driven by the Laplace operator (or, more generally, by uniformly elliptic operators) with homogeneous Dirichlet boundary conditions (see, for instance, [2, 13, 25]).

In the sequel saying that f is a *subcritical* function means that

$$|f(\xi)| \leq C(1 + |\xi|^\nu)$$

for some $1 \leq \nu \leq 2_s^* - 1$ and $C > 0$, where $2_s^* := 2n/(n - 2s)$ denotes the *fractional critical Sobolev exponent.*

The main result of the present paper gives a necessary and sufficient condition for the existence of solutions for problem (1.1), as stated here below (here by $\lambda_{1,s}$ we denote the first eigenvalue of $(-\Delta)^s$ with homogeneous Dirichlet boundary data, see Subsection 2.2):

Theorem 1. *Let $s \in (0, 1)$, $n > 2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. Further, let $f : [0, +\infty) \to \mathbb{R}$ be a function satisfying hypotheses* (1.3) *and* (1.4)*. Then, the following assertions are equivalent*:

(h_1) *h is not constant in $(0, b]$ for any $b > 0$;*

(h_2) *f is subcritical with $\lim\limits_{\xi \to 0^+} h(\xi) > 0$ and for each $r > 0$ there exists $\varepsilon_r > 0$ such that for every*

$$\lambda \in \left(\frac{\lambda_{1,s}}{2 \lim\limits_{\xi \to 0^+} h(\xi)}, \frac{\lambda_{1,s}}{2 \lim\limits_{\xi \to 0^+} h(\xi)} + \varepsilon_r \right),$$

the nonlocal problem (1.1) *has a weak solution $u_\lambda \in H^s(\mathbb{R}^n)$ such that $u_\lambda = 0$ in $\mathbb{R} \setminus \Omega$, and*

$$\left(\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u_\lambda(x) - u_\lambda(y)|^2}{|x - y|^{n+2s}} \, dxdy \right)^{1/2} < r. \tag{1.5}$$

Note that if $\lim\limits_{\xi \to 0^+} h(\xi) = +\infty$, condition ($\mathrm{h}_2$) assumes the simple form

(h_2') *f is subcritical and for each $r > 0$ there exists $\varepsilon_r > 0$ such that for every $\lambda \in (0, \varepsilon_r)$, the nonlocal problem* (1.1) *has a weak solution $u_\lambda \in H^s(\mathbb{R}^n)$ such that $u_\lambda = 0$ in $\mathbb{R} \setminus \Omega$, and* (1.5) *holds true.*

Theorem 1 is inspired by [25, Theorem 1], which is related to a two-point boundary value problem in the one-dimensional case. In the higher dimensional setting the method performed in [25] cannot be directly used for treating elliptic equations driven by the classical Laplacian operator. In such a case a different proof, again based on critical point methods, was developed in [2, Theorem 1]. The extension of the cited result to nonlocal spectral equations has been developed in [14]. We also notice that in [13] a similar variational approach was adopted in order to study elliptic equations involving the weak Laplacian operator defined on self-similar fractal domains, whose simple prototype is the Sierpiński gasket.

The proof of Theorem 1 is based on variational and topological techniques. Moreover, one of the main tool used along the proof is a regularity result for the first eigenfunction associated to a linear fractional problem (see [30, Theorem 1]), as well as an a-priori estimate for solutions of nonlocal equations in terms of the data, recently obtained in [20, Lemma 2.3], which extends the well known ones for the standard Laplacian case (see Proposition 3 and Lemma 2 in Section 2). We make also use of the Strong Maximum Principle for the fractional Laplacian operator, proved in [33, Proposition 2.2.8], and the properties of the spectrum $\sigma((-\Delta)^s)$ of the fractional Laplace operator obtained in [11, Proposition 2.1].

Our result extends to the nonlocal setting recent theorems got in the setting of ordinary and classical elliptic equations, as well as a characterization for elliptic problems on certain non-smooth domains (see the papers [2, 13, 14, 25]). In particular we notice that Theorem 1 can be viewed as the fractional counterpart of the classical elliptic case proved in [2, Theorem 1] by means of variational and topological methods.

For the sake of completeness we mention that in the literature there are many existence, non-existence and multiplicity results about nonlocal problems involving fractional Laplacian operators obtained with different methods and approaches (see, among others, the papers [3, 4, 16, 17, 18] and [19, 21, 22, 36], as well as the references therein).

The present paper is organized as follows. In Section 2 we collect some properties of the fractional Laplacian operator in a bounded domain, as well as we give some basic notions on the fractional Sobolev framework adopted here. Section 3 is devoted to the proof of Theorem 1. Finally, in Section 4 an example of an application is presented, together with some final comments.

2. Preliminaries

In this section we briefly recall some results useful along the paper. As we said in the Introduction, in order to give a variational formulation of problem (1.1), we need to consider a suitable functional space. Here we start by recalling the definition of the fractional functional space X^s_0, firstly introduced in [27, 28]. The reader familiar with this topic may skip this section and go directly to the next one.

The functional space X^s denotes the linear space of Lebesgue measurable functions g from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction of g to Ω belongs to $L^2(\Omega)$ and

$$(x,y)\mapsto \frac{g(x)-g(y)}{|x-y|^{\frac{n+2s}{2}}}\in L^2\big((\mathbb{R}^n\times\mathbb{R}^n)\setminus(\mathcal{C}\Omega\times\mathcal{C}\Omega),dxdy\big),$$

where $\mathcal{C}\Omega:=\mathbb{R}^n\setminus\Omega$. We denote by X^s_0 the following linear subspace of X^s

$$X^s_0:=\big\{g\in X^s : g=0 \text{ a.e. in } \mathbb{R}^n\setminus\Omega\big\}.$$

We remark that X^s and X^s_0 are non-empty, since $C^2_0(\Omega)\subseteq X^s_0$ by [27, Lemma 5.1]. Moreover, the space X^s is endowed with the norm defined as

$$\|g\|_{X^s}:=\|g\|_{L^2(\Omega)}+\Big(\int_Q\frac{|g(x)-g(y)|^2}{|x-y|^{n+2s}}dxdy\Big)^{1/2},$$

where $Q:=(\mathbb{R}^n\times\mathbb{R}^n)\setminus(\mathcal{C}\Omega\times\mathcal{C}\Omega)$. It is easily seen that $\|\cdot\|_{X^s}$ is a norm on X^s, see [28].

