SEMICLASSICAL ANALYSIS IN INFINITE DIMENSIONS: WIGNER MEASURES
ANALISI SEMICLASSICA IN DIMENSIONE INFINITA: MISURE DI WIGNER

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Abstract. We review some aspects of semiclassical analysis for systems whose phase space is of arbitrary (possibly infinite) dimension. An emphasis will be put on a general derivation of the so-called Wigner classical measures as the limit of states in a non-commutative algebra of quantum observables.

Sunto. In questo seminario si discutono alcuni aspetti dell'analisi semiclassica, per sistemi il cui spazio delle fasi ha dimensione arbitraria (eventualmente infinita). In particolare viene presentata una derivazione generale delle misure di Wigner come limite di stati in algebre non commutative di osservabili quantistiche.

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1. Introduction.

The Wigner (or semiclassical) measures have a long history, at least for finite dimensional phase spaces. They were first introduced, as Radon measures on $\mathbb{R}^d$, in the late eighties and early nighties as microlocal defect measures to study variational problems with loss of compactness [29, 30, 37, 20]. Almost at the same time, with the development of semiclassical analysis, the Wigner measures on (finite dimensional) symplectic spaces have been used to characterize the limit of quantum mechanical states [23, 19, 13, 31].

Motivated by the study of (bosonic) quantum field theories, as well as the mean field and thermodynamic limit of quantum mechanics, there have been interesting attempts

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to extend the Weyl pseudodifferential calculus to suitable infinite dimensional symplectic spaces by an inductive approach or using the structure of abstract Wiener spaces [25, 24, 22, 21, 5, 7, 6]. For the purpose of semiclassical characterization of states, it is not clear if all the classical phase space configurations are explored with these approaches. The projective approach introduced by Ammari and Nier [3] seems to be well adapted to study Wigner measures in the classical limit. For example, as it will be proved in Corollary 2.3, for any complex separable Hilbert space $\mathfrak{h}$, it is possible to realize every probability measure $\mu \in P(\mathfrak{h})$ as the classical limit of a suitable family of states $(\omega_\hbar)_{\hbar\in(0,1)}$ on the Weyl algebra associated to $\mathfrak{h}$. Cylindrical measures that are not probability measures can be reached as well (Example 4.3).

In these notes, we review some developments in the theory of infinite-dimensional Wigner measures. In particular, we characterize the semiclassical measures for general Weyl algebras. Since for infinite dimensional phase spaces there are infinitely many inequivalent representations of such algebras, we provide when possible results that are independent of the choice of representation (the results in [3] were obtained for the Weyl algebra over $L^2(\mathbb{R}^d)$, and in the Fock representation). To our knowledge, the method developed by Ammari and Nier is the most flexible to study the semiclassical limit of bosonic quantum field theories and the mean field limit of many bosons for general quantum states. Among the papers that utilize such approach, we mention some by the author [1, 2, 4]. The infinite dimensional Wigner measures have also been used — again studying the mean field limit of bosonic systems, but from a different point of view — by Lewin, Nam and Rougerie [26–28]. Additional results, complementary to the ones provided in these notes, can be found in [18].

2. Regular states on the Weyl algebra and promeasures.

Let $\mathrm{Symp}_{\mathbb{R}}$ be the collection of real symplectic spaces [1], and C^*-Alg the collection of C^*-algebras. We define a map between the two collections, called the Segal map. We remark that such map can be seen as a functor, if we introduce suitable morphisms on

[1]We will denote a real symplectic space by (V, σ), where V is a real vector space and $\sigma : V \times V \to \mathbb{R}$ a skew-symmetric, non-degenerate bilinear form.

the aforementioned collections. The Segal map is defined as[2]:

$$\mathbb{S}_\hbar : \mathrm{Symp}_{\mathbb{R}} \longrightarrow C^*\text{-Alg}$$

$$(V,\sigma) \longmapsto \mathscr{V}$$

where $\mathscr{V}$ is the smallest C^*-algebra containing the set

$$\big\{ W_\hbar(v), v \in V \big\} \,, \tag{1}$$

that satisfies the following three properties:

i) $(\forall v \in V) W_\hbar(v) \neq 0$;
ii) $(\forall v \in V) W_\hbar(-v) = W_\hbar(v)^*$;
iii) $(\forall v \in V)(\forall w \in V) W_\hbar(v) W_\hbar(w) = e^{-i\frac{\hbar}{2}\sigma(v,w)} W_\hbar(v+w)$.

We call $\mathscr{V}$ the Weyl algebra associated to (V,σ). In quantum systems, the Weyl algebra encodes the canonical commutation relations: the elements of (1) are the Weyl operators, and (V,σ) is the classical phase space. Therefore the Segal map is a quantization that associates to any phase space the corresponding algebra of canonical commutation relations. Given $(V,\sigma) \in \mathrm{Symp}_{\mathbb{R}}$, the Weyl algebra is unique up to $*$-isomorphisms:

Theorem 2.1 ([36]). *$\forall (V,\sigma) \in Symp_{\mathbb{R}}$; $\exists \mathscr{V}_1$ generated by $\big\{ W_\hbar^{(1)}(v), v \in V \big\}$ satisfying i), ii), and iii) $\Longleftrightarrow \exists! \xi : \mathscr{V} \to \mathscr{V}_1$, ξ $*$-isomorphism, $(\forall v \in V) \xi\big(W_\hbar(v)\big) = W_\hbar^{(1)}(v)$.*

On the Weyl algebra, we define the set of non-commutative probabilities (quantum states) as

$$P_{\mathscr{V}} = \big\{ \omega_\hbar \in \mathscr{V}^{\mathrm{dual}}, \omega_\hbar \geq 0, \|\omega_\hbar\|_{\mathscr{V}^{\mathrm{dual}}} = 1 \big\} \,. \tag{2}$$

For our purpose, a particular subset of states plays a very important role, the so-called *regular states*. As a matter of fact, we will show that the regular states are the ones that have a classical counterpart with probabilistic interpretation. They are defined as follows:

$$R_{\mathscr{V}} = \big\{ \omega_\hbar \in P_{\mathscr{V}}, (\forall v \in V) \omega_\hbar\big(W_\hbar(\,\cdot\, v)\big) \in C(\mathbb{R}, \mathbb{C}) \big\} \,; \tag{3}$$

where $\cdot\, v$ denotes the $\mathbb{R}$-action on V.

