NONCOMMUTATIVE FOURIER ANALYSIS ON INVARIANT SUBSPACES: FRAMES OF UNITARY ORBITS AND HILBERT MODULES OVER GROUP VON NEUMANN ALGEBRAS

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Abstract. This is a joint work with E. Hernández, J. Parcet and V. Paternostro.
We will discuss the structure of bases and frames of unitary orbits of discrete groups in invariant subspaces of separable Hilbert spaces. These invariant spaces can be characterized, by means of Fourier intertwining operators, as modules whose rings of coefficients are given by the group von Neumann algebra, endowed with an unbounded operator valued pairing which defines a noncommutative Hilbert structure. Frames and bases obtained by countable families of orbits have noncommutative counterparts in these Hilbert modules, given by countable families of operators satisfying generalized reproducing conditions. These results extend key notions of Fourier and wavelet analysis to general unitary actions of discrete groups, such as crystallographic transformations on the Euclidean plane or discrete Heisenberg groups.

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Discuteremo la struttura di basi e frames ottenute da orbite di rappresentazioni unitarie di gruppi discreti in sottospazi invarianti di spazi di Hilbert separabili. Tali spazi invarianti possono essere caratterizzati, attraverso intrallacciamenti, come moduli il cui anello dei coefficienti é dato dall'algebra di von Neumann del gruppo, e sono dotati inoltre di una mappa sesquilineare a valori in spazi di operatori di convoluzione densamente definiti, che definiscono una struttura di Hilbert. Si puó mostrare che i frames e le basi associate a famiglie numerabili di orbite hanno una controparte in queste strutture di Hilbert, che ammettono sistemi riproducenti. Questi risultati estendono nozioni chiave di analisi di Fourier e wavelets a sistemi pi generali che possono includere trasformazioni geometriche per gruppi cristallografici o rappresentazioni di gruppi di Heisenberg discreti.

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1. Introduction

1.1. **Background and motivation.** One of the basic results of Fourier analysis is Shannon sampling theorem: any function in the Paley-Wiener space

$$\mathcal{PW} = \left\{ f \in L^2(\mathbb{R}) \,|\, \mathrm{supp}(\widehat{f}) \subset \left[-\frac{1}{2}, \frac{1}{2} \right] \right\},$$

where $\widehat{f}(\xi) = \displaystyle\int_{\mathbb{R}} e^{-2\pi i \xi x} f(x) dx$ is the Fourier transform of f, can be perfectly recovered from a collection of integer samples $\{f(k)\}_{k\in\mathbb{Z}}$ as

$$f(x) = \sum_{k\in\mathbb{Z}} f(k) \frac{\sin(\pi(x-k))}{\pi(x-k)} \tag{1}$$

where convergence is in $L^p(\mathbb{R})$ for all $p \geq 2$. This result lies at the roots of the problem of sampling in Hilbert spaces of entire functions, and the Paley-Wiener space is the prototype of a *shift-invariant space*, being it invariant under integer translations:

$$f \in \mathcal{PW} \;\Rightarrow\; \mathrm{t}_k f \in \mathcal{PW} \quad \forall\, k \in \mathbb{Z}$$

where $\mathrm{t}_k f(x) = f(x-k)$. This comes as a consequence of the Fourier intertwining of translations with modulations

$$\widehat{\mathrm{t}_k f}(\xi) = e^{-2i\pi\xi k} \widehat{f}$$

since a phase multiplication does not change the support of $\widehat{f}$.

A simple proof of the Shannon sampling theorem in L^2 can then be obtained as follows. Denote with $S(x) = \dfrac{\sin \pi x}{\pi x}$, and observe that $\widehat{S} = \mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}$. Since $\mathcal{PW}$ functions are analytic, thus continuous, then

$$\langle f, \mathrm{t}_k S\rangle_{L^2(\mathbb{R})} = \langle \widehat{f}, \widehat{\mathrm{t}_k S}\rangle_{L^2(\mathbb{R})} = \langle \widehat{f}, e^{-2i\pi k \cdot}\rangle_{L^2([-\frac{1}{2},\frac{1}{2}])} = f(k).$$

The conclusion follows because, by classic Fourier analysis,

$$L^2\big(\big[-\frac{1}{2}, \frac{1}{2}\big]\big) = \overline{\mathrm{span}\{e^{-2i\pi k\cdot}\}_{k\in\mathbb{Z}}}$$

and complex exponentials form an orthonormal basis. Thus $\{\mathrm{t}_k S\}_{k\in\mathbb{Z}}$ is an orthonormal basis of $\mathcal{PW}$, and the sampling formula (1) reads

$$f = \sum_{k\in\mathbb{Z}} \langle f, \mathrm{t}_k S\rangle_{L^2(\mathbb{R})} \mathrm{t}_k S.$$

Orthonormal bases of translates are also crucial for *multiresolution analysis* [22, 17]. This is defined by means of (dyadic) dilations $\delta_2 f(x) = 2^{\frac{1}{2}} f(2x)$ in terms of a sequence of nested closed subspaces $\ldots \subset V_j \subset V_{j+1} \subset \ldots$ of $L^2(\mathbb{R})$, and of a $\varphi \in V_0$, satisfying

1. $f \in V_j \iff \delta_2 f \in V_{j+1}$ for all $j \in \mathbb{Z}$
2. $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$
3. $\{\mathrm{t}_k \varphi\}_{k\in\mathbb{Z}}$ is an orthonormal basis of V_0 .