By [28, Lemma 6 and Lemma 7] in the sequel we can take the function

$$X^s_0\ni v\mapsto \|v\|_{X^s_0}:=\left(\int_Q\frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}dxdy\right)^{1/2} \tag{2.1}$$

as a norm on X^s_0. Also, $\big(X^s_0,\|\cdot\|_{X^s_0}\big)$ is a Hilbert space with scalar product

$$\langle u,v\rangle_{X^s_0}:=\int_Q\frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{n+2s}}dxdy,$$

see [28, Lemma 7]. Note that in (2.1) (and in the related scalar product) the integral can be extended to all $\mathbb{R}^n\times\mathbb{R}^n$, since $v\in X^s_0$ (and so $v=0$ a.e. in $\mathbb{R}^n\setminus\Omega$).

Further, by [32, Lemma 7] the space X_0^s consists of all the functions in $H^s(\mathbb{R}^n)$ which vanish a.e. outside Ω, i.e.

$$X_0^s = \Big\{ u \in H^s(\mathbb{R}^n) : u = 0 \ \text{ in } \ \mathbb{R}^n \setminus \Omega \Big\},$$

where $H^s(\mathbb{R}^n)$ denotes the usual fractional Sobolev space endowed with the norm (the so-called *Gagliardo norm*)

$$\|g\|_{H^s(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)} + \Big(\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|g(x) - g(y)|^2}{|x-y|^{n+2s}} \, dxdy \Big)^{1/2}.$$

Finally, we recall the embedding properties of X_0^s into the usual Lebesgue spaces (see [28, Lemma 8]). The embedding $j : X_0^s \hookrightarrow L^\nu(\mathbb{R}^n)$ is continuous for any $\nu \in [1, 2_s^*]$, while it is compact whenever $\nu \in [1, 2_s^*)$.

For further details on the fractional Sobolev spaces we refer to [10] and to the references therein, while for other details on X^s and X_0^s we refer to [27], where these functional spaces were introduced, and also to [15, 26, 28, 29, 32], where various properties of these spaces were proved.

2.1. **Weak solutions.** For the sake of completeness, here we recall that if $g : \mathbb{R} \to \mathbb{R}$ is a subcritical continuous function, by a *weak solution* for the following nonlocal problem

$$\begin{cases} (-\Delta)^s u = g(u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{2.2}$$

we mean a function $u : \mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{cases} \displaystyle\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} \, dxdy = \int_\Omega g(u(x))\varphi(x) \, dx \quad \forall \, \varphi \in X_0^s \\ \\ u \in X_0^s. \end{cases}$$

We observe that problem (2.2) has a variational structure, indeed it is the Euler-Lagrange equation of the functional $\mathcal{J} : X_0^s \to \mathbb{R}$ defined as follows

$$\mathcal{J}(u) := \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dxdy - \int_\Omega \left(\int_0^{u(x)} g(t) dt \right) dx \, .$$

Notice that the functional $\mathcal{J}$ is well defined and Fréchet differentiable at $u \in X_0^s$ and its Fréchet derivative at u is given by

$$\mathcal{J}'(u)(\varphi) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} \, dxdy - \int_\Omega g(u(x))\varphi(x) \, dx \, ,$$

for any $\varphi \in X_0^s$. Thus, critical points of $\mathcal{J}$ are solutions to problem (2.2). By using the above remarks, in order to prove Theorem 1 we will use critical point methods and regularity arguments.

Finally, we recall that weak solutions of the equation

$$\begin{cases} (-\Delta)^s u = k(x) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

with $k \in L^q(\Omega)$, enjoy the natural L^q-estimates given in the following lemma (for a detailed proof we refer to [20, Lemma 2.3]):

Lemma 2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary and let $u \in X_0^s$ be such that*

$$\langle u, \varphi \rangle_{X_0^s} = \int_\Omega k(x)\varphi(x) dx$$

for every $\varphi \in X_0^s$, where $k \in L^q(\Omega)$ and $q > \dfrac{n}{2s}$.

Then, $u \in L^\infty(\Omega)$ *and*

$$\|u\|_{L^\infty(\Omega)} \le M_q \|k\|_{L^q(\Omega)}$$

for some positive constant $M_q = M(n, s, \Omega, q)$.

2.2. **Eigenfunctions and eigenvalues of $(-\Delta)^s$.** In order to prove the main result of the present paper, we need also to exploit the nonlocal eigenvalue problem

$$\begin{cases} (-\Delta)^s u = \lambda u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{2.3}$$

related to the operator $(-\Delta)^s$.

A spectral theory for general integrodifferential nonlocal operators was developed in [29, Proposition 9 and Appendix A]: see also [26] for further properties of the spectrum and its associated eigenfunctions. With respect to the eigenvalue problem (2.3), we recall that it possesses a divergent sequence of eigenvalues

$$0 < \lambda_{1,s} < \lambda_{2,s} \le \cdots \le \lambda_{k,s} \le \lambda_{k+1,s} \le \dots .$$

As usual, in what follows we will denote by $e_{k,s}$ the eigenfunction related to the eigenvalue $\lambda_{k,s}$, $k \in \mathbb{N}$. From [29, Proposition 9], we know that we can choose $\{e_{k,s}\}_{k\in\mathbb{N}}$ normalized in such a way that this sequence provides an orthonormal basis in $L^2(\Omega)$ and an orthogonal basis in X_0^s, so that for any $k, i \in \mathbb{N}$ with $k \ne i$

$$\langle e_{k,s}, e_{i,s}\rangle_{X_0^s} = 0 = \int_\Omega e_{k,s}(x) e_{i,s}(x)\, dx$$

and

$$\|e_{k,s}\|^2_{X_0^s} = \lambda_{k,s} \|e_{k,s}\|^2_{L^2(\Omega)} = \lambda_{k,s}.$$

Furthermore, by [29, Proposition 9 and Appendix A], we have the following characterization of the first eigenvalue $\lambda_{1,s}$:

$$\lambda_{1,s} = \min_{u \in X_0^s \setminus \{0\}} \frac{\displaystyle\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dxdy}{\displaystyle\int_\Omega |u(x)|^2 dx}, \tag{2.4}$$

see also [11] for related topics.

In the sequel it will be useful the following regularity result for the eigenfunctions of $(-\Delta)^s$, proved in [30, Theorem 1] (see also [26, Proposition 2.4]):

Proposition 3. *Let* $e \in X_0^s$ *and* $\lambda > 0$ *be such that*

$$\langle e, \varphi\rangle_{X_0^s} = \lambda \int_\Omega e(x)\varphi(x) dx,$$

for every $\varphi \in X_0^s$.