[2] The parameter $\hbar \in (0,1)$ plays the role of the semiclassical parameter, and the notation has a straightforward physical interpretation.

Following [35], given a quantum state $\omega_\hbar$, we define its generating functional $G_{\omega_\hbar} \in \mathbb{C}^V$ to be:

$$\begin{aligned} G_{\omega_\hbar} : V &\longrightarrow \mathbb{C} \\ v &\longmapsto \omega_\hbar\big(W_\hbar(v)\big) \end{aligned} \quad .$$

The generating functional satisfies the following crucial properties:

Lemma 2.1. $G_{\omega_\hbar}(V) \subset \{z \in \mathbb{C}, |z| \leq 1\}$.

Proof. $\omega_\hbar \in P_{\mathscr{V}} \Rightarrow (\forall v \in V)|\omega_\hbar\big(W_\hbar(v)\big)| \leq \|W_\hbar(v)\|_{\mathscr{V}}$. Now from the properties i), ii), and iii) of $\big\{W_\hbar(v), v \in V\big\}$ it is straightforward to conclude that $W(0) = \mathbb{1}$, $(\forall v \in V) W_\hbar(v)^* = W_\hbar(v)^{-1}$, and therefore $(\forall v \in V)\|W_\hbar(v)\|_{\mathscr{V}} = 1$. □

Theorem 2.2 ([35]). *$\omega_\hbar \in R_{\mathscr{V}}$ iff $G_{\omega_\hbar}$ satisfies the following properties:*

- $\Big(W \subset V$ *subspace and* $(\exists d \in \mathbb{N})\mathrm{dim} W = d\Big) \Longrightarrow G_{\omega_\hbar}\big|_W \in C(W, \mathbb{C})$;
- $G_{\omega_\hbar}(0) = 1$;
- $(\exists n \in \mathbb{N}) F = \{1, \dots, n\} \Longrightarrow \big(\forall (v_j)_{j\in F} \subset V\big)\big(\forall (\alpha_j)_{j\in F} \subset \mathbb{C}\big)$

$$\sum_{j,k\in F} G_{\omega_\hbar}(v_j - v_k) e^{i\frac{\hbar}{2}\sigma(v_j, v_k)} \bar{\alpha}_k \alpha_j \geq 0 \ .$$

The properties of the generating functional are similar to those of the Fourier transform of a promeasure[3] on a locally convex space, apart from the presence of the semiclassical parameter $\hbar$. This leads to the intuition that in the limit the regular states should behave like promeasures. This intuition becomes substantial by means of the corollary to the following theorem.

[3] We adopt here the terminology of [8]. These projective systems of measures are also known as *cylindrical measures* or *weak distributions*. Bochner's theorem for promeasures on a locally convex space $\mathcal{L}$ [8] ensures that there is a bijection (the Fourier transform) between promeasures and functions on $g \in \mathbb{C}^{\mathcal{L}^{\mathrm{dual}}}$ such that: g is continuous on each finite dimensional subspace, $g(0) = 1$, and $(\exists n \in \mathbb{N}) F = \{1, \dots, n\} \Longrightarrow \big(\forall (x_j)_{j\in F} \subset \mathcal{L}^{\mathrm{dual}}\big)\big(\forall (\alpha_j)_{j\in F} \subset \mathbb{C}\big)$

$$\sum_{j,k\in F} g(v_j - v_k) \bar{\alpha}_k \alpha_j \geq 0 \ .$$

Theorem 2.3. *Let $(\omega_\hbar)_{\hbar\in(0,1)}$, $\bigl(\forall\hbar\in(0,1)\bigr)\omega_\hbar\in R_{\mathscr{V}}$. In addition, suppose that for any subspace $W\subset V$ such that $(\exists d\in\mathbb{N})\mathrm{dim}W=d$, then*[4] $H_W=\{G_{\omega_\hbar}\bigr|_W\subset\mathbb{C}^W,\hbar\in(0,1)\}$ *is equicontinuous.*

Then for any net $(\hbar_\alpha)_{\alpha\in A}$, $\hbar_\alpha\to 0$, there exists a subnet $(\hbar_\beta)_{\beta\in B}$ such that $\exists g_\omega\in\mathbb{C}^V$ satisfying the following properties:

- $(\forall v\in V)g_\omega(v)=\lim_\beta G_{\omega_{\hbar_\beta}}(v)$ *(simple convergence);*
- $g_\omega(0)=1$*;*
- $(\exists n\in\mathbb{N})F=\{1,\dots,n\}\implies\bigl(\forall(v_j)_{j\in F}\subset V\bigr)\bigl(\forall(\alpha_j)_{j\in F}\subset\mathbb{C}\bigr)$
$$\sum_{j,k\in F}g_\omega(v_j-v_k)\bar{\alpha}_k\alpha_j\geq 0\ ;$$
- $\Bigl(W\subset V$ *subspace and* $(\exists d\in\mathbb{N})\mathrm{dim}W=d\Bigr)\implies C(W,\mathbb{C})\ni g_\omega\bigr|_W=\lim_\beta G_{\omega_{\hbar_\beta}}\bigr|_W$, *and the convergence holds uniformly on compact subsets.*

Proof. Define $H=\{G_{\omega_\hbar}\subset\mathbb{C}^V,\hbar\in(0,1)\}$. Then Lemma 2.1 yields
$$(\forall v\in V)H(v)=\{G_{\omega_\hbar}(v)\subset\mathbb{C},\hbar\in(0,1)\}\subset\{z\in\mathbb{C},|z|\leq 1\}\ .$$
It follows that $(\forall v\in V)H(v)$ is relatively compact in $\mathbb{C}$. Hence H is precompact with respect to the uniform structure of the simple convergence, and the first point is proved. From the first property, and Theorem 2.2, it immediately follows that the second and third properties are also true.