Multiresolution analysis is a standard construction for obtaining *orthonormal wavelets*. They are defined as systems of dilations and translations $\{\psi_{j,k} = \delta_2^j \mathrm{t}_k \psi\}_{j,k\in\mathbb{Z}}$ of a template

$$\psi \in L^2(\mathbb{R}) \ : \ \widehat{\psi}(2\xi) = e^{2\pi i \xi}\, \overline{m_0(\xi + \frac{1}{2})}\, \widehat{\varphi}(\xi)$$

where m_0 is the low pass filter associated to φ (1-periodic such that $\widehat{\varphi}(2\xi) = m_0(\xi)\widehat{\varphi}(\xi)$). The translates of ψ form an orthonormal basis of $V_0^\perp \cap V_1$, so that its wavelets system provides an orthonormal basis of $L^2(\mathbb{R})$. Such classical wavelets has been used for several purposes of pure and applied nature. For example, they allow to obtain an elegant proof of the so-called $T(1)$ theorem, by showing that the wavelet matrix representation $\langle \psi_{j,k}, T\psi_{j',k'}\rangle$ of a singular integral operator T is almost diagonal [23]. They also provide characterizations of several function spaces: for example, $f \in L^2(\mathbb{R})$ belongs to the *Sobolev space* $W^{s,2}(\mathbb{R})$ if and only if (see e.g. [17, Th. 6.18]) the weighted wavelet coefficients

$$\left\{ 2^{js} \langle f, \psi_{j,k}\rangle_{L^2(\mathbb{R})} \right\}_{j,k\in\mathbb{Z}}$$

belong to $\ell_2(\mathbb{Z}^2)$ for a smooth bandlimited mother wavelet ψ, and similar characterizations are available for Besov or Hardy spaces. But wavelets are widely used also for applications, especially in signal and image processing, data compression and inverse problems.

In many situations, it has turned out to be useful to weaken the condition of having orthonormal bases. In particular, one may have to deal with, or to look for, dictionaries which contain redundancies, in the sense that their elements are not necessarily linearly independent. This has led to the notion of *frames* [9]. Given a separable Hilbert space $\mathcal{H}$, a countable set $\Psi = \{\psi_j\}_{j\in\mathcal{I}} \subset \mathcal{H}$ is a frame if there exist two constants $0 < A \leq B < \infty$ such that

$$A\|f\|_{\mathcal{H}}^2 \leq \sum_{j\in\mathcal{I}} |\langle f, \psi_j\rangle_{\mathcal{H}}|^2 \leq B\|f\|_{\mathcal{H}}^2 \quad \forall\ f \in \overline{\operatorname{span}\Psi}. \tag{2}$$

A frame defines thus a quasi-isometry between the Hilbert space it generates and the ℓ_2 space of its indices (parameters), by means of the *analysis operator*

$$\mathrm{T}^*_\Psi f = \{\langle f, \psi_j\rangle_{\mathcal{H}}\}_{j\in\mathcal{I}}.$$

Its adjoint operator, called *synthesis operator*, can be defined on the dense space of finite sequences $c = \{c_j\}_{j\in\mathcal{I}} \in \ell_0(\mathcal{I})$ as

$$\mathrm{T}_\Psi c = \sum_{j\in\mathcal{I}} c_j \psi_j$$

and whenever Ψ is a frame one can define a *frame operator* as

$$\mathfrak{F}_\Psi f = \mathrm{T}_\Psi \mathrm{T}^*_\Psi = \sum_{j\in\mathcal{I}} \langle f, \psi_j\rangle_{\mathcal{H}} \psi_j.$$

Of course orthonormal systems are frames, and in that case the frame operator is simply the identity. In general, one has that the frame condition (2) is equivalent to

$$A\mathbb{I} \leq \mathfrak{F}_\Psi \leq B\mathbb{I}\,,$$

which implies in particular that the frame operator is invertible with a bounded inverse. This provides a canonical way to reconstruct an element $f \in \overline{\mathrm{span}\Psi}$ starting from the "generalized samples" provided by the analysis operator, as

$$f = \sum_{j\in\mathcal{I}} \langle f, \psi_j\rangle_{\mathcal{H}} \mathfrak{F}_\Psi^{-1} \psi_j = \sum_{j\in\mathcal{I}} \mathrm{T}^*_\Psi f(j)\, \mathring{\psi}_j.$$

For this reason, given a frame $\Psi = \{\psi_j\}_{j\in\mathcal{I}}$, the frame $\mathring{\Psi} = \{\mathring{\psi}_j = \mathfrak{F}_\Psi^{-1}\psi_j\}$ is called the *canonical dual frame*. Observe, however, that the coefficients of these reconstruction formulas are not unique, because frames are not bases[1]. Moreover, the inversion of the frame operator may be a too expensive task: in these cases it may be convenient to work with a tight frame, i.e. if $A = B$, since the associated frame operator is a multiple of the identity.

Finally, it is worth mentioning that multiresolution analysis, as well as many tools of standard linear decompositions, have been extended to frames, but frame theory has also several connections with other branches of mathematics [10, 21].

[1]A frame $\{\psi_j\}_{j\in\mathcal{I}}$ is an orthonormal basis if and only if $A = B = 1$ and $\|\psi_j\| = 1$ for all $j \in \mathcal{I}$.

These arguments have motivated a considerable amount of results aimed to settle a general structure theory for frames and bases in shift-invariant spaces [24, 6]. A key tool in the development of this program is a linear isometric isomorphism, called *fiberization mapping*:

$$\begin{array}{rccl} \mathcal{T}: & L^2(\mathbb{R}) & \to & L^2([0,1],\ell_2(\mathbb{Z})) \\ & f & \mapsto & \left\{\widehat{f}(\cdot+k)\right\}_{k\in\mathbb{Z}}. \end{array} \tag{3}$$

One of the fundamental results of this approach is the following. (Cf. [6]).