Then, $e \in C^{0,\alpha}(\bar\Omega)$ *for some* $\alpha \in (0,1)$*, i.e. the function* e *is Hölder continuous up to the boundary.*

Finally, we recall that in [31, Corollary 8] the authors proved that the first eigenfunction $e_{1,s} \in X_0^s$ is strictly positive in Ω.

3. Proof of Theorem 1

This section is devoted to the proof of the main result of the present paper. At this purpose we have to show that assertions (h_1) and (h_2) are equivalent. For this, let

$$\bar a := \sup\Big\{\eta > 0 : \ h \text{ is non-increasing in } (0, \eta]\Big\} \in (0, +\infty].$$

3.1. **Condition** (h_1) **is sufficient for** (h_2)**.** We consider separately the two different cases $\bar{a} = +\infty$ and $\bar{a} < +\infty$.

Case 1: $\bar{a} = +\infty$**.** In this setting h is non-increasing in the half-line $(0, +\infty)$. Let us denote by σ_1 and σ_2 the following limits

$$\lim_{\xi \to 0^+} h(\xi) = \sigma_1 \quad \text{and} \quad \lim_{\xi \to +\infty} h(\xi) = \sigma_2 \,. \tag{3.1}$$

Since h is non-negative, $\sigma_1 \geq 0$ and $\sigma_2 \geq 0$. Moreover, since condition (h_1) holds, it follows that $\sigma_1 > \sigma_2$ and $\sigma_2 < +\infty$. Hence, $\sigma_1 > 0$. Therefore, one has that the interval

$$I := \left(\frac{\lambda_{1,s}}{2\sigma_1}, \frac{\lambda_{1,s}}{2\sigma_2} \right) \neq \emptyset.$$

Now, let us to show that for every $\lambda \in I$ problem (1.1) has a weak solution in the Hilbert space X_0^s. To this end, we first extend f to the whole real axis by putting $f(t) = 0$ for each $t \in (-\infty, 0)$. After that, fix $\lambda \in I$ and define the functional $\mathcal{J}_\lambda : X_0^s \to \mathbb{R}$ as follows

$$\mathcal{J}_\lambda(u) := \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dxdy - \lambda \int_\Omega F(u(x))dx \,. \tag{3.2}$$

Since the function h is non-increasing in $(0, +\infty)$, the functional $\mathcal{J}_\lambda$ is well defined, sequentially weakly lower semicontinuous, coercive and belongs to $C^1(X_0^s)$, as we will prove here below.

- **The energy functional $\mathcal{J}_\lambda$ is well defined.** By definition of σ_2 and the continuity of F, for any $\varepsilon > 0$ there exists $\sigma_\varepsilon > 0$ such that

$$F(\xi) \leq (\sigma_2 + \varepsilon)|\xi|^2 + \sigma_\varepsilon, \tag{3.3}$$

for every $\xi \in \mathbb{R}$. Hence, as a consequence of (3.3) and of the embedding properties of X_0^s into $L^2(\Omega)$, we get that $\mathcal{J}_\lambda$ is well defined.

Also, note that, bearing in mind that h is non-increasing in $(0, +\infty)$ and $f \equiv 0$ in $(-\infty, 0]$,

$$\xi f(\xi) \leq 2F(\xi) \quad \text{for all} \quad \xi \in \mathbb{R} \,. \tag{3.4}$$

By using the growth condition (3.3) and (3.4), we have

$$\xi f(\xi) \leq 2F(\xi) \leq 2(\sigma_2 + \varepsilon)|\xi|^2 + 2\sigma_\varepsilon, \;\; \text{for all} \;\; \xi \in \mathbb{R} \,,$$

so that, taking again into account that $f \equiv 0$ in $(-\infty, 0)$, we get

$$f(\xi) \leq 2(\sigma_2 + \varepsilon)|\xi| + \frac{2\sigma_\varepsilon}{|\xi|} \;\; \text{for all} \;\; \xi \in \mathbb{R} \setminus \{0\}.$$

Hence, fixing $\xi_0 > 0$, by using the above inequality, it follows that

$$f(\xi) \leq 2(\sigma_2 + \varepsilon)|\xi| + \frac{2\sigma_\varepsilon}{\xi_0} \;\; \text{for all} \;\; |\xi| \geq \xi_0 \,,$$

so that, in conclusion, since f is continuous in $\mathbb{R}$, one has

$$f(\xi) \leq 2(\sigma_2 + \varepsilon)|\xi| + \gamma \;\; \text{for all} \;\; \xi \in \mathbb{R} \,, \tag{3.5}$$

where

$$\gamma := \max_{|\xi| \leq \xi_0} f(\xi) + \frac{2\sigma_\varepsilon}{\xi_0} \,.$$

Now, by (3.5) and the embedding properties of X_0^s into the Lebesgue spaces, standard arguments easily show that $\mathcal{J}_\lambda$ is Gâteaux differentiable at $u \in X_0^s$ with continuous Gâteaux derivative given by

$$\mathcal{J}_\lambda'(u)(\varphi) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} dxdy - \lambda \int_\Omega f(u(x))\varphi(x)dx$$

for any $\varphi \in X_0^s$. Hence, the functional $\mathcal{J}_\lambda$ is of class C^1 in X_0^s.

- **Weakly lower semicontinuity of $\mathcal{J}_\lambda$.** First of all we claim that the functional

$$u \mapsto \int_\Omega F(u(x))dx$$

is continuous in the weak topology of X_0^s. Indeed, let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in X_0^s such that

$$u_j \rightharpoonup u \ \text{ weakly in } \ X_0^s$$

as $j \to +\infty$. Then, by using Sobolev embedding results and [7, Theorem IV.9], up to a subsequence still denoted by $\{u_j\}_{j\in\mathbb{N}}$, we have that

$$u_j \to u \ \text{ strongly in } \ L^\nu(\Omega)$$

and

$$u_j \to u \ \text{ a.e. in } \ \Omega$$

as $j \to +\infty$, and it is dominated by some function $h_\nu \in L^\nu(\Omega)$, i.e.

$$|u_j(x)| \le h_\nu(x) \quad \text{a.e. } x \in \Omega \ \text{ for any } j \in \mathbb{N} \tag{3.6}$$

for any $\nu \in [1, 2_s^*)$.