To prove the final property, consider a subspace $W\subset V$ such that $(\exists d\in\mathbb{N})\mathrm{dim}W=d$. We endow W with the usual topology. By aid of Lemma 2.1 and Theorem 2.2 the following properties are easily verified:

- W is locally compact;
- $H_W\subset C(W,\mathbb{C})$;
- H_W equicontinuous;
- $(\forall v\in V)H_W(w)$ is relatively compact in $\mathbb{C}$.

[4] H_W equicontinuous $\Leftrightarrow(\forall w\in W)(\forall\epsilon>0)\exists U_\epsilon(w)$ neighbourhood of w such that $\bigl(\forall u\in U_\epsilon(w)\bigr)\bigl(\forall G\in H_W\bigr)|G(w)-G(u)|<\epsilon$.

Therefore it follows that H_W is relatively compact in $C_c(W, \mathbb{C})$ [9], where $C_c(W, \mathbb{C})$ denotes the space of continuous functions endowed with the compact-open topology induced by the uniform structure of compact convergence. In addition, on H_W the uniform structures of simple and compact convergence are equivalent [9].

Now, since $G_{\omega_{\hbar_\beta}}\big|_W \to g_\omega\big|_W$ simply, then it follows that it converges also in $C_c(W, \mathbb{C})$. $\square$

Corollary 2.1. *Let there exist a locally convex space $\mathcal{L}$ such that $V = \mathcal{L}^{\mathrm{dual}}$, and let the hypotheses of Theorem 2.3 be satisfied.*

Then for any net $(\omega_{\hbar_\alpha})_{\alpha \in A}$ of regular states, $\hbar_\alpha \to 0$, there exist a subnet $(\omega_{\hbar_\beta})_{\beta \in B}$ and a unique promeasure[5] *$\mu_\omega \in \Psi(\mathcal{L})$ such that*

$$\omega_{\hbar_\beta} \to \mu_\omega \Longleftrightarrow \hat{\mu}_\omega = g_\omega = \lim_{\beta \in B} G_{\omega_{\hbar_\beta}} \ . \tag{4}$$

Proof. A straightforward application of Bochner's theorem [8]. $\square$

This theorem shows that promeasures are the natural classical counterpart of regular states of the Weyl algebra. Some remarks are in order at this point. The first remark concerns the classical phase space (V, σ). In order to interpret the classical states as promeasures emerging from the non-commutative quantum probabilities, we have to identify the phase space with a topological symplectic space that is dual to a locally convex space. This "duality property" of the phase space is not uncommon in classical mechanics. It is in fact usual to consider the phase space to be the cotangent bundle $T^*\mathcal{M}$ of some smooth manifold $\mathcal{M}$, i.e. the fiberwise dual of the tangent bundle $T\mathcal{M}$ (that is naturally endowed

[5] We denote by $\Psi(\mathcal{L})$ the set of promeasures on $\mathcal{L}$. We recall the following basic facts on promeasures. Let $F(\mathcal{L}) = \{\mathcal{M} \subset \mathcal{L} \text{ subspace and } (\exists d \in \mathbb{N})\mathrm{codim}\mathcal{M} = d\}$. Then $\mu = \{\mu_{\mathcal{M}}\}_{\mathcal{M} \in F(\mathcal{L})} \in \Psi(\mathcal{L})$ iff: $(\forall \mathcal{M} \in F(\mathcal{L}))\mu_{\mathcal{M}} \in P(\mathcal{L}/\mathcal{M})$, where $P(\mathcal{L}/\mathcal{M})$ is the set of (Borel) probability measures on the finite dimensional space $\mathcal{L}/\mathcal{M}$, and $(\mathcal{M} \supset \mathcal{N}) \Rightarrow \mu_{\mathcal{M}} = p_{\mathcal{M}\mathcal{N}}(\mu_{\mathcal{N}})$, where $p_{\mathcal{M}\mathcal{N}} : \mathcal{L}/\mathcal{N} \to \mathcal{L}/\mathcal{M}$ is obtained from $\mathrm{id}_{\mathcal{L}} : \mathcal{M} \to \mathcal{N}$ passing to the quotients.

Let $\mu \in \Psi(\mathcal{L})$. Then its Fourier transform $\hat{\mu} : \mathcal{L}^{\mathrm{dual}} \to \mathcal{C}$ is defined by

$$(\forall x' \in \mathcal{M}^0)\hat{\mu}(x') = \int_{\mathcal{L}/\mathcal{M}} e^{i\langle x', x\rangle} d\mu_{\mathcal{M}}(x) \ ;$$

where $\mathcal{M}^0 \subset \mathcal{L}^{\mathrm{dual}}$ is the orthogonal to $\mathcal{M}$.

with a symplectic structure). For any Hilbert space $\mathfrak{h}$, seen as a Hilbert manifold, it is easy to see that

$$T^*\mathfrak{h} = \bigl(T\mathfrak{h}\bigr)^{\mathrm{dual}} \; .$$

Therefore the phase-space generating functional $G_{\omega_\hbar}$ defines, in the limit $\hbar \to 0$, a promeasure in the "Lagrangian environment" of coordinates and velocities.