Theorem 1. *Let $\{\phi_j\}_{j\in\mathcal{I}}$ be a countable family in $L^2(\mathbb{R})$. Then the system of translates $E_\phi = \{\mathrm{t}_k\phi_j\}_{\substack{k\in\mathbb{Z}\\ j\in\mathcal{I}}}$ is a frame with bounds $0 < A \leq B < \infty$ if and only if the system $\Phi(\xi) = \{\mathcal{T}\phi_j(\xi)\}_{j\in\mathcal{I}} \subset \ell_2(\mathbb{Z})$ is a frame with bounds $0 < A \leq B < \infty$ for a.e. $\xi \in [0,1]$.*

This result allows to get rid of translations and reduce to families on $\ell_2(\mathbb{Z})$ with the same cardinality as $\{\phi_j\}_{j\in\mathcal{I}}$, parametrized by the compact space $[0,1]$. Its proof relies basically on the Fourier analysis implemented by the isometry $\mathcal{T}$, which satisfies the intertwining property

$$\mathcal{T}\mathrm{t}_k f(\xi) = e^{-2\pi i k\xi}\mathcal{T}f(\xi). \tag{4}$$

When only one generator is considered, as in the case of the Paley-Wiener space, frames of translates can be characterized (cf. [5]) in terms of the *bracket map*

$$\begin{array}{rccl} [\cdot,\cdot]: & L^2(\mathbb{R})\times L^2(\mathbb{R}) & \to & L^1([0,1]) \\ & (\varphi,\psi) & \mapsto & [\varphi,\psi](\xi) = \sum_{k\in\mathbb{Z}} \widehat{\varphi}(\xi+k)\overline{\widehat{\psi}(\xi+k)}, \end{array} \tag{5}$$

a sesquilinear map whose Fourier coefficients are the matrix elements of the translation operator

$$\int_0^1 [\varphi,\psi](\xi)e^{2\pi i k\xi}d\xi = \langle\varphi,\mathrm{t}_k\psi\rangle_{L^2(\mathbb{R})}. \tag{6}$$

Theorem 2. *Given $\phi \in L^2(\mathbb{R})$, the system of translates $\{\mathrm{t}_k\phi\}_{k\in\mathbb{Z}}$ is a frame with bounds $0 < A \leq B < \infty$ if and only if*

$$A\mathbf{1}_{\Omega_\phi} \leq [\phi,\phi] \leq B\mathbf{1}_{\Omega_\phi} \qquad \textit{for a.e. } \xi \in [0,1]$$

where $\Omega_\phi = \{\xi \in [0,1] \,:\, [\phi,\phi](\xi) \neq 0\}$.

While there are direct proofs of this result, it actually can also be deduced from the previous theorem by observing that

$$[\varphi, \psi](\xi) = \langle \mathcal{T}\varphi(\xi), \mathcal{T}\psi(\xi)\rangle_{\ell_2(\mathbb{Z})}.$$

1.2. **Abstract setting and examples.** Shift invariant spaces and systems of translates can be considered as special cases of the following setting.

> Let Γ be a discrete group, $\mathcal{H}$ a separable Hilbert space, and $\Pi : \Gamma \to \mathcal{U}(\mathcal{H})$ a unitary representation. A closed subspace $V \subset \mathcal{H}$ is (Γ, Π)-invariant if
>
> $$f \in V \ \Rightarrow\ \Pi(\gamma) f \in V \quad \forall\, \gamma \in \Gamma.$$
>
> Given a countable set $\{\phi_j\}_{j\in\mathcal{I}} \subset \mathcal{H}$, denote its family of orbits by
>
> $$E^\Gamma_\phi = \{\Pi(\gamma)\phi_j\}_{\substack{\gamma\in\Gamma\\ j\in\mathcal{I}}}$$
>
> and denote by $S^\Gamma_\phi = \overline{\mathrm{span} E^\Gamma_\phi}^{\mathcal{H}}$ the (Γ, Π)-invariant they generate.

Motivated by the previous discussion we seek to characterize such invariant spaces in terms of some multiplier property, and to characterize the families E^Γ_ϕ which give rise to frames.

In order to exploit the invariance with respect to the group action, an effective Fourier analysis is needed[2]. When Γ is Abelian, this is provided by Pontryagin duality and the group of characters. For non-Abelian groups, a different notion of duality is needed. The one based on the *group von Neumann algebra* is adequate to deal with all discrete groups and, with minor changes, it can be extended to unimodular groups [20]. This duality consists of, roughly speaking, *associating to a function on the group the (spectrum of the) group convolution operator by that function*. When the group is Abelian, its group von Neumann algebra is isomorphic (via Pontryagin duality) to the algebra of essentially bounded functions on its dual group, and the spectrum of the convolution

[2]The Fourier transform is a fundamental tool to exploit group symmetries. In the typical Abelian setting, an operator that is invariant under translations, and thus can be realized as a convolution with a – possibly singular – integral kernel (see e.g. a sharp statement in [25, Th. 3.16]), can be studied as a multiplier by the Fourier transform of the kernel. However, when the symmetry is related to a non-Abelian group, Fourier duality produces noncommutative objects, so multipliers themselves are operator-valued.

operator coincides with the ordinary Fourier transform. A sketch of this construction will be given in §2, together with some facts about noncommutative L^p spaces, the analogues of Plancherel theorem and the L^1-uniqueness of Fourier coefficients.

Some examples of relevant unitary representations of non-Abelian discrete groups can be useful to get an idea of possibly related problems.

