ANALYTIC HYPOELLIPTICITY AND THE TREVES CONJECTURE
IPOELLITTICITÀ ANALITICA E CONGETTURA DI TREVES

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ABSTRACT. We are concerned with the problem of the analytic hypoellipticity; precisely, we focus on the real analytic regularity of the solutions of sums of squares with real analytic coefficients. Treves conjecture states that an operator of this type is analytic hypoelliptic if and only if all the strata in the Poisson-Treves stratification are symplectic. We discuss a model operator, P, (firstly appeared and studied in [3]) having a single symplectic stratum and prove that it is not analytic hypoelliptic. This yields a counterexample to the sufficient part of Treves conjecture; the necessary part is still an open problem.

SUNTO. Questo articolo riguarda il problema dell'ipoelliticità analitica; precisamente, si intende studiare la regolarità analitica reale delle soluzioni di somme di quadrati di campi a coefficienti reali analitici. La congettura di Treves afferma che un siffatto operatore è ipoellitico analitico se e solo se tutti i suoi strati di Poisson-Treves risultano essere simplettici. In questo articolo si presenta un operatore modello P (introdotto e studiato in [3]) avente uno strato simplettico singolo e si prova che non è analitico ipoellittico, contraddicendo la parte sufficiente della congettura di Treves. La parte necessaria risulta essere ancora un problema aperto.

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1. Introduction

This paper is concerned with the problem of the analytic hypoellipticity of a sum of squares operator

$$P = \sum_{j=1}^{N} X_j(x, D)^2, \tag{1}$$

where $X_j(x, D)$ is a vector field with real analytic coefficients defined in an open set $\Omega \subset \mathbb{R}^n$. Precisely, we are interested in studying the analytic regularity of the distribution solutions to the equation

$$Pu = \sum_{j=1}^{N} X_j(x, D)^2 u = f, \tag{2}$$

where u is a distribution in Ω and $f \in C^\omega(\Omega)$, the space of all real analytic functions in Ω.

We say that P is analytic hypoelliptic in Ω if P preserves the analytic singular support; namely, if for every $u \in \mathscr{D}'(\Omega)$ and every open set $V \subset \Omega$,

$$Pu \in C^\omega(V) \Longrightarrow u \in C^\omega(V).$$

The problem of the $C^\infty(\Omega)$ hypoellipticity of (2) has been solved completely by L. Hörmander in 1967, [20], by proving that P is hypoelliptic if the vector fields defining it verify the Hörmander condition

(H) The Lie algebra generated by the vector fields and their commutators has dimension n, equal to the dimension of the ambient space.

We point out that, if the X_j in (1) are C^∞ vector fields, the condition (H) is only sufficient but not necessary in order for P to be C^∞ hypoelliptic (see Fedii [14], Morimoto [27]). However, if the coefficients of the X_j in (1) are analytic, as in the present case, M. Derridj, [13], showed that then the condition (H) is also necessary for the C^∞ hypoellipticity of P.

Therefore, the analytic setting seems to be a better choice if we are interested in studying the geometric properties of a sum of squares operator.

As a further step in the analysis of the hypoellipticity of P one may ask if, when condition

(H) is satisfied, it is analytic hypoelliptic, i.e. if $Pu = f$, $f \in C^\omega(\Omega)$, for a certain distribution $u \in \mathscr{D}'(\Omega)$, implies that actually $u \in C^\omega(\Omega)$.

In 1972 M. S. Baouendi and C. Goulaouic [4] produced an example of a sum of squares satisfying condition (H)—and hence C^∞ hypoelliptic—which is not analytic hypoelliptic. Precisely, consider in $\mathbb{R}^3_{x,y,t}$ the operator

$$B = D_x^2 + x^2 D_y^2 + D_t^2$$

and, for a positive ϵ, the function

$$u(x,y,t) = \int_0^\infty e^{isy+\sqrt{s}t-\epsilon\sqrt{s}} \cdot e^{-s\frac{x^2}{2}}\, ds.$$