For any locally convex space $\mathcal{L}$, $\Psi(\mathcal{L}) \supseteq P(\mathcal{L})$, i.e. any probability measure is a promeasure. If $(\exists d \in \mathbb{N})$, $\dim\mathcal{L} = d$, then $\Psi(\mathcal{L}) = P(\mathcal{L})$. One is therefore tempted to ask whether only the probability measures, and not all the promeasures, are physically relevant. The answer is that we can indeed find states of physical interest for which the classical counterpart is not a probability measure. An interesting example are the grand-canonical Gibbs states of free Hamiltonians in second quantization, that give rise in a suitable thermodynamic/mean-field limit to Gaussian promeasures [28]. These Gaussian promeasures — also known as free Gibbs measures — as well as their interacting counterpart, play an important role in the analysis of nonlinear Schrödinger equations with rough initial data [10, 11, 14, 16]. In [28], these promeasures are not probability measures in the phase space $L^2(\Omega)$, $\Omega \subset \mathbb{R}^d$ bounded, when $d \geq 2$. We will discuss Gibbs states and Gaussian promeasures in more detail in Section 4.

We conclude this section by constructing the promeasures associated to a special class of states on the Weyl algebra, the so-called (squeezed) coherent states. Let $(V,\sigma) \in \mathrm{Symp}_{\mathbb{R}}$, such that $\forall w \in V$, the application

$$\begin{aligned} \sigma_w : V &\longrightarrow \mathbb{R} \\ v &\longmapsto \sigma(v,w) \end{aligned}$$

is continuous when restricted to any finite dimensional subspace $W \subset V$. Let $Q_\sigma : V \to \mathbb{R}^+$ be any positive non-degenerate quadratic form on V, that is continuous on any finite dimensional subspace $W \subset V$ and such that for any $\hbar \in (0,1)$: $(\exists n \in \mathbb{N}) F = \{1,\dots,n\}$ $\Longrightarrow \bigl(\forall (v_j)_{j\in F} \subset V\bigr)\bigl(\forall (\alpha_j)_{j\in F} \subset \mathbb{C}\bigr)$

$$\sum_{j,k\in F} e^{-\hbar\bigl(Q_\sigma(v_j-v_k)-\frac{i}{2}\sigma(v_j,v_k)\bigr)}\bar{\alpha}_k\alpha_j \geq 0 \; .$$

Then we denote by $\gamma_\hbar^{Q_\sigma} \in P_{\mathscr{V}}$ the regular state on the Weyl algebra defined by the generating functional

$$G_{\gamma_\hbar^{Q_\sigma}}(v) = e^{-\hbar Q_\sigma(v)} \ .$$

Let also $w \in V = \mathcal{L}^*$, $\mathcal{L}$ locally convex space. The squeezed coherent state $c^w_{Q_\sigma,\hbar}$ corresponding to the quadratic form Q_σ is defined by

$$(\forall A \in \mathscr{V}) c^w_{Q_\sigma,\hbar}(A) = \gamma_\hbar^{Q_\sigma}\big(W_\hbar(w/\hbar)^* A W_\hbar(w/\hbar)\big) \ .$$

Theorem 2.4.

$$c^w_{Q_\sigma,\hbar} \to \delta_w \ ,$$

as $\hbar \to 0$, where $\delta_w \in \Psi(\mathcal{L})$ is the promeasure with Fourier transform $\hat{\delta}_w = e^{i\sigma(v,w)}$.

Proof. Let $v \in V$. Then the generating functional of $c^w_{Q_\sigma,\hbar}$ takes the form

$$G_{c^w_{Q_\sigma,\hbar}}(v) = \gamma_\hbar^{Q_\sigma}\big(W_\hbar(w/\hbar)^* W_\hbar(v) W_\hbar(w/\hbar)\big) \ .$$

Using the properties ii) and iii) of the Weyl algebra, and the definition of $\gamma_\hbar^{Q_\sigma}$, we obtain

$$\begin{aligned} G_{c^w_{Q_\sigma,\hbar}}(v) &= e^{i\frac{\hbar}{2}(\sigma(-w/\hbar,v)+\sigma(-w/\hbar,w/\hbar)+\sigma(v,w/\hbar))}\gamma_\hbar^{Q_\sigma}\big(W_\hbar(v)\big) \\ &= e^{i\sigma(v,w)}e^{-\hbar Q_\sigma(v)} \ . \end{aligned}$$

Now the limit $\hbar \to 0$ is trivial, yielding the expected result. $\square$

Corollary 2.2. *Let $\mathfrak{h}$ be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle_\mathfrak{h}$, and identify $(V,\sigma) \equiv (\mathfrak{h}^\mathbb{R}, \mathrm{Im}\langle\cdot,\cdot\rangle_\mathfrak{h})$, where $\mathfrak{h}^\mathbb{R}$ is $\mathfrak{h}$ considered as a real Hilbert space with scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{h}^\mathbb{R}} = \mathrm{Re}\langle\cdot,\cdot\rangle_\mathfrak{h}$.*

Then the promeasure $\delta_{iw} \in P(\mathfrak{h}^\mathbb{R})$ of Theorem 2.4 associated to any quadratic form Q_σ is the point measure concentrated at $w \in \mathfrak{h}^\mathbb{R}$.

Using the quadratic form $Q_\mathfrak{h}(\cdot) = \frac{\|\cdot\|_\mathfrak{h}^2}{2}$, we see that the coherent state $c^{iw}_{Q_\mathfrak{h},\hbar}$ constructed on the Fock vacuum $\Omega_{F,\hbar} = \gamma_\hbar^{Q_\mathfrak{h}}$ converges to the point measure δ_{iw}.