The discrete Heisenberg group

The discrete Heisenberg group $\mathbb{H}^n_d$ is the subgroup of $\mathbb{H}^n$ with underlying set $\mathbb{Z}^n \times \mathbb{Z}^n \times \mathbb{Z}$ and composition law

$$(j, k, l) \cdot (j', k', l') = (j + j', k + k', l + l' + k.j').$$

It can be considered with respect to the representation given by left discrete translations on $L^2(\mathbb{H}^n)$, which was studied in [1], for example with the purpose of combining it with homogeneous dilations. Another typical unitary representation is the Schrödinger representation on $L^2(\mathbb{R}^n)$

$$\pi_{a,b}(j, k, l)f(x) = e^{-2\pi iabl}e^{2\pi iaj.x}f(x - bk)$$

which lies at the heart of *Gabor systems*. When $f \in L^2(\mathbb{R}^n)$ is a localized function, such as a Gaussian, a linear decomposition along its $\pi_{a,b}$ orbit defines (up to the central variable) a localized Fourier transform.

Crystallographic groups

For $x \in \mathbb{R}^n$ and $r \in O(n)$ we can define a composition law on $\mathbb{G} = \mathbb{R}^n \times O(n)$ as

$$(x, r) \cdot (x', r') = (x + rx', rr').$$

Let B be a full rank lattice of $\mathbb{R}^n$ and let R be a finite subgroup of $O(n)$ such that $rB = B$ for all $r \in R$. Then $\Gamma = B \rtimes R$ is a discrete subgroup of $\mathbb{G}$ that is called a crystallographic group. These are transformation groups that can be studied in terms of the so-called quasiregular representation on $L^2(\mathbb{R}^n)$ given by

$$\pi(b, r)\psi(x) = \psi(r^{-1}(x - b)),$$

and linear decompositions along their orbits can be used to extract information contained in anisotropic data such as images.

2. Fourier analysis in terms of the group von Neumann algebra

If $\mathbb{G}$ is a locally compact Abelian (LCA) group and $f \in L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$, the Fourier-Pontryagin transform is defined via the characters $\widehat{\mathbb{G}} = \{\alpha : \mathbb{G} \overset{\text{homo}}{\to} \mathbb{C},\, |\alpha| = 1\}$ by

$$\mathcal{F}f(\alpha) := \int_G f(g)\alpha(g)dg$$

where dg is the Haar measure of $\mathbb{G}$. If we denote by L_f convolution with f in $L^2(\mathbb{G})$ and by $M_{\mathcal{F}f}$ multiplication by $\mathcal{F}f$ in $L^2(\widehat{\mathbb{G}})$, then L_f and M_f are bounded on $L^2(\mathbb{G})$ and $L^2(\widehat{\mathbb{G}})$, respectively, and

$$M_{\mathcal{F}f} = \mathcal{F}L_f\mathcal{F}^{-1}.$$

Analogously, one can define the Fourier transform of any measurable f for which L_f is a closed densely defined operator on $L^2(\mathbb{G})$. Then Pontryagin theory allows one to prove many classical results of Fourier analysis such as Plancherel theorem or L^1 uniqueness in terms of this abstract correspondence.

On general locally compact groups Pontryagin duality is not available. Instead one can consider the convolution operator L_f as a Fourier transform of f, paying attention to the fact that for non-Abelian groups convolution is not commutative. Given a discrete group Γ, the space of bounded left convolution operators on $\ell_2(\Gamma)$ forms a C*-algebra that is called the left von Neumann algebra of Γ, denoted by $\mathscr{L}(\Gamma)$. The notion of Fourier duality for discrete (and more general locally compact) groups in terms of their von Neumann algebra was introduced in several contexts [20, 14, 13, 11].

Recall that, in general, a von Neumann algebra $\mathcal{M}$ (see e.g. [19, Vol.1, Chapt. 5]) is a unital weak-operator closed C*-algebra[3]. A von Neumann algebra $\mathcal{M}$ is equivalently characterized as a subalgebra of the algebra of bounded operators over a Hilbert space $\mathscr{B}(\mathcal{H})$ satisfying $\mathcal{M}'' = \mathcal{M}$, where $\mathcal{M}' = \{a \in \mathscr{B}(\mathcal{H}) \,|\, ax = xa \;\forall\, x \in \mathcal{M}\}$ denotes the commutant (see e.g. [12, Th. 6.4, Def. 7.1]).

[3] A C*-algebra $\mathcal{M}$ is a Banach algebra, i.e. an associative algebra endowed with a norm with respect to which it is a Banach space and that satisfies $\|xy\| \leq \|x\|\,\|y\| \;\forall\; x, y \in \mathcal{M}$, that has an involution $* : \mathcal{M} \to \mathcal{M}$ such that $\|x^*x\| = \|x\|^2 \;\forall\; x \in \mathcal{M}$. Given a Hilbert space $\mathcal{H}$, the space of its bounded operators $\mathscr{B}(\mathcal{H})$ is a C^*-algebra where an involution is given by the operation of taking the adjoint. Actually every C^*-algebra can be realized as a subalgebra of $\mathscr{B}(\mathcal{H})$ for some $\mathcal{H}$ (see e.g. [12, Th. 5.17]).