Note that u is a C^∞ function near the origin and an easy computation shows that

$$Bu(x,y,t) = \int_0^\infty e^{isy+\sqrt{s}t} e^{-\epsilon\sqrt{s}} \left((D_x^2 + x^2 s^2 - s) e^{-s\frac{x^2}{2}} \right) ds = 0,$$

but $u(x,y,t)$ is not analytic at the origin, being

$$D_y^k u(0,0,0) = \int_0^\infty s^k e^{-\epsilon\sqrt{s}}\, ds = 2\epsilon^{-2(k+1)} \int_0^\infty q^{(2k+2)-1} e^{-q}\, dq =$$

$$2\epsilon^{-2(k+1)}\Gamma(2k+2) = 2\epsilon^{-2(k+1)}(2k+1)! \geq 2\epsilon^{-2(k+1)}(k!)^2.$$

In 1996, [40], F. Treves stated a conjecture for the sums of squares of vector fields that takes into account all the cases known to this date (see [3] for a brief survey on this). The conjecture requires some work to be stated precisely; see to this end the papers [40], [10], [41]. In what follows we give a brief, sketchy account of how to formulate it.

Let P be as in (2). Then the characteristic variety of P is $\mathrm{Char}(P) = \{(x,\xi) \mid X_j(x,\xi) = 0, j = 1,\dots,N\}$. This is a real analytic variety and, as such, it can be stratified in real analytic manifolds, Σ_i, for i in a family of indices, having the property that for $i \neq i'$, either $\Sigma_i \cap \overline{\Sigma}_{i'} = \varnothing$ or, if $\Sigma_i \cap \overline{\Sigma}_{i'} \neq \varnothing$, then $\Sigma_i \subset \partial\Sigma_{i'}$.

Next one examines the rank of the restriction of the symplectic form to the analytic strata Σ_i. If there is a change of rank of the symplectic form on a stratum, we may add to the equations of the stratum the equations of the subvariety where there is a change in rank and stratify the so obtained variety into strata which are real analytic manifolds where the symplectic form has constant rank.

In the final step one considers the multiple Poisson brackets of the symbols of the vector fields. Denote by $X_j(x,\xi)$ the symbol of the j-th vector field. Let $I = (i_1, i_2, \ldots, i_r)$, where $i_j \in \{1, \ldots, N\}$. Write $|I| = r$ and define

$$X_I(x,\xi) = \{X_{i_1}(x,\xi), \{X_{i_2}(x,\xi), \{\cdots \{X_{i_{r-1}}(x,\xi), X_{i_r}(x,\xi)\} \cdots\}\}\}.$$

r is called the length of the multiple Poisson bracket $X_I(x,\xi)$. For each stratum previously obtained, say Σ_{ik}, we want that all brackets of length lesser than a certain integer, say ℓ_{ik} vanish, but that there is at least one bracket of length ℓ_{ik} which is non zero on Σ_{ik}. One can show that this makes sense and defines a stratification.

Thus the strata obtained are real analytic manifolds where the symplectic form has constant rank and where all brackets of the vector fields vanish if their length is $< \ell_{ik}$, and there is at least one microlocally elliptic bracket of length ℓ_{ik}, ℓ_{ik} depending on the stratum. ℓ_{ik} is also called the depth of the stratum.

We now state Treves' conjecture:

Conjecture 1 (Treves, [40])**.** *The operator P in* (2) *is analytic hypoelliptic if and only if every stratum in the above described stratification is symplectic.*

We remark that the above statement is in agreement with a number of known results. We note that Baouendi-Goulaouic operator does not have a symplectic characteristic manifold and so one might expect it not to be analytic hypoelliptic. We just would like to mention that a number of results have been published over the last fifteen years in agreement with the conjecture. As a non exhaustive and certainly incomplete list we mention the papers [11], [12], [15], [16], [17], [38], [39] as well as [2], [9], [7], [6], [31].
In [3] Albano-Bove-M. prove that the sufficient part of the Treves conjecture is actually false by showing a counterexample based on an operator whose stratification has just a single symplectic stratum. The study of that operator requires a precise semiclassical analysis of the spectral properties of suitable anharmonic Schrödinger operators.
The purpose of this paper is to discuss a simplified version P of the counterexample in [3]; this choice lead us to consider harmonic oscillators whose spectral properties are explicitly known. Taking advantage of this, the proof of the non analytic hypoellipticity of P will

be a little bit shorter. However we refer the reader to [3] for a more general and detailed discussion of the problem.