Corollary 2.3. *Let $\mathfrak{h}$ be a complex* separable *Hilbert space with inner product $\langle\cdot,\cdot\rangle_\mathfrak{h}$, and identify $(V,\sigma) \equiv (\mathfrak{h}^\mathbb{R}, \mathrm{Im}\langle\cdot,\cdot\rangle_\mathfrak{h})$, where $\mathfrak{h}^\mathbb{R}$ is $\mathfrak{h}$ considered as a real Hilbert space with scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{h}^\mathbb{R}} = \mathrm{Re}\langle\cdot,\cdot\rangle_\mathfrak{h}$.*

Then for any $\mu \in P(\mathfrak{h})$, there is a family of states $(\omega_\hbar)_{\hbar\in(0,1)}$ such that

$$\omega_\hbar \to \mu \ .$$

Proof. The proof of this corollary follows immediately from Corollary 2.2 identifying $\mathfrak{h}^{\mathbb{R}}$ with $\mathfrak{h}$ in the natural way. In fact, by Corollary 2.2 we can infer that any measure with finite support can be obtained in the limit by a suitable convex combination of squeezed coherent states. Since for any separable metric space M the measures supported in finite subsets of M are dense in $P(M)$, endowed with the weak topology [33], the result follows immediately. More precisely, let $(F_j)_{j\in\mathbb{N}} \subset \mathfrak{h}^{\mathbb{R}}$ be a sequence of finite subsets of V, $(k_w)_{w\in F_j} \subset \mathbb{C}$ such that $\sum_{w\in F_j} k_w = 1$ uniformly with respect to $j \in \mathbb{N}$. Now let $\mu \in P(\mathfrak{h}^{\mathbb{R}})$ be the measure defined as the (weak) limit

$$\mu = \lim_{j\to\infty} \sum_{w\in F_j} k_w \delta_{iw} \ .$$

Then we define the family $(\omega_\hbar)_{\hbar\in(0,1)}$ by

$$\bigl(\forall \hbar \in (0,1)\bigr)\omega_\hbar = \lim_{j\to\infty} \sum_{w\in F_j} k_w c^{iw}_{Q_{\mathfrak{h}},\hbar} \ ,$$

where the limit is taken in the $\sigma(\mathscr{V}^{\mathrm{dual}}, \mathscr{V})$ topology.

Therefore for any $v \in \mathfrak{h}^{\mathbb{R}}$,

$$\begin{aligned}\Bigl|G_{\omega_\hbar}(v) - \int_{\mathfrak{h}} e^{i\mathrm{Re}\langle v,z\rangle_{\mathfrak{h}}} d\mu(z)\Bigr| &\leq \Bigl|G_{\omega_\hbar}(v) - \sum_{w\in F_j} k_w G_{c^{iw}_{Q_{\mathfrak{h}},\hbar}}(v)\Bigr| \\ &+ \Bigl|\sum_{w\in F_j} k_w G_{c^{iw}_{Q_{\mathfrak{h}},\hbar}}(v) - \sum_{w\in F_j} k_w \int_{\mathfrak{h}} e^{i\mathrm{Re}\langle v,z\rangle_{\mathfrak{h}}} d\delta_{iw}\Bigr| \\ &+ \Bigl|\sum_{w\in F_j} k_w \int_{\mathfrak{h}} e^{i\mathrm{Re}\langle v,z\rangle_{\mathfrak{h}}} d\delta_{iw} - \int_{\mathfrak{h}} e^{i\mathrm{Re}\langle v,z\rangle_{\mathfrak{h}}} d\mu(z)\Bigr| \ .\end{aligned}$$

It is now straightforward to verify that the right-hand side converges to zero in the limit $j \to \infty$, $\hbar \to 0$. $\square$

3. Fock normality and measures.

In this section, we would like to discuss a sufficient condition on quantum states such that they converge to probability measures. For that purpose, we will restrict to phase spaces with a separable Hilbert structure. Classical probability measures are crucial in order to study the limit dynamics corresponding to the unitary quantum evolution. This dynamics is usually generated by the flow solving some non-linear partial differential equation. In order to have such flow acting as a continuous deformation of (pro)measures, we need a rich structure: usually it is only defined on a suitable subspace of probability measures, and not on the whole set of promeasures. An explicit example is given by the 2-Wasserstein space, that is often used to study dynamical flows and transport equations.

Example 3.1 (2-Wasserstein space). *Let $\mathfrak{h}^{\mathbb{R}}$ be a real Hilbert space. Then $P_2(\mathfrak{h}^{\mathbb{R}}) \subset P(\mathfrak{h})$ is the set of probability measures μ such that $\int_{\mathfrak{h}^{\mathbb{R}}} \|x\|^2_{\mathfrak{h}^{\mathbb{R}}} d\mu(x) < \infty$. If $\mathfrak{h}^{\mathbb{R}}$ is separable, $P_2(\mathfrak{h}^{\mathbb{R}})$ becomes a complete and separable metric space with the 2-Wasserstein distance W_2 defined by*

$$W_2^2(\mu,\nu) = \min\left\{ \int_{\mathfrak{h}^{\mathbb{R}}\times\mathfrak{h}^{\mathbb{R}}} \|x_1 - x_2\|^2_{\mathfrak{h}^{\mathbb{R}}} d\mu(x_1,x_2)\ ;\ (\Pi_j)_*\mu = \mu_j \right\} ,$$

where $\Pi_j : \mathfrak{h}^{\mathbb{R}} \times \mathfrak{h}^{\mathbb{R}} \to \mathfrak{h}^{\mathbb{R}}$, $j = 1, 2$ is the natural projection.