In order to construct the left von Neumann algebra $\mathscr{L}(\Gamma)$ of a discrete group Γ as a subalgebra of $\mathscr{B}(\ell_2(\Gamma))$, observe that given $f \in \ell_0(\Gamma)$, i.e. with finite support, left convolution with f defines a bounded operator on $\ell_2(\Gamma)$ satisfying

$$L_f v(\gamma_0) = f * v(\gamma_0) = \sum_{\gamma\in\Gamma} f(\gamma) v(\gamma^{-1}\gamma_0) = \sum_{\gamma\in\Gamma} f(\gamma)\lambda(\gamma)v(\gamma_0)$$

where $\lambda : \Gamma \to \mathcal{U}(\ell_2(\Gamma))$ is the left regular representation. Thus we can write

$$L_f = \sum_{\gamma\in\Gamma} f(\gamma)\lambda(\gamma) \ \in \ \mathrm{span}\{\lambda(\gamma)\}_{\gamma\in\Gamma} \subset \mathscr{B}(\ell_2(\Gamma)).$$

In analogy to classical Fourier analysis on $\mathbb{Z}$, we will call *trigonometric polynomials* the operators obtained by finite linear combinations of the left regular representation. The *left von Neumann Algebra of* Γ is then defined to be the closure of the trigonometric polynomials in the weak operator topology of $\ell_2(\Gamma)$, i.e. by

$$\mathscr{L}(\Gamma) := \overline{\mathrm{span}\{\lambda(\gamma)\}_{\gamma\in\Gamma}}^{\mathrm{WOT}}.$$

We will denote by $\tau : \mathscr{L}(\Gamma) \to \mathbb{C}$ the Haar trace, given by

$$\tau(F) = \langle F\delta_{\mathrm{e}}, \delta_{\mathrm{e}}\rangle_{\ell_2(\Gamma)}$$

where e denotes the identity element of Γ and $\{\delta_\gamma\}_{\gamma\in\Gamma}$ is the canonical basis of $\ell_2(\Gamma)$.

Note that the functional $\tau : \mathscr{L}(\Gamma) \to \mathbb{C}$ is

i. tracial, i.e. $\tau(F_1F_2) = \tau(F_2F_1)$ for all $F_1, F_2 \in \mathscr{L}(\Gamma)$

ii. normal, i.e. for all $\{F_n\}_{n\in\mathbb{N}} \subset \mathscr{L}(\Gamma)$ that converges to $F \in \mathscr{L}(\Gamma)$, then $\tau(F_n) \to \tau(F)$

iii. finite, i.e. $\tau(F^*F) < \infty$ for all $F \in \mathscr{L}(\Gamma)$

iv. faithful, i.e. $\tau(F^*F) = 0 \Rightarrow F = 0$ for all $F \in \mathscr{L}(\Gamma)$.

These properties are easily checked for trigonometric polynomials, and thus extend by density. To see traciality, it suffices to observe that

$$\tau(\lambda(\gamma_1)\lambda(\gamma_2)) = \delta_{\gamma_1,\gamma_2}, \tag{7}$$

which implies that $\tau(F_1F_2) = \sum_{\gamma\in\Gamma} f_1(\gamma)f_2(\gamma) = \tau(F_2F_1)$, for $F_i = \sum_{\gamma\in\Gamma} f_i(\gamma)\lambda(\gamma)$, $i = 1, 2$. Normality is a direct consequence of the WOT closure of $\mathscr{L}(\Gamma)$, while finiteness and faithfulness are direct consequences of $\tau(F^*F) = \|F\delta_{\mathrm{e}}\|^2_{\ell_2(\Gamma)}$, by definition of τ.

A normal, (semi)finite, faithful trace endows a von Neumann algebra with the structure of a *noncommutative measure space*. This statement can be intuitively understood as a consequence of the following definition. In analogy to classical Fourier analysis, we define the *Fourier coefficients* a given $F \in \mathscr{L}(\Gamma)$ by

$$\widehat{F}(\gamma) = \tau(\lambda(\gamma)^* F) \tag{8}$$

so that F has a Fourier series $F = \sum_{\gamma\in\Gamma} \widehat{F}(\gamma)\lambda(\gamma)$. The trace of F is thus

$$\tau(F) = \widehat{F}(\mathrm{e}),$$

which in the Abelian setting coincides with the definition of integral[4].

Moreover, for $1 \leq p < \infty$ we can define the norms

$$\|F\|_p = \tau(|F|^p)$$

for trigonometric polynomials F, where the modulus is defined as the selfadjoint operator $|F| = \sqrt{F^*F}$. This provides a notion of *noncommutative L^p spaces* on $\mathscr{L}(\Gamma)$, as

$$L^p(\mathscr{L}(\Gamma)) = \overline{\operatorname{span}\{\lambda(\gamma)\}_{\gamma\in\Gamma}}^{\|\cdot\|_p} \quad 1 \leq p < \infty,$$

while for $p = \infty$ we can set $L^\infty(\mathscr{L}(\Gamma)) = \mathscr{L}(\Gamma)$. For $p < \infty$ the elements of $L^p(\mathscr{L}(\Gamma))$ are closed and densely defined (possibly unbounded) operators on $\ell_2(\Gamma)$. Moreover, by Hölder inequality (which still holds for these norms) and the finiteness of τ we have that $L^q(\mathscr{L}(\Gamma)) \subset L^p(\mathscr{L}(\Gamma))$ for all $1 \leq q < p \leq \infty$, and the Fourier coefficients (8) are well defined for any $F \in L^p(\mathscr{L}(\Gamma))$. The following theorem summarizes fundamental consequences of this noncommutative Fourier duality (see e.g. [2, Lem 2.1, 2.2]).

Theorem 3.

Uniqueness: *Let $F \in L^1(\mathscr{L}(\Gamma))$. If $\widehat{F}(\gamma) = 0$ for all $\gamma \in \Gamma$, then $F = 0$.*

Plancherel: *Let $F \in L^2(\mathscr{L}(\Gamma))$. Then $\widehat{F} \in \ell_2(\Gamma)$ and $\|F\|_2 = \|\widehat{F}\|_{\ell_2(\Gamma)}$. Moreover, if $f \in \ell_2(\Gamma)$ then $\sum_{\gamma\in\Gamma} f(\gamma)\lambda(\gamma)$ converges in $L^2(\mathscr{L}(\Gamma))$ to an operator F such that $\widehat{F}(\gamma) = f(\gamma)$ for all $\gamma \in \Gamma$.*

[4]Think of it in $\mathbb{Z}$: by definition $\widehat{f}(k) = \displaystyle\int_0^1 f(\xi)e^{2\pi i k\xi}d\xi$, so $\widehat{f}(0)$ is the integral of f.