Although the operator P we shall consider here is less general than the one presented in [3], it is enough to get that a symplectic stratification does not imply analytic hypoellipticity, at least if the dimension of the stratum is ≥ 4. The necessary part of the conjecture, as far as we know, is still an open problem: If there is a non symplectic stratum, so that Hamilton leaves appear, then the operator P is not analytic hypoelliptic.

Here is the structure of the paper. In Section 2 we state the result by considering an operator having a single simplectic stratum.

Section 3 is devoted to the proof of the optimality of the s_0 Gevrey regularity. We construct a solution to $P_1 u = 0$ which is not better than Gevrey $s_0 > 1$; hence P is not analytic hypoelliptic. To obtain u we have to discuss a semiclassical spectral problem for a stationary Schrödinger equation with a symmetric double well potential depending on two parameters.

It is known that, since the bottom of the well is quadratic, for very small values of the Planck constant h the eigenvalues, which are simple and positive, behave like the eigenvalues of a harmonic oscillator.

2. Statement of the Result

The object of this section is to state the optimal Gevrey regularity result for the operator

$$P(x, D) = D_1^2 + D_2^2 + x_1^2 \left(D_3^2 + D_4^2 \right) + x_2^2 D_3^2 + x_2^4 D_4^2, \tag{1}$$

First of all we remark that both P is a sum of squares of vector fields with real analytic coefficients satisfying Hörmander bracket condition, i.e. the whole ambient space is generated when we take iterated commutators of the vector fields in the definition of P. In particular P is C^∞ hypoelliptic at the origin. This means that there exists an open neighborhood of the origin, Ω, such that for every open set $V \Subset \Omega$, $0 \in V$, we have,

$$Pu \in C^\infty(V) \Rightarrow u \in C^\infty(V),$$

for every distribution $u \in \mathscr{D}'(\Omega)$.

The characteristic manifold of P is the real analytic manifold

$$\mathrm{Char}(P) = \left\{(x,\xi) \in T^*\mathbb{R}^4 \setminus \{0\} \mid \ \xi_1 = \xi_2 = 0, x_1 = x_2 = 0, \xi_3^2 + \xi_4^2 > 0\right\}. \tag{2}$$

According to Treves conjecture one has to look at the strata associated with P.

The stratification associated with P is made up of a symplectic single stratum

$$\Sigma_1 = \left\{(0,0,x_3,x_4;0,0,\xi_3,\xi_4) \ |\xi_3^2 + \xi_4^2 > 0 \ \right\} = \mathrm{Char}(P).$$

According to the conjecture we would expect local real analyticity near the origin for the distribution solutions, u, of $P_1 u = f$, with a real analytic right hand side.

We are ready to state the theorem that is proved in the next section.

Theorem 2.1. *P is not analytic hypoelliptic near the origin.*

As a consequence of Theorem 2.1 we have

Corollary 2.1. *The sufficient part of Treves conjecture does not hold.*

We note however that for a single symplectic stratum of codimension 2 the conjecture is true (see [12]).

3. Proof of Theorem 2.1

In this section we construct a solution to the equation $Pu = 0$ which is not Gevrey s for any $s < s_0 = \frac{4}{3}$ and is defined in a neighborhood of the origin. This proves Theorem 2.1.