We start by introducing the Fock representation of the Weyl algebra. Let $\mathfrak{h}$ be a complex Hilbert space, and let $(\mathfrak{h}^{\mathbb{R}}, \sigma_{\mathfrak{h}} = \mathrm{Im}\langle \cdot, \cdot \rangle_{\mathfrak{h}}) \in \mathrm{Symp}_{\mathbb{R}}$ be the corresponding real symplectic space already introduced in Corollaries 2.2 and 2.3 of Section 2. Using the Segal map, we obtain the associated Weyl algebra $\mathfrak{H} = \mathbb{S}_{\hbar}(\mathfrak{h}^{\mathbb{R}}, \sigma_{\mathfrak{h}})$. A well-known irreducible representation of such Weyl algebra is the so-called Fock representation $\big(\Gamma_s(\mathfrak{h}), \pi_\Gamma\big)$. The Hilbert space $\Gamma_s(\mathfrak{h})$ is called the symmetric Fock space and it is constructed as follows. Let $\mathfrak{h}_0 = \mathbb{C}$, and $\mathfrak{h}_n$, $n \geq 1$, be the n-fold symmetric tensor copy of $\mathfrak{h}$:

$$(\forall n \geq 1)\mathfrak{h}_n = \otimes_s^n \mathfrak{h} \ .$$

Then the Fock space $\Gamma_s(\mathfrak{h})$ is the direct sum of the $\mathfrak{h}_n$, for $n \in \mathbb{N}$:

$$\Gamma_s(\mathfrak{h}) = \bigoplus_{n\in\mathbb{N}} \mathfrak{h}_n \ .$$

It is a Hilbert space with scalar product

$$\langle \phi, \psi \rangle_\Gamma = \sum_{n\in\mathbb{N}} \langle \phi, \psi \rangle_{\mathfrak{h}_n} \ .$$

On the Fock space, there are three unbounded operators that play a very important role: the self-adjoint number operator N, the annihilation operator-valued map[6] $a : \mathfrak{h} \to \mathrm{ClOp}\bigl(\Gamma_s(\mathfrak{h})\bigr)$, and the creation operator-valued map $a^* : \mathfrak{h} \to \mathrm{ClOp}\bigl(\Gamma_s(\mathfrak{h})\bigr)$. We will not discuss them in detail here, the interested reader may consult e.g. [17], [34], or any textbook on mathematical methods of modern physics.

From the annihilation and creation operators, we can construct the self-adjoint field operator $\varphi : \mathfrak{h} \to \mathrm{SelfAdj}\bigl(\Gamma_s(\mathfrak{h})\bigr)$ defined as

$$\varphi_\hbar(\cdot) = \sqrt{\tfrac{\hbar}{2}}\bigl(a^*(\cdot) + a(\cdot)\bigr)\ .$$

The field operator generates a family of unitary operators $\{e^{i\varphi_\hbar(f)}, f \in \mathfrak{h}\}$. We define the representation map[7] $\pi_\Gamma : \mathfrak{H} \to \mathcal{B}\bigl(\Gamma_s(\mathfrak{h})\bigr)$ by

$$(\forall f \in \mathfrak{h})\pi_\Gamma\bigl(W_\hbar(f)\bigr) = e^{i\varphi_\hbar(f)}\ .$$

We also recall the following notions. Let $\mathfrak{j}$ be a Hilbert space, a von Neumann algebra $\mathscr{A} \subset \mathcal{B}(\mathfrak{j})$ is a C^* algebra such that it is equal to its bi-commutant[8] $\mathscr{A}''$. Every von Neumann algebra $\mathscr{A}$ has a predual $\mathscr{A}_{\mathrm{pred}}$, and we define the set of normal states on $\mathscr{A}$ as

$$N_{\mathscr{A}} = \{\varrho \in \mathscr{A}_{\mathrm{pred}}, \varrho \geq 0, \|\varrho\|_{\mathscr{A}_{\mathrm{pred}}} = 1\}\ .$$

Now let $\omega_\hbar \in P_{\mathfrak{H}}$ be a state of the Weyl algebra. We say that $\omega_\hbar$ is π_Γ-normal (Fock-normal) iff there exists $\varrho_{\omega_\hbar} \in N_{\pi_\Gamma(\mathfrak{H})''}$ such that

$$(\forall X \in \mathfrak{H})\omega_\hbar(X) = \varrho_{\omega_\hbar}\bigl(\pi_\Gamma(X)\bigr)\ .$$

For π_Γ-normal states, we can give a simple sufficient condition for the corresponding classical promeasures to be probability measures. The precise result is stated in the following theorem.

Theorem 3.1 ([3])**.** *Let $\mathfrak{h}$ be a complex* separable *Hilbert space, and $\mathfrak{H}$ the associated Weyl algebra. Furthermore, let $(\omega_\hbar)_{\hbar\in(0,1)}$ such that: $\bigl(\forall \hbar \in (0,1)\bigr)\omega_\hbar \in R_{\mathfrak{H}}$ and $\omega_\hbar$ is*

[6] Here $\mathrm{ClOp}\bigl(\Gamma_s(\mathfrak{h})\bigr)$ stands for the space of closed densely defined operators.

[7] We denote by $\mathcal{B}\bigl(\Gamma_s(\mathfrak{h})\bigr)$ th bounded operators on $\Gamma_s(\mathfrak{h})$.

[8] The commutant of a von Neumann algebra $\mathscr{A}$ is defined as $\mathscr{A}' = \{X \in \mathcal{B}(\mathfrak{j}), (\forall A \in \mathscr{A})[X, A] = 0\}$. The bi-commutant $\mathscr{A}''$ is obviously the commutant of the commutant of $\mathscr{A}$.

π_Γ*-normal (denote the corresponding Fock state by* $\varrho_{\omega_\hbar}$*). In addition, suppose there exists a* $\delta > 0$ *and a* $C > 0$ *such that* $\varrho_{\omega_\hbar}\big((\hbar N)^\delta\big) \leq C$.