While the Plancherel-type result is a simple consequence of $\{\lambda(\gamma)\}_{\gamma\in\Gamma}$ being an orthonormal basis of $L^2(\mathscr{L}(\Gamma))$ by (7), the uniqueness result is somewhat deeper and based on the fact that $\mathscr{L}(\Gamma)$ is the Banach dual of $L^1(\mathscr{L}(\Gamma))$. The predual of $\mathscr{L}(\Gamma)$ was introduced in [14] as a function space, called the *Fourier algebra* defined by

$$\mathcal{A}(\Gamma) = \{f : \Gamma \to \mathbb{C} \,:\, \exists \psi_1, \psi_2 \in \ell_2(\Gamma) \text{ such that } f(\gamma) = \langle \psi_1, \lambda(\gamma)\psi_2\rangle_{\ell_2(\Gamma)}\}$$

which is isomorphic to $L^1(\mathscr{L}(\Gamma))$ because of a von Neumann algebra has a unique predual.

3. The bracket and the Helson maps

The bracket map (5) is a central tool for the study of shift-invariant spaces. By the $L^1(\mathbb{T})$ uniqueness of Fourier coefficients, it can be characterized in terms of condition (6). This approach was adopted in [16], where a generalized bracket map was defined in terms of the Fourier coefficients of a unitary representation of an Abelian group. When the group is not necessarily Abelian, an effective notion is the following [2].

Definition 4. *Let* Π *be a unitary representation of a discrete group* Γ *on a separable Hilbert space* $\mathcal{H}$*. We say that* Π *is* dual integrable *if there exists a sesquilinear map* $[\cdot,\cdot] : \mathcal{H}\times\mathcal{H} \to L^1(\mathscr{L}(\Gamma))$*, called* bracket map*, satisfying*

$$\langle \varphi, \Pi(\gamma)\psi\rangle_{\mathcal{H}} = \tau([\varphi,\psi]\lambda(\gamma)^*) \quad \forall\, \varphi,\psi \in \mathcal{H}\,,\ \forall\, \gamma \in \Gamma.$$

In such a case we will call $(\Gamma, \Pi, \mathcal{H})$ *a* dual integrable triple*.*

By [2, Prop. 3.2] and [4, Prop. 21] the bracket map satisfies the properties.

Proposition 5. *Let* $(\Gamma, \Pi, \mathcal{H})$ *be a dual integrable triple. Then*

i. $[\psi,\psi] \geq 0$*, and* $\|[\psi,\psi]\|_1 = \|\psi\|^2_{\mathcal{H}}$.
ii. $[\psi_1,\psi_2]^* = [\psi_2,\psi_1]$.
iii. *Given* $F \in \mathscr{L}(\Gamma)$*, denote by* $\mathcal{P}_F$ *the bounded operator on* $\mathcal{H}$ *given by*

$$\mathcal{P}_F = \sum_{\gamma\in\Gamma} \widehat{F}(\gamma)\Pi(\gamma)\,.$$

Then

$$[\mathcal{P}_F\psi_1,\psi_2] = F[\psi_1,\psi_2]\ ,\quad [\psi_1,\mathcal{P}_F\psi_2] = [\psi_1,\psi_2]F^*$$

for all $\psi,\psi_1,\psi_2 \in \mathcal{H}$*.*

Sketch of the proof. To see iii. consider the special case of F having only one nonzero Fourier coefficient $\widehat{F}(\gamma_0) = 1$. In this case $\mathcal{P}_F = \Pi(\gamma_0)$, and by traciality of τ we get

$$\begin{aligned}\tau([\psi_1, \Pi(\gamma_0)\psi_2]\lambda(\gamma)^*) &= \langle \psi_1, \Pi(\gamma)\Pi(\gamma_0)\psi_2\rangle_{\mathcal{H}} = \langle \psi_1, \Pi(\gamma\gamma_0)\psi_2\rangle_{\mathcal{H}}\\ &= \tau([\psi_1, \psi_2]\lambda(\gamma\gamma_0)^*) = \tau([\psi_1, \psi_2]\lambda(\gamma_0)^*\lambda(\gamma)^*)\,, \quad \forall\, \gamma \in \Gamma.\end{aligned}$$

By the $L^1(\mathscr{L}(\Gamma))$-uniqueness theorem, this implies $[\psi_1, \Pi(\gamma_0)\psi_2] = [\psi_1, \psi_2]\lambda(\gamma_0)^*$. □

Dual integrability is equivalent to square integrability, which is a minimal request for having a reproducing system. Moreover, the existence of an L^1 bracket map is equivalent to the existence of a map that generalizes the fiberization mapping (3), intertwining the representation π with the left regular representation, as in (4). The following result is contained in [2, Th. 4.1], [4, Th. 4] and [4, Prop. 21].