In fact we look for a function $u(x,y,t)$ defined on $\mathbb{R}_x \times \mathbb{R}_y \times (\mathbb{R}_t^+ \cup \{0\})$ and such that

$$P(x,D)A(u) = 0,$$

where

$$A(u)(x) = \int_{M_u}^{+\infty} e^{-i\rho x_4 + x_3 z(\rho)\rho^\theta - \rho^\theta} u(\rho^{\frac{1}{2}} x_1, \rho^\mu x_2, \rho) d\rho, \tag{1}$$

with

$$\theta = \frac{3}{4},$$

$\mu > 0$, $z(\rho)$ and $M_u > 0$ are to be determined. Here we assume that $x \in U$, a suitable neighborhood of the origin whose size will ultimately depend on the upper estimate for $z(\rho)$.

We have

$$
\begin{aligned}
&P(x, D)A(u)(x) \\
&\qquad = \int_{M_u}^{+\infty} e^{-i\rho x_4 + x_3 z(\rho)\rho^\theta - \rho^\theta} \Big[- \rho\partial_{x_1}^2 u - x_1^2 (z(\rho))^2 \rho^{\frac{3}{2}} u \\
&\qquad\qquad + x_1^2\rho^2 u - \rho^{2\mu}\partial_{x_2}^2 u - x_2^2(z(\rho))^2\rho^{\frac{3}{2}} u + x_2^4\rho^2 u \Big] d\rho.
\end{aligned}
$$

Rewriting the r.h.s. of the above relation in terms of the variables $y_1 = \rho^{\frac{1}{2}} x_1$, $y_2 = \rho^\mu x_2$, we obtain

$$
\begin{aligned}
P(x, D)A(u)(x) = &\int_{M_u}^{+\infty} e^{-i\rho x_4 + x_3 z(\rho)\rho^\theta - \rho^\theta} \Big[- \rho\partial_1^2 u - y_1^2 (z(\rho))^2 \rho^{\frac{1}{2}} u \\
&+ y_1^2 \rho u - \rho^{2\mu}\partial_2^2 u - y_2^2 (z(\rho))^2 \rho^{\frac{3}{2} - 2\mu} u + y_2^4 \rho^{2-4\mu} u \Big]_{\substack{y_1 = \rho^{1/2} x_1 \\ y_2 = \rho^\mu x_2}} d\rho.
\end{aligned}
$$

Choose now $\mu = \frac{1}{3}$. Then the above relation becomes

$$
\begin{aligned}
P_1(x, D)A(u)(x) = &\int_{M_u}^{+\infty} e^{-i\rho x_4 + x_3 z(\rho)\rho^\theta - \rho^\theta} \Big[- \rho \left(\partial_1^2 - y_1^2 \left(1 - (z(\rho))^2 \rho^{-\frac{1}{2}} \right) \right) u \\
&+ \rho^{\frac{2}{3}} \left(-\partial_2^2 - y_2^2 (z(\rho))^2 \rho^{\frac{1}{6}} + y_2^4 \right) u \Big]_{\substack{y_1 = \rho^{1/2} x_1 \\ y_2 = \rho^{\frac{1}{3}} x_2}} d\rho.
\end{aligned}
$$

We make the Ansatz that

$$
|z(\rho)| < M_u^{\frac{1}{4}}. \tag{2}
$$

We shall see that condition (2) will be satisfied.

Set $\tau(\rho) = \left(1 - (z(\rho))^2 \rho^{-\frac{1}{2}} \right)^{\frac{1}{4}}$. We note that, due to condition (2), the quantity in parentheses is positive. Choose

$$
u(y_1, y_2, \rho) = e^{-(\tau(\rho) y_1)^2/2} u_2(y_2, \rho), \tag{3}
$$

in such a way we have

$$
\left(-\partial_1^2 + y_1^2 \tau(\rho)^4 \right) e^{-(\tau(\rho) y_1)^2/2} = \tau(\rho)^2 e^{-(\tau(\rho) y_1)^2/2}. \tag{4}
$$