Then for any sequence $(\hbar_k)_{k\in\mathbb{N}}$, $\hbar_k \to 0$*, there exists a subsequence* $(\hbar_{k_j})_{j\in\mathbb{N}}$ *and* $\mu_\omega \in P(\mathfrak{h})$ *such that*

$$\omega_{\hbar_{k_j}} \to \mu_\omega \ . \tag{5}$$

This result shows that families of regular, Fock-normal probabilities of the Weyl algebra (for which the evaluation of the density of particles is uniform in $\hbar$) converge to classical probabilities.

4. Some concrete examples on $\Gamma_s(L^2(\mathbb{R}))$.

In this section we illustrate the results of the preceding sections for some specific family of states of the Weyl algebra $\mathfrak{H}_{L^2} = \mathbb{S}_\hbar\big((L^2(\mathbb{R}))^{\mathbb{R}}, \sigma_{L^2}\big)$ in the Fock representation $\big(\Gamma_s(L^2(\mathbb{R})), \pi_\Gamma\big)$.

Example 4.1 (Squeezed coherent states)**.** *Let* $\Omega \in \Gamma_s(L^2(\mathbb{R}))$ *be the Fock vacuum, i.e.* $\Omega = (1, 0, \dots, 0, \dots)$*. The state* $\gamma_\hbar \in R_{\mathfrak{H}_{L^2}}$ *associated*[9] *to* Ω *has generating functional*

$$G_{\gamma_\hbar}(f) = e^{-\frac{\hbar}{2}\|f\|_2^2} \ .$$

It then follows from Corollary 2.2 that the squeezed coherent state $c_\hbar^{if}$ *associated to* $e^{i\varphi_\hbar(if/\hbar)}\Omega$ *converges to the point measure* $\delta(f) \in P\big(L^2(\mathbb{R})\big)$ *concentrated in* $f \in L^2(\mathbb{R})$ *in the limit* $\hbar \to 0$.

As it is expected from a physical standpoint, the states of less indeterminacy (squeezed coherent states) yield the classical trajectory in the limit $\hbar \to 0$*. In fact, their corresponding classical probability is concentrated at a single point of the phase space.*

[9]Let $\mathscr{W}$ be a C^*-algebra, let $(\mathfrak{j}, \pi)$ be a representation of $\mathscr{W}$, and let $\psi \in \mathfrak{j}$. Then the state $\omega \in P_{\mathscr{W}}$ associated to $\psi \in \mathfrak{j}$ is the functional defined by:

$$(\forall X \in \mathscr{W})\omega(X) = \langle \psi, \pi(X)\psi\rangle_{\mathfrak{j}} \ .$$

Example 4.2 (Factor states)**.** *The following class of vectors is important when considering non-relativistic many body boson theories. Let $f \in L^2(\mathbb{R})$, let us denote by $(\eta_{\hbar_k})_{k\in\mathbb{N}}$, $\hbar_k = (k+1)^{-1}$, the sequence of states associated for any $k \in \mathbb{N}$ to the vector*[10]

$$f^{k+1}(x_1, \dots, x_{k+1}) = f(x_1)f(x_2)\cdots f(x_{k+1}) \in L^2_s(\mathbb{R}^{k+1}) \; .$$

These states are called factor states, and for each $k \in \mathbb{N}$, represent $k+1$ bosons, with each boson in the same single-particle state.

The sequence $(\eta_{\hbar_k})_{k\in\mathbb{N}}$ converges in the limit $k \to \infty$ to the measure $\frac{1}{2\pi}\int_0^{2\pi} \delta(e^{i\theta} f) d\theta \in P(L^2(\mathbb{R}))$. Therefore the classical probability corresponding to factor states has not finite support.

Example 4.3 (Gibbs states)**.** *With a carefully chosen semiclassical scaling, the Gibbs states provide a physical context on which promeasures that are not measures emerge. We recall here some basic fact about Gibbs states (on $\Gamma_s(L^2(\mathbb{R}))$); the reader interested in details may consult [12]. First of all, we recall that for any $H \in \mathit{SelfAdj}\big(L^2(\mathbb{R})\big)$, we define its second quantization $d\Gamma(H) \in \mathit{SelfAdj}\big(\Gamma_s(L^2(\mathbb{R}))\big)$ by $d\Gamma(H)\Omega = 0$ and*

$$\big(\forall n \geq 1\big)\big(\forall \psi_n \in L^2_s(\mathbb{R}^n)\big) d\Gamma(H)\psi_n(x_1, \dots, x_n) = \sum_{j=1}^{n} H_j \psi_n(x_1, \dots, x_n) \; ;$$

where H_j is H acting on the j-th variable.

Now, let $H_0 \in \mathit{SelfAdj}\big(L^2(\mathbb{R})\big)$, and $(\beta_k)_{k\in\mathbb{N}}, (\mu_k)_{k\in\mathbb{N}}$ be two sequences of (positive) numbers such that $(\forall k \in \mathbb{N}) e^{-\beta_k H_0}$ is trace class and $\beta_k(H_0 - \mu_k) > 0$. We define the Gibbs state on $\mathfrak{H}_{L^2}$ by

$$(\forall A \in \mathfrak{H}_{L^2}) \omega_k(A) = \frac{\mathrm{Tr}\big[z_k e^{-\beta_k d\Gamma(H_0)} A\big]}{\mathrm{Tr}\big[z_k e^{-\beta_k d\Gamma(H_0)}\big]} \; ;$$

where $z_k = e^{\beta_k \mu_k}$. The hypotheses above ensure that $z_k e^{-\beta_k d\Gamma(H_0)}$ is trace class. In this context, we interpret $k+1 \sim \hbar^{-1}$ to be the (inverse of) the semiclassical parameter, β_k to be a k-dependent thermodynamic beta (roughly speaking, the inverse of temperature), and μ_k a k-dependent chemical potential.