Theorem 6. *Let Π be a unitary representation of the discrete group Γ on the Hilbert space $\mathcal{H}$. The following are equivalent:*

i. *$(\Gamma, \Pi, \mathcal{H})$ is a dual integrable triple.*

ii. *Π is square integrable, i.e. there exists a dense subspace $\mathcal{D}$ of $\mathcal{H}$ such that*

$$\big\{\langle \varphi, \Pi(\gamma)\psi\rangle_{\mathcal{H}}\big\}_{\gamma\in\Gamma} \in \ell_2(\Gamma) \quad \forall\, \varphi \in \mathcal{H}\,,\ \forall\, \psi \in \mathcal{D}.$$

iii. *There exist a σ-finite measure space $(\mathcal{M}, \nu)$ and an isometry*

$$\mathcal{T} : \mathcal{H} \to L^2((\mathcal{M}, \nu), L^2(\mathscr{L}(\Gamma)))\ ,$$

called Helson map*, satisfying*

$$\mathcal{T}[\mathcal{P}_F\varphi] = F\mathcal{T}[\varphi] \quad \forall\, F \in \mathscr{L}(\Gamma),\ \forall\, \varphi \in \mathcal{H}. \tag{9}$$

A Helson map can actually be given via a concrete construction, which we sketch here. For $\psi \in \mathcal{H}$, let $\langle\psi\rangle_\Gamma = \overline{\mathrm{span}\{\Pi(\gamma)\psi\}_{\gamma\in\Gamma}}$ be the space generated by its orbit. It is easy to construct a countable family $\{\psi_j\}_{j\in\mathcal{I}}$ such that $\mathcal{H}$ has the orthogonal decomposition

$$\mathcal{H} = \bigoplus_{j\in\mathcal{I}} \langle\psi_j\rangle_\Gamma.$$

For each orbit, we define a map $S_j : \sum_\gamma f(\gamma)\Pi(\gamma)\psi_j \mapsto \sum_\gamma f(\gamma)\lambda(\gamma)$ which provides an isometric isomorphism of $\langle\psi_j\rangle_\Gamma$ onto the noncommutative weighted space

$$L^2(\mathscr{L}(\Gamma), [\psi_j, \psi_j]) = \overline{\mathscr{L}(\Gamma)/\{\|\cdot\|_{2,[\psi_j,\psi_j]} = 0\}}^{\|\cdot\|_{2,[\psi_j,\psi_j]}}\ , \quad \|F\|_{2,[\psi_j,\psi_j]} = \tau\big(|F|^2[\psi_j, \psi_j]\big)^{\frac{1}{2}}.$$

A realization of a Helson map (with $\mathcal{M} = \mathcal{I}$) can then be obtained as

$$\begin{array}{rcl} \mathcal{T}: \ \mathcal{H} & \to & \ell_2(\mathcal{I}, L^2(\mathscr{L}(\Gamma))) \\ \varphi & \mapsto & \left\{ S_j[\mathbb{P}_{\langle\psi_j\rangle_\Gamma}\varphi]\,[\psi_j,\psi_j]^{\frac{1}{2}} \right\}_{j\in\mathcal{I}}. \end{array} \tag{10}$$

This satisfies, as a special case of (9), the Fourier intertwining relation

$$\begin{aligned} \mathcal{T}[\Pi(\gamma)\varphi] &= \left\{ S_j[\mathbb{P}_{\langle\psi_j\rangle_\Gamma}\Pi(\gamma)\varphi]\,[\psi_j,\psi_j]^{\frac{1}{2}} \right\}_{j\in\mathcal{I}} = \left\{ S_j[\Pi(\gamma)\mathbb{P}_{\langle\psi_j\rangle_\Gamma}\varphi]\,[\psi_j,\psi_j]^{\frac{1}{2}} \right\}_{j\in\mathcal{I}} \\ &= \left\{ \lambda(\gamma) S_j[\mathbb{P}_{\langle\psi_j\rangle_\Gamma}\varphi]\,[\psi_j,\psi_j]^{\frac{1}{2}} \right\}_{j\in\mathcal{I}} = \lambda(\gamma)\mathcal{T}[\varphi]. \end{aligned}$$

Moreover, a Helson map provides an expression for the bracket map via

$$[\varphi,\psi] = \int_{\mathcal{M}} \mathcal{T}[\varphi]^*(x)\mathcal{T}[\psi](x)d\nu(x). \tag{11}$$

The notion of Helson map is not unique for a given dual integrable representation, and the map (10) is not the only possible realization. When the representation Π is given by a measurable group action over a measure space, satisfying a tiling condition, another realization can be obtained in terms of a generalized Zak transform [2, 3, 4]. However, the bracket map is unique at the level of Fourier coefficients. The object defined by (10) is thus independent of the choice of Helson map.

In the case of integer translations on $L^2(\mathbb{R})$, it is possible to obtain the map (3) via Construction (10) by choosing a Shannon family $\{\psi_j\}_{j\in\mathbb{Z}}$ defined by

$$\widehat{\psi}_j = \mathbf{1}_{[j,j+1]}.$$

Then, the bracket map defined by (11) coincides (up to intertwining with the ordinary Fourier transform) with the one in (5).

4. Invariant spaces and Hilbert modules

A Helson map endows a dual integrable triple with an isometry that maps $\mathcal{H}$ into a fibered Hilbert space $L^2(\mathcal{M},\nu)\overline{\otimes}L^2(\mathscr{L}(\Gamma))$, where the image of each (Γ,Π)-invariant space generated by an orbit is a closed subspace of $L^2(\mathscr{L}(\Gamma))$ (so it is isometrically isomorphic to a closed subspace of $\ell_2(\Gamma)$ via the Plancherel theorem).

A remarkable consequence of Condition (9), expressed by the next theorem, is that any (Γ,Π)-invariant subspace of $\mathcal{H}$ can be characterized as a closed subspace of $L^2((\mathcal{M},\nu), L^2(\mathscr{L}(\Gamma)))$

that is invariant under some noncommutative multiplier provided by the group von Neumann algebra. This extends Abelian characterizations such as those in [7]. Thus, it is equivalent to say that the latter space is a *module* whose algebra of coefficients is $\mathscr{L}(\Gamma)$.