Then, by the continuity of F and (3.3) it follows that

$$F(u_j(x)) \to F(u(x)) \ \text{ a.e. } x \in \Omega$$

as $j \to \infty$ and

$$\begin{aligned} |F(u_j(x))| &\le (\sigma_2 + \varepsilon)|u_j(x)|^2 + \sigma_\varepsilon \\ &\le (\sigma_2 + \varepsilon)|h_2(x)|^2 + \sigma_\varepsilon \in L^1(\Omega) \end{aligned}$$

a.e. $x \in \Omega$ and for any $j \in \mathbb{N}$. Thus, thanks to the Lebesgue Dominated Convergence Theorem in $L^1(\Omega)$, we have that

$$\int_\Omega F(u_j(x))\, dx \to \int_\Omega F(u(x))\, dx$$

as $j \to \infty$, that is the map

$$u \mapsto \int_\Omega F(u(x))dx$$

is continuous from X_0^s with the weak topology into $\mathbb{R}$, as claimed.

On the other hand, the map

$$u \mapsto \|u\|^2_{X_0^s} := \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dxdy$$

is lower semicontinuous in the weak topology of X_0^s. Thus, the functional $\mathcal{J}_\lambda$ is lower semicontinuous in the weak topology of X_0^s.

- **Coercivity of $\mathcal{J}_\lambda$.** Since $\lambda \in I$, we clearly have

$$\lambda < \frac{\lambda_{1,s}}{2\sigma_2}.$$

Therefore, we can find $\varrho > \sigma_2$ such that

$$\frac{\lambda_{1,s}}{2\sigma_1} < \lambda < \frac{\lambda_{1,s}}{2\varrho}. \tag{3.7}$$

Now, by (3.3) with $\varepsilon = \varrho - \sigma_2 > 0$ and (3.7), we get that for some positive constant β independent of λ (for instance take $\beta := \dfrac{\sigma_\varepsilon \lambda_{1,s}}{2\varrho}$), we have

$$\begin{aligned}\lambda \int_\Omega F(u(x))dx &\le \lambda\varrho \int_\Omega |u(x)|^2 dx + \lambda\sigma_\varepsilon \\ &\le \frac{\lambda\varrho}{\lambda_{1,s}} \|u\|^2_{X_0^s} + \frac{\sigma_\varepsilon \lambda_{1,s}}{2\varrho} \\ &= \frac{\lambda\varrho}{\lambda_{1,s}} \|u\|^2_{X_0^s} + \beta\,,\end{aligned}$$

and so

$$\mathcal{J}_\lambda(u) \ge \left(\frac{1}{2} - \frac{\lambda\varrho}{\lambda_{1,s}}\right) \|u\|^2_{X_0^s} - \beta \tag{3.8}$$

for every $u \in X_0^s$.

In view of (3.7), by (3.8) it follows that

$$\mathcal{J}_\lambda(u) \to +\infty, \tag{3.9}$$

as $\|u\|_{X_0^s} \to +\infty$. Hence, the functional $\mathcal{J}_\lambda$ is coercive in X_0^s.

- **Existence of a (nontrivial) global minimum $u_\lambda \in X_0^s$ for the functional $\mathcal{J}_\lambda$.** Since $\mathcal{J}_\lambda$ is coercive and weakly lower semicontinuous in X_0^s, it is easy to see that it admits a global minimum $u_\lambda \in X_0^s$.

Let us show that u_λ is nontrivial. At this purpose, we observe that the first eigenfunction $e_{1,s}$ of $(-\Delta)^s$ is in X_0^s and it is positive in Ω (see [31, Corollary 8]). Moreover, by Proposition 3, one has that $e_{1,s} \in C^{0,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$. Let us put

$$\bar{e}_{1,s} := \max_{x\in\bar{\Omega}} e_{1,s}(x)\,,$$

so that, thanks to (h_1), for every $t > 0$ we have

$$h(te_{1,s}(x)) > h(t\bar{e}_{1,s}) \quad \text{for all } x \in \Omega_0\,,$$

where $\Omega_0 \subseteq \Omega$ is a set of positive Lebesgue measure.

Thus, for any $t > 0$ we get

$$\begin{aligned}\mathcal{J}_\lambda(te_{1,s}) &= \frac{t^2}{2}\|e_{1,s}\|^2_{X_0^s} - \lambda \int_\Omega F(te_{1,s}(x))dx \\ &= \frac{t^2}{2}\|e_{1,s}\|^2_{X_0^s} - \lambda \int_\Omega h(te_{1,s}(x))(te_{1,s}(x))^2 dx \\ &< \frac{t^2}{2}\|e_{1,s}\|^2_{X_0^s} - \lambda t^2 h(t\bar{e}_{1,s}) \int_\Omega |e_{1,s}(x)|^2 dx \\ &= t^2\|e_{1,s}\|^2_{X_0^s}\left(\frac{1}{2} - \frac{\lambda}{\lambda_{1,s}} h(t\bar{e}_{1,s})\right).\end{aligned} \tag{3.10}$$

Now, suppose that $\sigma_1 < +\infty$ (see (3.1) for the definition of σ_1). Then, for every $\varepsilon > 0$ there exists a positive constant δ_ε such that for every $\zeta \in (0, \delta_\varepsilon]$ we have

$$\sigma_1 - \varepsilon < \frac{F(\zeta)}{\zeta^2} < \sigma_1 + \varepsilon.$$

By (3.7) we know that $\dfrac{\lambda_{1,s}}{2\lambda} < \sigma_1$. Hence, there exists $\bar{\varepsilon} > 0$ such that

$$\frac{\lambda_{1,s}}{2\lambda} < \sigma_1 - \bar{\varepsilon} < \sigma_1,$$

and so

$$\frac{F(\zeta)}{\zeta^2} > \sigma_1 - \bar{\varepsilon} > \frac{\lambda_{1,s}}{2\lambda}$$

for every $\zeta \in (0, \delta_{\bar{\varepsilon}}]$.

On the other hand, if $\sigma_1 = +\infty$, one has

$$\frac{F(\zeta)}{\zeta^2} > \frac{\lambda_{1,s}}{2\lambda}$$

for ζ sufficiently small. In both cases, there exists $\bar{t} > 0$ sufficiently small such that

$$h(\bar{t}\bar{e}_{1,s}) := \frac{F(\bar{t}^2\bar{e}_{1,s})}{\bar{t}^2\bar{e}_{1,s}} > \frac{\lambda_{1,s}}{2\lambda}.$$

Consequently, by this and (3.10) we get that $\mathcal{J}_\lambda(\bar{t}e_{1,s}) < 0$ and so

$$\inf_{u \in X_0^s} \mathcal{J}_\lambda(u) < 0\,.$$

This yields that $u_\lambda \not\equiv 0$ in X_0^s and so the existence of a nontrivial global minimum for $\mathcal{J}_\lambda$ in X_0^s is proved.