We then obtain

$$
\begin{aligned}
&P(x,D)A(u)(x)\\
&\qquad= \int_{M_u}^{+\infty} e^{-i\rho x_4 + x_3 z(\rho)\rho^\theta - \rho^\theta} u_1(\tau(\rho)\rho^{\frac{1}{2}} x_1) \Big[\Big\{\rho\left(1-(z(\rho))^2\rho^{-\frac{1}{2}}\right)^{\frac{1}{2}}\\
&\qquad\qquad + \rho^{\frac{2}{3}}\left(-\partial_2^2 - y_2^2 (z(\rho))^2 \rho^{\frac{1}{6}} + y_2^4\right)\Big\} u_2(y_2,\rho)\Big]_{y_2=\rho^{\frac{1}{3}}x_2} d\rho.
\end{aligned}
$$

Our next step is to find u_2 as a solution of the differential equation

$$
\left(1-(z(\rho))^2\rho^{-\frac{1}{2}}\right)^{\frac{1}{2}} \lambda u + \rho^{-\frac{1}{3}}\left(-\partial_2^2 - y_2^2(z(\rho))^2\rho^{\frac{1}{6}} + y_2^4\right)\Big) u = 0, \tag{5}
$$

where we wrote u instead of u_2 for the sake of simplicity. (5) becomes

$$
\left(1-(z(\rho))^2\rho^{-\frac{1}{2}}\right)^{\frac{1}{2}} \lambda u + \rho^{-\frac{1}{3}}\left(-\partial_2^2 + y_2^4\right) u - (z(\rho))^2\rho^{-\frac{1}{6}} y_2^2 u = 0,
$$

We set

$$
t = \rho^{-\frac{1}{6}}, \tag{6}
$$

so that the above equation becomes

$$
\left(1-(z_1(t))^2 t^3\right)^{\frac{1}{2}} \lambda u + t^2\left(-\partial_2^2 + y_2^4\right) u - (z_1(t))^2 t y_2^2 u = 0,
$$

where $z_1(t) = z(\rho)$. The latter equation can be turned into a stationary semiclassical Schrödinger equation if we perform the canonical dilation

$$
y_2 = y t^{-\frac{1}{2}} :
$$

$$
\left(1-(z_1(t))^2 t^3\right)^{\frac{1}{2}} u - t^3 \partial_y^2 u + y^4 u - (z_1(t))^2 y^2 u = 0.
$$

Set

$$
h = t^{\frac{3}{2}}. \tag{7}
$$

Note that t, h are small and positive for large ρ. Thus we may rewrite the above equation as

$$
\Big[\left(1-(z_2(h))^2 h^2\right)^{\frac{1}{2}} - h^2\partial_y^2 + y^4 - (z_2(h))^2 y^2\Big] u = 0, \tag{8}
$$

where $z_2(h) = z_1(t)$.

We want to show that there are countably many choices for the function $z_2(h)$ in such a way that equation (8) has a non zero solution in $L^2(\mathbb{R})$, which actually turns out to be a smooth rapidly decreasing function.

Since the term $(1-(z_2(h))^2h^2)^{\frac{1}{2}}$ is a scalar quantity commuting with the other terms, we consider first the operator

$$-h^2\partial_y^2 + y^4 - (z_2(h))^2y^2.$$

This is a Schrödinger operator with a symmetric double well potential. The latter is not positive everywhere; in order to work with a positive double well potential we subtract (and add) its minimum. This is

$$-\frac{1}{4}z_2^4.$$

Equation (8) becomes

$$\left[\left(1-(z_2(h))^2h^2\right)^{\frac{1}{2}} - \frac{1}{4}z_2(h)^4 - h^2\partial_y^2 + y^4 - (z_2(h))^2y^2 + \frac{1}{4}z_2(h)^4\right]u = 0, \tag{9}$$

Let us make the Ansatz that z_2 is a positive valued function. We make the canonical dilation

$$y = xz_2.$$

Then (9) becomes

$$\begin{aligned}\left[\left(1-(z_2(h))^2h^2\right)^{\frac{1}{2}} - \frac{1}{4}z_2(h)^4 - h^2z_2(h)^{-2}\partial_x^2\right.\\ \left. + z_2(h)^4x^4 - (z_2(h))^4x^2 + \frac{1}{4}z_2(h)^4\right]u = 0,\end{aligned} \tag{10}$$

whence

$$\left[\left(1-(z_2(h))^2h^2\right)^{\frac{1}{2}} z_2(h)^{-4} - \frac{1}{4} - h^2z_2(h)^{-6}\partial_x^2 + x^4 - x^2 + \frac{1}{4}\right]u = 0, \tag{11}$$