Theorem 7. *Let* $(\Gamma, \Pi, \mathcal{H})$ *be a dual integrable triple, let* V *be a closed subspace of* $\mathcal{H}$*, and let* $\mathrm{M} = \mathcal{T}(V)$*. Then* V *is* (Γ, Π)*-invariant if and only if*

$$F\mathrm{M} \subset \mathrm{M} \quad \forall\, F \in \mathscr{L}(\Gamma).$$

Modules are linear structures that allow many of the constructions associated to vector spaces by taking linear combinations with coefficients in associative algebras. In this case, it is also possible to define, on any submodule M of $\mathcal{T}(\mathcal{H})$, an operator-valued inner product that can be obtained from the bracket map via

$$\begin{array}{rcl} \{\cdot,\cdot\} = [\cdot,\cdot] \circ \mathcal{T}^{-1} : \quad \mathrm{M} \times \mathrm{M} & \to & L^1(\mathscr{L}(\Gamma)) \\ (\Phi, \Psi) & \mapsto & \{\Phi, \Psi\} = [\mathcal{T}^{-1}\Phi, \mathcal{T}^{-1}\Psi]. \end{array}$$

It is in fact a positive definite sesquilinear map satisfying

$$\{F\Phi, \Psi\} = F\{\Phi, \Psi\}\ , \quad \{\Phi, F\Psi\} = \{\Phi, \Psi\}F^*. \tag{12}$$

All $\mathscr{L}(\Gamma)$-modules given as image under a Helson map of an invariant subspace are then endowed with a so-called $L^2(\mathscr{L}(\Gamma))$*-Hilbert module* structure, in the sense of [18]. Such Hilbert modules are also Hilbert spaces, with scalar product

$$\langle \Phi, \Psi \rangle = \tau(\{\Phi, \Psi\}). \tag{13}$$

Condition (12) ensures that the noncommutative inner product $\{\cdot,\cdot\}$ is compatible with an algebra of coefficients that incorporates the group action, in the sense of the intertwining (9) implemented by the Helson map. This is precisely the setting in which to look for a result like Theorem 1. The significance of such a result would, however, be highly dependent on the availability of a sufficiently rich notion of frames in this noncommutative environment.

A notion of modular frames in general $L^2(\mathscr{L}(\Gamma))$-Hilbert modules is developed in [4], and it can be summarized as follows.

Theorem 8. *Let* M *be an* $L^2(\mathscr{L}(\Gamma))$*-Hilbert module, let* $\Phi = \{\Phi_j\}_{j\in\mathcal{I}} \subset \mathrm{M}$ *be a countable family and denote by*

$$\mathrm{M}_\Phi = \overline{\mathrm{span}_{\mathscr{L}(\Gamma)}\{\Phi_j\}}$$

the closed submodule it generates, where the closure is taken with respect to the norm induced by (13). Suppose Φ *satisfies the condition*

$$A\{\Psi,\Psi\} \leq \sum_{j\in\mathcal{I}} |\{\Phi_j,\Psi\}|^2 \leq B\{\Psi,\Psi\} \qquad \forall\ \Psi \in \mathrm{M}_\Phi$$

for two constants $0 < A \leq B < \infty$*. Then the operator* $\widetilde{\mathfrak{F}}_\Phi$ *given by*

$$\widetilde{\mathfrak{F}}_\Phi \Psi = \sum_{j\in\mathcal{I}} \{\Psi,\Phi_j\}\Phi_j \tag{14}$$

is well-defined, bounded and invertible on M_Φ*, and there exists a countable family*

$$\mathring{\Phi} = \{\mathring{\Phi}_j = \widetilde{\mathfrak{F}}_\Phi^{-1}\Phi_j\}_{j\in\mathcal{I}}$$

such that

$$\Psi = \sum_{j\in\mathcal{I}} \Phi_j\{\Psi,\mathring{\Phi}_j\} = \sum_{j\in\mathcal{I}} \mathring{\Phi}_j\{\Psi,\Phi_j\} \quad \forall\, \Psi \in \mathrm{M}_\Phi.$$

One of the main issues of dealing with such modular frames is that linear combinations such as the ones defining the modular frame operator (14) need to incorporate coefficients arising from the inner product. A theory of frames in C^*-Hilbert modules, whose inner product takes values in the algebra (and hence is bounded) was developed in [15]. However, in this case the inner product is $L^1(\mathscr{L}(\Gamma))$-valued, so linear combinations such as in (14) are not standard modular combinations. These coefficients are not bounded operators on $\ell_2(\Gamma)$, and do not constitute an algebra. A proper definition of such linear combinations require a limiting process which heuristically corresponds to a topological closure over a group orbit. Such a construction was developed in [4].

By the following theorem, proved in [4], we can see that this notion of noncommutative frames is the correct one to treat unitary group actions. It describes the main structure of reproducing systems in group-invariant Hilbert spaces and provides a full generalization of the Euclidean results obtained in [6] and of their counterparts in LCA groups obtained in [8].

Theorem 9. *Let $(\Gamma, \Pi, \mathcal{H})$ be a dual integrable triple with Helson map $\mathcal{T}$. For a countable family $\{\phi_j\}_{j\in\mathcal{I}} \subset \mathcal{H}$, let E^Γ_ϕ be the system of orbits*

$$E^\Gamma_\phi = \{\Pi(\gamma)\phi_j : \gamma \in \Gamma, j \in \mathcal{I}\} \subset \mathcal{H}$$

and let Φ be the modular system

$$\Phi = \{\mathcal{T}[\phi_j] : j \in \mathcal{I}\} \subset \mathcal{T}(\mathcal{H}).$$

Then the system E^Γ_ϕ is a frame sequence if and only if Φ is a modular frame sequence with the same frame bounds.

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