[10] Precisely, we mean the vector $\psi_{k+1} \in \Gamma_s(L^2(\mathbb{R}))$ defined as $(0, \dots, 0, f_{k+1}, 0, \dots)$, where f_{k+1} occupies the $k+1$-th spot.

The generating functional $G_{\omega_k} : L^2(\mathbb{R}) \to \mathbb{C}$ of the Gibbs state — keeping in mind the relation $k+1 \sim \hbar^{-1}$, i.e. substituting every $\hbar$ in the definition for $(k+1)^{-1}$ — has the following simple form:

$$G_{\omega_k}(f) = \exp\bigl(\tfrac{1}{4(k+1)}\|f\|_2^2\bigr) \exp\Bigl(-\tfrac{1}{2}\bigl\langle f, \tfrac{z_k}{k+1} e^{-\beta_k H_0}\bigl(1 - z_k e^{-\beta_k H_0}\bigr)^{-1} f\bigr\rangle_2\Bigr) \,.$$

Now suppose that for any $k \in \mathbb{N}$, $K_k = \frac{z_k}{k+1} e^{-\beta_k H_0}\bigl(1 - z_k e^{-\beta_k H_0}\bigr)^{-1} \in \mathcal{B}\bigl(L^2(\mathbb{R})\bigr)$. In addition, suppose there exists a strictly positive and self-adjoint $K_\infty \in \mathcal{B}\bigl(L^2(\mathbb{R})\bigr)$ such that $K_k \rightharpoonup K_\infty$ (weak topology). It then follows that $G_{\omega_k} \to g_\omega$, with

$$g_\omega(f) = e^{-\frac{1}{2}\langle f, K_\infty f\rangle_2} = e^{-\frac{1}{2} Q_{K_\infty}(f)} \;;$$

where Q_{K_∞} is a positive non-degenerate quadratic form on $L^2(\mathbb{R})$. Therefore we have proved that there exists a unique Gaussian promeasure $\mu_{G,K_\infty} \in \Psi\bigl(L^2(\mathbb{R})\bigr)$ such that $\omega_k \to \mu_{G,K_\infty}$. In addition, a theorem by Cameron and Martin [15] ensures that $\mu_{G,K_\infty} \notin P\bigl(L^2(\mathbb{R})\bigr)$ whenever K_∞ is not Hilbert-Schmidt.

We conclude this example by showing that we can nevertheless construct a probability measure on the space of tempered distributions $\mathscr{S}'(\mathbb{R})$ that extends μ_{G,K_∞} and whose support lies outside of $L^2(\mathbb{R})$. First of all, it is straightforward to prove that $g_\omega \in C\bigl(L^2(\mathbb{R}), \mathbb{R}\bigr)$. In addition, the associated promeasure $\mu_{G,K_\infty} \in \Psi\bigl(L^2(\mathbb{R})\bigr)$ can be extended to a promeasure $\tilde{\mu}_{G,K_\infty} \in \Psi\bigl(\mathscr{S}'(\mathbb{R})\bigr)$ in such a way that $\widehat{(\tilde{\mu}_{G,K_\infty})}\big|_{L^2} = g_\omega$. However, by Minlos' theorem [32], $\tilde{\mu}_{G,K_\infty} \in P\bigl(\mathscr{S}'(\mathbb{R})\bigr)$ since $\widehat{(\tilde{\mu}_{G,K_\infty})}\big|_{\mathscr{S}}$ is continuous. It also follows that $\tilde{\mu}_{G,K_\infty}$ is concentrated outside of $L^2(\mathbb{R})$ if K_∞ is not Hilbert-Schmidt, and inside $L^2(\mathbb{R})$ if K_∞ is Hilbert-Schmidt.

Example 4.4. *In this last example, we show that there are families of states for which the subnet (or in this case subsequence) extraction is necessary, in order to have convergence; and that different subnets may lead to different limits.*

Let $g_1, g_2 \in L^2(\mathbb{R})$, $(g_{2,k})_{k\in\mathbb{N}} \subset L^2(\mathbb{R})$ such that $g_{2,k} \rightharpoonup g_2$ (weak convergence). Let us define the sequence $(f_k)_{k\in\mathbb{N}} \subset L^2(\mathbb{R})$ as follows:

$$f_k = \begin{cases} g_1 \text{ if } k \text{ is even or zero} \\ g_{2,k} \text{ if } k \text{ is odd} \end{cases} \,.$$

Therefore the sequence $(f_k)_{k\in\mathbb{N}}$ is bounded, it does not converge, and the subsequence $(f_{2j+1})_{j\in\mathbb{N}}$ converges weakly to g_2 (while the sequence $(f_{2j})_{j\in\mathbb{N}}$ converges strongly to g_1).

As in the previous example, we identify $k+1 \sim \hbar^{-1}$, and consider the sequence of squeezed coherent states $(c_k^{if_k})_{k\in\mathbb{N}}$ introduced in Example 4.1. Its generating functional takes then the form

$$G_{c_k^{if_k}}(f) = e^{i\mathrm{Re}\langle f,f_k\rangle_2} e^{-\frac{1}{2(k+1)}\|f\|_2^2} .$$

Therefore it does not converge in the limit $k\to\infty$. However, after the extraction of the subsequence $(c_{2j+1}^{if_{2j+1}})_{j\in\mathbb{N}}$, then

$$G_{c_{2j+1}^{if_{2j+1}}}(f) = e^{i\mathrm{Re}\langle f,g_{2,j}\rangle_2} e^{-\frac{1}{4j+4}\|f\|_2^2} \longrightarrow e^{i\mathrm{Re}\langle f,g_2\rangle_2} .$$

It follows that $c_{2j+1}^{if_{2j+1}} \to \delta(g_2)$. Analogously, it can be shown that $c_{2j}^{if_{2j}} \to \delta(g_1)$.

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