Let us consider the one dimensional Schrödinger operator

$$-\left(hz_2(h)^{-3}\right)^2\partial_x^2 + \left(x^2 - \frac{1}{2}\right)^2. \tag{12}$$

This kind of anharmonic oscillators play a deep role in the study of hypoelliptic problems (see, for instance, [24],[29], [30], [32]).

By [5] (Chapter 2, Theorem 3.1) the above Schrödinger operator has a discrete simple spectrum depending in a real analytic way on the parameter $hz_2(h)^{-3}$, for $h > 0$. Let us denote by

$$E\left(\frac{h}{z_2(h)^3}\right)$$

an eigenvalue. Let $u = u(x,h)$ be the corresponding eigenfunction. Then (11) becomes

$$\left(1-(z_2(h))^2h^2\right)^{\frac{1}{2}} z_2(h)^{-4} - \frac{1}{4} + E\left(\frac{h}{z_2(h)^3}\right) = 0. \tag{13}$$

Next we are going to solve the above equation w.r.t. z_2 as a function of h, for small positive values of h.

Proposition 3.1. *There is $h_0 > 0$ such that equation* (13) *implicitly defines a function $z_2 \in C([0,h_0[) \cap C^\omega(]0,h_0[)$. In particular*

$$z_2(h) \to \tilde{z} = \sqrt{2} > 0$$

when $h \to 0+$. Therefore we may always assume that

$$z_2(h) \in \left[\frac{1}{2}\tilde{z}, \frac{3}{2}\tilde{z}\right], \tag{14}$$

for $h \in [0,h_0[$.

Proof. The operator in (12) has a symmetric non negative double well potential with two non degenerate minima and unbounded at infinity. From Theorem 1.1 in [34] we deduce that

$$\lim_{\mu\to 0+} \frac{E(\mu)}{\mu} = e^* > 0. \tag{15}$$

We may then continue the function E, by setting $E(0) = 0$, as a function in $C([0,+\infty[) \cap C^\omega(]0,+\infty[)$.

Set

$$f(h,z) = \left(1-z^2h^2\right)^{\frac{1}{2}} z^{-4} - \frac{1}{4} + E\left(\frac{h}{z^3}\right). \tag{16}$$

Note that $f(0,\tilde{z}) = 0$. We want to show that the equation $f(h,z) = 0$ can be uniquely solved w.r.t. z for $h \in [0,h_0]$, for a suitable h_0.

We need the

Lemma 3.1. *[Feynmann-Hellman Formula] For every $\mu_0 > 0$ we have that $\partial_\mu E(\mu)$ exists and is bounded for $0 \leq \mu \leq \mu_0$.*

Proof of Lemma 3.1. Let

$$Q_\mu(x, \partial_x) = -\mu^2 \partial_x^2 + x^4 - x^2 + \frac{1}{4}. \tag{17}$$

From $Q_\mu v = E(\mu) v$ we get

$$\langle Q_\mu \partial_\mu v, v\rangle + 2\mu \|\partial_x v\|^2 = E(\mu)\langle \partial_\mu v, v\rangle + (\partial_\mu E(\mu))\|v\|^2.$$

Due to the self adjointness of Q_μ the first terms on both sides of the above identity are equal, so that

$$\partial_\mu E(\mu) = 2\mu \|\partial_x v\|^2 \geq 0,$$

for every $\mu > 0$, provided v is normalized. Again from $Q_\mu v = E(\mu) v$ we deduce that

$$\mu^2 \|\partial_x v\|^2 \leq \langle Q_\mu v, v\rangle = E(\mu). \tag{18}$$

Hence

$$0 \leq \partial_\mu E(\mu) \leq 2\frac{E(\mu)}{\mu} \to 2e^*$$

for $\mu \to 0+$. The existence of the right derivative in $\mu = 0$ is a consequence of (15). □