- **Asymptotic behaviour of u_λ for λ sufficiently small.** We claim that

$$\lim_{\lambda \to \mu_0^+} \|u_\lambda\|_{X_0^s} = 0, \tag{3.11}$$

where, from now on, for simplicity, we set

$$\mu_0 := \frac{\lambda_{1,s}}{2\sigma_1}$$

($\mu_0 := 0$ if $\sigma_1 = +\infty$). This, of course, completes the proof of (h_2) in Case 1.

To prove (3.11), let us take a real sequence $\{\lambda_j\}_{j\in\mathbb{N}}$ such that

$$\{\lambda_j\}_{j\in\mathbb{N}} \subset \left(\frac{\lambda_{1,s}}{2\sigma_1}, \frac{\lambda_{1,s}}{2\varrho}\right),$$

where ϱ is as in (3.7), and

$$\lim_{j\to\infty} \lambda_j = \frac{\lambda_{1,s}}{2\sigma_1}. \tag{3.12}$$

Also, let us denote by u_{λ_j} the nontrivial global minimum for $\mathcal{J}_\lambda$ in X_0^s when $\lambda = \lambda_j$.

For each $j \in \mathbb{N}$, we have $\mathcal{J}_{\lambda_j}(u_{\lambda_j}) < 0$. Hence, by this and in view of (3.8), we get

$$\|u_{\lambda_j}\|^2_{X_0^s} < \frac{\beta}{\left(\dfrac{1}{2} - \dfrac{\lambda_j \varrho}{\lambda_{1,s}}\right)}. \tag{3.13}$$

Observing that, by (3.13),

$$\lim_{j\to+\infty} \frac{\beta}{\left(\dfrac{1}{2} - \dfrac{\lambda_j \varrho}{\lambda_{1,s}}\right)} = \frac{\beta}{\left(\dfrac{1}{2} - \dfrac{\varrho}{2\sigma_1}\right)} \in (0, +\infty),$$

we have that the sequence $\{u_{\lambda_j}\}_{j\in\mathbb{N}}$ is bounded in X_0^s. Thus, up to a subsequence still denoted by $\{u_{\lambda_j}\}_{j\in\mathbb{N}}$, we deduce that there exists $u_\infty \in X_0^s$ such that

$$u_j \rightharpoonup u_\infty \ \text{ weakly in } \ X_0^s \tag{3.14}$$

and

$$u_{\lambda_j} \to u_\infty \ \text{ strongly in } \ L^\nu(\Omega) \tag{3.15}$$

as $j \to +\infty$, for every $\nu \in [1, 2_s^*)$.

We claim that $u_\infty \equiv 0$ in X_0^s. Indeed, arguing by contradiction, assume that $u_\infty \not\equiv 0$ in X_0^s. Now, note that for each $\varphi \in X_0^s$ and $j \in \mathbb{N}$, one has

$$\begin{aligned} 0 = \mathcal{J}'_{\lambda_j}(u_{\lambda_j})(\varphi) = \int_{\mathbb{R}^n\times\mathbb{R}^n} & \frac{(u_{\lambda_j}(x)-u_{\lambda_j}(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}dxdy \\ & - \lambda_j \int_\Omega f(u_{\lambda_j}(x))\varphi(x)dx. \end{aligned} \tag{3.16}$$

Assume that σ_1 is finite (the case $\sigma_1 = +\infty$ is similar). Taking into account inequality (3.5), (3.14) and (3.15), passing to the limit in (3.16) as $j \to +\infty$ we have

$$\begin{aligned} 0 = \mathcal{J}'_{\mu_0}(u_\infty)(\varphi) = \int_{\mathbb{R}^n\times\mathbb{R}^n} & \frac{(u_\infty(x)-u_\infty(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}dxdy \\ & - \frac{\lambda_{1,s}}{2\sigma_1}\int_\Omega f(u_\infty(x))\varphi(x)dx, \end{aligned} \tag{3.17}$$

for every $\varphi \in X_0^s$. Therefore, u_∞ is a nontrivial critical point of $\mathcal{J}_{\mu_0}$, that is u_∞ is a weak solution of the nonlocal problem (1.1) for $\lambda = \mu_0$.

Testing equation (3.17) with $\varphi = u_\infty$, by using inequality $\xi f(\xi) \leq 2F(\xi)$ for all $\xi \in \mathbb{R}$, we obtain

$$\begin{aligned} 0 &= \|u_\infty\|^2_{X_0^s} - \frac{\lambda_{1,s}}{2\sigma_1}\int_\Omega f(u_\infty(x))u_\infty(x)dx \\ &\geq \|u_\infty\|^2_{X_0^s} - \frac{\lambda_{1,s}}{\sigma_1}\int_\Omega F(u_\infty(x))dx \\ &= \|u_\infty\|^2_{X_0^s} - \frac{\lambda_{1,s}}{\sigma_1}\int_\Omega h(u_\infty(x))|u_\infty(x)|^2dx. \end{aligned} \tag{3.18}$$

Taking into account (h_1), relation (3.18) yields

$$\begin{aligned} 0 &\geq \|u_\infty\|^2_{X_0^s} - \frac{\lambda_{1,s}}{\sigma_1}\int_\Omega h(u_\infty(x))|u_\infty(x)|^2dx \\ &> \|u_\infty\|^2_{X_0^s} - \lambda_{1,s}\int_\Omega |u_\infty(x)|^2dx\,, \end{aligned} \tag{3.19}$$

that is

$$\lambda_{1,s} > \frac{\displaystyle\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|u_\infty(x)-u_\infty(y)|^2}{|x-y|^{n+2s}}dxdy}{\displaystyle\int_\Omega |u_\infty(x)|^2dx}$$

in contradiction with (2.4). Therefore, it must be $u_\infty \equiv 0$ in X_0^s.

Finally, choosing $\varphi = u_{\lambda_j}$ in (3.17), we have

$$\|u_{\lambda_j}\|^2_{X_0^s} = \lambda_j \int_\Omega f(u_{\lambda_j}(x))u_{\lambda_j}(x)dx \tag{3.20}$$

for each $j \in \mathbb{N}$. Now, note that, by (3.5), (3.12) and the fact that $u_\infty \equiv 0$, the right hand side in (3.20) converges to 0 as $j \to +\infty$. Thus

$$\lim_{j\to\infty}\|u_{\lambda_j}\|_{X_0^s} = 0,$$

so that (3.11) is proved. This fact concludes the proof of Case 1.

Case 2: $\bar{a} < +\infty$**.** First of all, note that $h'(\bar{a}) = 0$. Thus, the function $h_0 : (0,+\infty) \to \mathbb{R}$ given by

$$h_0(\xi) := \begin{cases} h(\xi) & \text{if } \xi \in (0,\bar{a}] \\ h(\bar{a}) & \text{if } \xi \in (\bar{a},+\infty), \end{cases}$$

is of class C^1 and non-increasing in $(0,+\infty)$.

Denote by F_0 the function defined as

$$F_0(\xi) := \begin{cases} 0 & \text{if } \xi \in (-\infty, 0] \\ h_0(\xi)\xi^2 & \text{if } \xi \in (0, +\infty). \end{cases}$$

Then, F_0 is a C^1 function and $F_0(\xi) = F(\xi)$ for every $\xi \in (-\infty, \bar{a}]$.

Now, consider the truncated problem

$$\begin{cases} (-\Delta)^s u = \lambda f_0(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{3.21}$$

where

$$f_0(\xi) := F_0'(\xi) = \begin{cases} 0 & \text{if } \xi \in]-\infty, 0] \\ f(\xi) & \text{if } \xi \in (0, \bar{a}] \\ 2h_0(\bar{a})\xi & \text{if } \xi \in (\bar{a}, +\infty). \end{cases}$$

Note that in the setting of problem (3.21) we have that

$$\sup\Big\{\eta > 0 : \ h_0 \text{ is non-increasing in } (0, \eta]\Big\} = +\infty\,.$$

- **Existence of one weak solution of the truncated problem** (3.21)**.** By using what we did in Case 1, for any $r > 0$, we can find an open interval

  $$J := (\mu_0, \mu_0 + \varepsilon_0)\,, \quad \varepsilon_0 > 0$$

  such that for every $\lambda \in J$ there exists a weak solution $u_\lambda \in X_0^s$ of (3.21), that is for every $\varphi \in X_0^s$

  $$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u_\lambda(x) - u_\lambda(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} dxdy = \lambda \int_\Omega f_0(u_\lambda(x))\varphi(x)dx\,, \tag{3.22}$$

  which satisfies $u_\lambda = 0$ in $\mathbb{R}^n \setminus \Omega$ and

  $$\|u_\lambda\|_{X_0^s} < r\,. \tag{3.23}$$

- **A weak solution of problem** (1.1)**.** Our aim consists in proving that u_λ is, in fact, solution of problem (1.1) for suitable values of the parameter λ. At this purpose, we claim that for any $\tau > 0$ there exists $K_\tau > 0$ such that

  $$f_0(\xi) \le K_\tau |\xi| + \tau \tag{3.24}$$

  for any $\xi \in \mathbb{R}$. Indeed, let $\tau \ge \max_{\xi \in [0,\bar{a}]} f_0(\xi)$: in this case the claim is obvious by the definition of f_0. Now, assume that $\tau < \max_{\xi \in [0,\bar{a}]} f_0(\xi)$ and denote by

  $$\xi_0 := \min\{\xi > 0 : f(\xi) = \tau\}\,.$$

  Then, by the continuity of f_0 and the fact that $f_0(0) = 0$ we deduce that

  $$f_0(\xi) < \tau \ \text{ for all } \ \xi \in [0, \xi_0)$$

  and

  $$f_0(\xi) \le H_\tau \xi \ \text{ for all } \ \xi \in [\xi_0, \bar{a}]$$

  for a suitable positive constant H_τ. Hence, by these inequalities and the definition of f_0 we get that the claim holds true.

  By [5, Proposition 2.2] and (3.24) we have that

  $$u_\lambda \in L^\infty(\Omega)\,. \tag{3.25}$$

  Now, fix $q > \dfrac{n}{2s}$ and apply Lemma 2, with $k(x) := \lambda f_0(u_\lambda(x))$, to problem (3.21). This choice of k is admissible, thanks to (3.24) and (3.25). Thus, we have that there exists $M_q > 0$ such that

  $$\|u_\lambda\|_{L^\infty(\Omega)} \le M_q \|\lambda f_0(u_\lambda)\|_{L^q(\Omega)}. \tag{3.26}$$

Choosing $\tau := \dfrac{\bar{a}}{2(\mu_0+\varepsilon_0)M_q|\Omega|^{1/q}}$ in (3.24) and using (3.26) we easily obtain

$$\begin{aligned}\|u_\lambda\|_{L^\infty(\Omega)} &\leq \lambda M_q K_\tau \|u_\lambda\|_{L^q(\Omega)} + \frac{\bar{a}}{2}\\ &\leq (\mu_0+\varepsilon_0) M_q K_\tau \|u_\lambda\|_{L^q(\Omega)} + \frac{\bar{a}}{2}\,,\end{aligned} \tag{3.27}$$

for every $\lambda \in J$.

Fix $\mu \in (0,1)$ such that $q\mu < 2^*_s$. We have that

$$\begin{aligned}\|u_\lambda\|_{L^q(\Omega)} &:= \left(\int_\Omega |u_\lambda(x)|^q dx\right)^{1/q}\\ &= \left(\int_\Omega |u_\lambda(x)|^{q(1-\mu)}|u_\lambda(x)|^{q\mu} dx\right)^{\mu/(q\mu)}\\ &\leq \left(\int_\Omega \|u_\lambda\|^{q(1-\mu)}_{L^\infty(\Omega)}|u_\lambda(x)|^{q\mu} dx\right)^{\mu/(q\mu)}\\ &= \|u_\lambda\|^{1-\mu}_{L^\infty(\Omega)}\|u_\lambda\|^{\mu}_{L^{\mu q}(\Omega)}.\end{aligned} \tag{3.28}$$