Now

$$\frac{\partial f}{\partial z}(h, z) = -\left(1 - z^2 h^2\right)^{-\frac{1}{2}} z^{-3} h^2 - 4\left(1 - z^2 h^2\right)^{\frac{1}{2}} z^{-3} - 3E'\left(\frac{h}{z^3}\right)\frac{h}{z^3} z^{-1}.$$

The above quantity is strictly negative if $(h, z) \in [0, h_0[\times [\tilde{z} - \delta, \tilde{z} + \delta]$, for a suitable choice of small h_0, δ.

Because of the definition of $\tilde{z}$ and (16), $f(h, \tilde{z} - \delta) > 0$, $f(h, \tilde{z} + \delta) < 0$ possibly taking a smaller h_0, δ, for $0 \leq h \leq h_0$. Since f is continuous and strictly decreasing on the h-lines there is a unique zero of the equation $f(h, z(h)) = 0$ with $z(h) \in [\tilde{z} - \delta, \tilde{z} + \delta]$ for $0 \leq h \leq h_0$.

For positive h trivially $z(h)$ is real analytic. Let us show that $z(h) \in C([0, h_0[)$. Arguing by contradiction assume that $z(h) \not\to \tilde{z}$ for $h \to 0+$. Then there is a sequence $h_k \to 0+$ such that $z(h_k) \to \hat{z} \neq \tilde{z}$. Then $0 = f(h_k, z(h_k)) \to f(0, \hat{z})$ which is false since $\tilde{z}$ is the only zero of $f(0, z) = 0$. □

Remark 3.1. *Let* h_0 *be the quantity define in Proposition 3.1. Set* $h_0 = \rho_0^{-\frac{1}{4}}$*. Choosing* $M_u \geq \max\{\rho_0, (\frac{3}{2}\tilde{z})^4\}$ *we have that the function* z_2 *is defined for* $\rho \geq M_u$ *and that* (2) *is satisfied, so that* $1 - z(\rho)^2\rho^{-\frac{1}{2}} > 0$.

We recall here the following result concerning the boundedness of the eigenfuntions of (17), which is a direct consequence of the Agmon estimates (see [19] or the book [18]).

Lemma 3.2. *Let* $v(x, \mu)$ *denote the* $L^2(\mathbb{R})$ *normalized eigenfunction corresponding to* $E(\mu)$*.* v *is rapidly decreasing w.r.t.* x *and satisfies the estimates*

$$|v^{(j)}(x, \mu)| \leq C_j \mu^{-(j+1)/2}, \tag{19}$$

for $x \in \mathbb{R}$*,* $C_j > 0$ *independent of* $0 < \mu < \mu_0$*,* $j = 0, 1, 2$*,* μ_0 *suitably small.*

Remark 3.2. *Note that because of Lemma 3.2 the formal method of sliding the differential operator* P *under the integration sign becomes legitimate, since a power singularity does no harm to the convergence of the integral in* ρ*.*

The integral $A(u)$ in (1) is a convergent integral since $u = e^{-(\tau(\rho)y_1)^2/2}u_2$, u_2 is a real analytic function of x_2, $e^{-(\tau(\rho)y_1)^2/2}$ is rapidly decreasing, while u_2 is rapidly decreasing w.r.t. x_2 and satisfies (19) with $\mu = \mathscr{O}\left(\rho^{\left(-\frac{1}{4}\right)}\right)$.

Then (11) holds for $0 \leq h \leq h_0$ and hence for $\rho \geq \rho_0$.