By (3.28) and using the Sobolev embedding $X_0^s \hookrightarrow L^{q\mu}(\Omega)$, one has

$$\begin{aligned}\|u_\lambda\|_{L^q(\Omega)} &\leq \|u_\lambda\|^{1-\mu}_{L^\infty(\Omega)}\|u_\lambda\|^{\mu}_{L^{\gamma q}(\Omega)}\\ &\leq c_{q\mu}\|u_\lambda\|^{1-\mu}_{L^\infty(\Omega)}\|u_\lambda\|^{\mu}_{X_0^s},\end{aligned}$$

for some positive constant $c_{q\mu}$, so that, combining this inequality with (3.27), we obtain

$$\|u_\lambda\|_{L^\infty(\Omega)} \leq K\|u_\lambda\|^{1-\mu}_{L^\infty(\Omega)}\|u_\lambda\|^{\mu}_{X_0^s} + \frac{\bar{a}}{2} \tag{3.29}$$

for any $\lambda \in J$, for some positive constant K independent of λ. Hence, (3.11) and (3.29) give that

$$\lim_{\lambda\to\mu_0^+} \|u_\lambda\|_{L^\infty(\Omega)} \leq \frac{\bar{a}}{2}.$$

As a consequence of this, there exists some $\varepsilon_1 \in (0,\varepsilon_0)$ such that

$$u_\lambda(x) \leq \bar{a} \ \text{ a.e. } x \in \Omega \tag{3.30}$$

for every $\lambda \in J' := (\mu_0, \mu_0+\varepsilon_1)$.

In conclusion, thanks to (3.30) and the definition of f_0, the function $u_\lambda \in X_0^s$ is a weak solution of problem (1.1) for every $\lambda \in J'$. Finally, note that u_λ satisfies (3.23) for any $\lambda \in J'$, since $\varepsilon_1 < \varepsilon_0$. This completes the proof of Theorem 1.

3.2. **Condition** (h_1) **is necessary for** (h_2)**.** In order to prove our result we argue by contradiction. We assume that h is constant in some interval $[0,b]$, that is there exists two positive constants b and c such that

$$F(\xi) = c\xi^2$$

for every $\xi \in [0,b]$. Consequently, one has

$$f(\xi) = 2c\xi$$

for every $t \in [0,b]$ and assumptions (1.3) and (1.4) are satisfied.

Let $\{r_j\}_{j\in\mathbb{N}} \subset (0,+\infty)$ be a sequence such that $\lim\limits_{j\to\infty} r_j = 0$. Then, thanks to (h_2), for every $j \in \mathbb{N}$ there exists $\varepsilon_j > 0$ such that for every

$$\lambda \in J_j := (\mu_0, \mu_0+\varepsilon_j)$$

problem (1.1) has a weak solution $u_{\lambda,j} \in X_0^s$ satisfying $\|u_{\lambda,j}\|_{X_0^s} < r_j$. In particular, we have

$$\lim_{j\to\infty} \sup_{\lambda\in J_j} \|u_{\lambda,j}\|_{X_0^s} = 0. \tag{3.31}$$

Now, since f is subcritical and [5, Proposition 2.2] holds true, we get that $u_{\lambda,j} \in L^\infty(\Omega)$. Moreover, arguing as in Subsection 3.1, we can find a positive constant K (independent of j and λ) and μ sufficiently small such that

$$\|u_{\lambda,j}\|_{L^\infty(\Omega)} \le K\|u_{\lambda,j}\|^{1-\mu}_{L^\infty(\Omega)}\|u_{\lambda,j}\|^{\mu}_{X_0^s} + \frac{b}{2} \tag{3.32}$$

for every $\lambda \in J_j$ and $j \in \mathbb{N}$.

From (3.31) and (3.32), we deduce that

$$\lim_{j\to\infty} \sup_{\lambda\in J_j} \|u_{\lambda,j}\|_{L^\infty(\Omega)} \le \frac{b}{2}.$$

In particular, we can fix $j_0 \in \mathbb{N}$ such that

$$\|u_{\lambda,j_0}\|_{L^\infty(\Omega)} \le b$$

for every $\lambda \in J_{j_0}$. Consequently, for every $\lambda \in (\mu_0, \mu_0 + \varepsilon_{j_0})$ we get that

$$0 \le u_{\lambda,j_0}(x) \le b \ \text{ a.e. } \ x \in \Omega \tag{3.33}$$

and so u_{λ,j_0} solves the following problem

$$\begin{cases} (-\Delta)^s u = 2\lambda c u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,. \end{cases} \tag{3.34}$$

This is a contradiction, since problem (3.34) has solution only for countably many positive value of the parameter λ (see [11, Proposition 2.1]). The proof of Theorem 1 is now complete.

4. An example and some final comments

In conclusion, in this section we present a direct application of our main result. As a model for f we can take the nonlinearity

$$f(\xi) := \sqrt{\xi} \ \text{ for all } \xi \in [0, +\infty).$$

Indeed, the real function

$$h(\xi) := \frac{\displaystyle\int_0^\xi \sqrt{t}\, dt}{\xi^2} = \frac{2}{3\sqrt{\xi}} \quad \text{for any } \xi \in (0, +\infty)$$

is strictly decreasing in $(0, +\infty)$.

Hence, Theorem 1 ensures that for each $r > 0$ there exists $\varepsilon_r > 0$ such that for every $\lambda \in (0, \varepsilon_r)$ the nonlocal problem

$$\begin{cases} (-\Delta)^s u = \lambda\sqrt{u} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{4.1}$$

admits a weak solution $u_\lambda \in H^s(\mathbb{R}^n)$, such that $u_\lambda = 0$ in $\mathbb{R}^n \setminus \Omega$, and

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u_\lambda(x) - u_\lambda(y)|^2}{|x-y|^{n+2s}}\, dxdy < r^2.$$

Remark 4. It is easily seen that the statements of Theorem 1 are still true if, instead of (1.1), we consider the following nonlocal problem

$$\begin{cases} (-\Delta)^s u = \lambda\alpha(x)f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{4.2}$$

where $\alpha : \bar{\Omega} \to \mathbb{R}$ is continuous and positive. In such a case, if we set

$$\lambda_{1,s}^{(\alpha)} := \min_{u \in X_0^s \setminus \{0\}} \frac{\displaystyle\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dxdy}{\displaystyle\int_\Omega \alpha(x)|u(x)|^2 dx},$$

condition (h_2) assumes the form

($\mathrm{h}_{2,\alpha}$) *f is subcritical with $\lim_{\xi\to 0^+} h(\xi) > 0$ and for each $r > 0$ there exists $\varepsilon_r > 0$ such that for every*

$$\lambda \in \left(\frac{\lambda_{1,s}^{(\alpha)}}{2 \lim_{\xi \to 0^+} h(\xi)}, \frac{\lambda_{1,s}^{(\alpha)}}{2 \lim_{\xi\to 0^+} h(\xi)} + \varepsilon_r \right),$$

the problem (4.2) *has a weak solution $u_\lambda \in X_0^s$, satisfying $\|u_\lambda\|_{X_0^s} < r$.*

We think that an interesting open problem is to study of an analogous version of Theorem 1 for problem (4.2) assuming that α is continuous and sign-changing.

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References

[1] A. Ambrosetti and P. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381 .

[2] G. Anello, *A characterization related to the Dirichlet problem for an elliptic equation*, Funkcialaj Ekvacioj **59** (2016), 113–122.

[3] G. Autuori and P. Pucci, *Existence of entire solutions for a class of quasilinear elliptic equations*, NoDEA Nonlinear Differential Equations Appl. **20** (2013), 977–1009.

[4] G. Autuori and P. Pucci, *Elliptic problems involving the fractional Laplacian in $\mathbb{R}^N$*, J. Differential Equations **255** (2013), 2340–2362.

[5] B. Barrios, E. Colorado, R. Servadei and F. Soria, *A critical fractional equation with concave-convex power nonlinearities*, Ann. Inst. H. Poincaré Anal. Non Linéaire **32** (2015), 875–900.

[6] C. Brändle, E. Colorado, A. de Pablo, and U. Sánchez, *A concave-convex elliptic problem involving the fractional Laplacian*, Proc. Roy. Soc. Edinburgh Sect. A **143** (2013), 39–71.

[7] H. Brezis, Analyse Fonctionelle. Théorie et Applications, *Masson*, Paris, 1983.

[8] X. Cabré and J. Tan, *Positive solutions of nonlinear problems involving the square root of the Laplacian*, Adv. Math. **224** (2010), 2052–2093.

[9] L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32** (2007), 1245–1260.

[10] E. Di Nezza, G. Palatucci, and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), no. 5, 521–573.

[11] A. Fiscella, R. Servadei, and E. Valdinoci, *A resonance problem for non-local elliptic operators*, Z. Anal. Anwendungen **32** (2013), 411–431.

[12] R. Musina and A. Nazarov, *On fractional Laplacians*, Commun. Partial Differential Equations **39** (2014), 1780–1790.

[13] G. Molica Bisci and V. Rădulescu, *A characterization for elliptic problems on fractal sets*, Proc. Amer. Math. Soc. **143** (2015), 2959–2968.
[14] G. Molica Bisci and V. Rădulescu, *A sharp eigenvalue theorem for fractional elliptic equations*, to appear in Israel Journal of Math.
[15] G. Molica Bisci, V. Rădulescu, and R. Servadei, Variational Methods for Nonlocal Fractional Problems. With a Foreword by Jean Mawhin, *Encyclopedia of Mathematics and its Applications*, *Cambridge University Press* **162**, Cambridge, 2016. ISBN 9781107111943.
[16] G. Molica Bisci and D. Repovš, *Fractional nonlocal problems involving nonlinearities with bounded primitive*, J. Math. Anal. Appl. **420** (2014), 167–176.
[17] G. Molica Bisci and D. Repovš, *On doubly nonlocal fractional elliptic equations*, Rend. Lincei Mat. Appl. **26** (2015), 161–176.
[18] G. Molica Bisci and D. Repovš, *Existence and localization of solutions for nonlocal fractional equations*, Asymptot. Anal. **90** (2014), 367–378.
[19] G. Molica Bisci and R. Servadei, *A Brezis-Nirenberg splitting approach for nonlocal fractional equations*, Nonlinear Anal. **119** (2015), 341–353.
[20] S. Mosconi, K. Perera, M. Squassina, and Y. Yang, *The Brezis-Nirenberg problem for the p–Laplacian*, to appear in Calc. Var. Partial Differential Equations.
[21] P. Pucci and S. Saldi, *Multiple solutions for an eigenvalue problem involving non-local elliptic p-Laplacian operators*, in Geometric Methods in PDE's - Springer INdAM Series - Vol. 11, G. Citti, M. Manfredini, D. Morbidelli, S. Polidoro, F. Uguzzoni Eds., pages 16.
[22] P. Pucci and S. Saldi, *Critical stationary Kirchhoff equations in $\mathbb{R}^N$ involving nonlocal operators*, to appear in Rev. Mat. Iberoam., pages 23.
[23] P.H. Rabinowitz, Some critical point theorems and applications to semilinear elliptic partial differential equations, Ann. Scuola Norm. Sup. Pisa Cl. Sci., **5** (1978), 215–223.
[24] P.H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, *CBMS Reg. Conf. Ser. Math.*, 65, *American Mathematical Society*, Providence, RI (1986).
[25] B. Ricceri, *A characterization related to a two-point boundary value problem*, J. Nonlinear Convex Anal. **16** (2015), 79–82.
[26] R. Servadei, *The Yamabe equation in a non-local setting*, Adv. Nonlinear Anal. **2** (2013), 235–270.
[27] R. Servadei and E. Valdinoci, *Lewy-Stampacchia type estimates for variational inequalities driven by nonlocal operators*, Rev. Mat. Iberoam. **29**, no. 3, (2013), 1091–1126.
[28] R. Servadei and E. Valdinoci, *Mountain Pass solutions for non-local elliptic operators*, J. Math. Anal. Appl. **389** (2012), 887–898.
[29] R. Servadei and E. Valdinoci, *Variational methods for non-local operators of elliptic type*, Discrete Contin. Dyn. Syst. **33**, 5 (2013), 2105–2137.
[30] R. Servadei and E. Valdinoci, *On the spectrum of two different fractional operators*, Proc. Roy. Soc. Edinburgh Sect. A **144** (2014), 1–25.
[31] R. Servadei and E. Valdinoci, *Weak and viscosity solutions of the fractional Laplace equation*, Publ. Mat. **58** (2014), 133–154.
[32] R. Servadei and E. Valdinoci, *The Brezis-Nirenberg result for the fractional Laplacian*, Trans. Amer. Math. Soc. **367** (2015), 67–102.
[33] L. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Comm. Pure Appl. Math., **60** (2007), no. 1, 67–112.
[34] M. Struwe, Variational methods, Applications to nonlinear partial differential equations and Hamiltonian systems, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, 3, *Springer Verlag*, Berlin–Heidelberg (1990).
[35] M. Willem, Minimax theorems, *Progress in Nonlinear Differential Equations and their Applications*, 24, *Birkhäuser*, Boston (1996).
[36] B. Zhang, G. Molica Bisci, and R. Servadei, *Superlinear nonlocal fractional problems with infinitely many solutions*, Nonlinearity **28** (2015), 2247–2264.

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