Going back to (1) we see that

$$P(x, D)A(u) = 0.$$

Before concluding the proof of the sharpness of the Gevrey $s_0 = 4/3$ regularity for $A(u)$, we need to make sure that the function $u = e^{-(\tau(\rho)y_1)^2/2}u_2$ does not have any effect on the convergence of the integral at infinity as well as on the Gevrey behavior of $A(u)$. As far as $u_1 = e^{-(\tau(\rho)y_1)^2/2}$ is concerned, this is obvious, since u_1 is a rapidly decreasing function of $\tau(\rho)\rho^{\frac{1}{2}}x_1$, where $\tau(\rho)$ is defined before equation (3), and, computing this function at the origin—as we need to do—will not affect the exponential in $A(u)$. We are thus left with $u_2 = u_2(\rho^{\frac{1}{3}}x_2, \rho)$. Note that u_2 is rapidly decreasing w.r.t. $\rho^{\frac{1}{3}}x_2$, and, due to Lemma 3.2, that u_2 is polynomially bounded in ρ, uniformly for x_2 in a neighborhood of the origin.

We have now to show that $u_2(0,\rho)$ can be bounded from below in order to be sure that $u_2(0,\rho)$ does not vanish with so high a speed to compromise the Gevrey 4/3 regularity.

Let us then choose $u_2 = u_2(x,\hbar)$. It is the ground state of the operator

$$Q_\hbar(x,\partial_x) = -\hbar^2\partial_x^2 + x^4 - x^2 + \frac{1}{4}, \tag{20}$$

where, by (11),

$$\hbar = \frac{h}{z_2(h)^3}. \tag{21}$$

Note that $\hbar$ tends to zero if and only if h tends to zero. u_2 is an eigenfunction of (20) corresponding to the lowest eigenvalue $E(\hbar)$

$$Q_\hbar(x,\partial_x)u_2 = E(\hbar)u_2. \tag{22}$$

It is wellknown that $E(\hbar)$ is a simple eigenvalue and u can be chosen strictly positive. Since the Schrödinger operator $Q_\hbar$ has a symmetric potential, so that its eigenfunctions are either even or odd functions w.r.t. the variable x_2. Therefore, u is an even function. Furthermore u_2 satisfies the following bound from below:

$$u_2(0,\hbar) \geq ce^{-\frac{C_1}{\hbar}},$$

where c, C_1 are positive constant independent of $\hbar$. This type of tunneling estimate could be deduced from the results of Helffer and Sjöstrand, [19] (see also the book [18], section 4.5). Another way of deriving such an estimate as a consequence of [19] is using the paper [25] by Martinez (see also [3], section 3).

We are now in a position to conclude the proof of Theorem 2.1. We recall that

$$\hbar = \mathscr{O}\left(\rho^{-\frac{1}{4}}\right) \quad \text{and} \quad \theta = \frac{3}{4}.$$

We compute

$$\begin{aligned}(-D_{x_4})^k A(u)(0) = \int_{M_u}^{+\infty} e^{-\rho^{\frac{3}{4}}}\rho^k u_1(0)u_2(0,\rho)d\rho \\ \geq u_1(0)C\int_{M_u}^{+\infty} e^{-\rho^{\frac{3}{4}}-C_1\rho^{\frac{1}{4}}}\rho^k d\rho \geq C_2^{k+1}k!^{\frac{4}{3}}.\end{aligned}$$

The last inequality above holds since

$$\begin{aligned}\int_{M_u}^{+\infty} e^{-\rho^{\frac{3}{4}}-C_1\rho^{\frac{1}{4}}}\rho^k d\rho &\geq C_\tau \int_{M_u}^{+\infty} e^{-c\rho^{\frac{3}{4}}}\rho^k d\rho \\ &= -C_\tau \int_0^{M_u} e^{-c\rho^{\frac{3}{4}}}\rho^k d\rho + C_2^{k+1}k!^{\frac{4}{3}} \\ &\geq C_2^{k+1}k!^{\frac{4}{3}}\left(1 - C_\tau C_2^{-(k+1)} M_u e^{-cM_u^{\frac{3}{4}}}\frac{M_u^k}{k!^{\frac{4}{3}}}\right) \geq C_3^{k+1}k!^{\frac{4}{3}},\end{aligned}$$

if k is suitably large and C_3 is suitable.

In particular, this shows that $A(u)$ is not an analytic function.

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