ON KP MULTI–SOLITON SOLUTIONS ASSOCIATED TO RATIONAL DEGENERATIONS OF REAL HYPERELLIPTIC CURVES
SUI MULTI–SOLITON KP ASSOCIATI A DEGENERAZIONI RAZIONALI DI CURVE REALI IPERELLITTICHE

SIMONETTA ABENDA

Abstract. Using the technique introduced in [1], we explain the relations between the description of KP–multisolitons in the Sato Grassmannian and in finite–gap theory in the special cases $Gr^{\mathrm{TP}}(1,M)$ and $Gr^{\mathrm{TP}}(M-1,M)$ where the multisolitons may be associated to Krichever data on rational degenerations of regular hyperelliptic M–curves of genus $M-1$.

Sunto. Usando la tecnica introdotta in [1], spieghiamo le relazioni fra la descrizione dei multi-solitoni KP nell'ambito della Grassmanniana di Sato e della teoria finite–gap nei casi particolari $Gr^{\mathrm{TP}}(1,M)$ e $Gr^{\mathrm{TP}}(M-1,M)$, dove i multisolitoni possono essere associati a dati di Krichever su degenerazioni razionali di M–curve iperellittiche di genere $M-1$.

2010 MSC. 37K40; 37K20, 14H50, 14H70.

Keywords. Total positivity, KP equation, real solitons, M-curves, hyperelliptic curves, duality of Grassmann cells via space–time transformations

1. Introduction

The KP-II equation

$$(-4u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0, \tag{1}$$

was originally proposed by Kadomtsev and Petshiavili [14] to study the stability of soliton solutions to the Korteweg de Vries equation under a weak transverse perturbation. It was soon realized that such equation is associated to a completely integrable system with

Bruno Pini Mathematical Analysis Seminar, Vol. 1 (2015) pp. 138–157

Dipartimento di Matematica, Università di Bologna

ISSN 2240-2829.

remarkably rich mathematical structure (see for instance the monographs [4, 6, 13, 21, 24]) and it is nowadays considered the prototype model of the integrable nonlinear dispersive wave equations in $2+1$–dimensions.

The KP soliton solutions are a class of solutions to (1) which may be associated to points in the infinite–dimensional Sato Grassmannian. In a series of recent papers, Chakravarty–Kodama [3] and Kodama–Williams [15, 16] have classified the asymptotics of a particular class of KP soliton solutions associated to finite–dimensional reductions of the Sato Grassmannian - the so–called regular bounded $(M-N, N)$-line solitons, - using the combinatorial classification of the totally non–negative part of the real Grassmannian, $Gr^{\mathrm{TNN}}(N, M)$ (see [23] for necessary definitions). In particular in [16] it has been determined the connection between the tropical limit of the KP-soliton graphs and the theory of cluster algebras of Fomin and Zelevinsky [8, 9].

The class of KP solutions considered in [3, 15, 16] are, in principle, also associated to Krichever data on rational degenerations of $\mathtt{M}$–curves. Indeed, on one side, by a theorem of Dubrovin and Natanzon[7], real regular finite–gap solutions are parameterized by Krichever data on regular $\mathtt{M}$–curves and on the other side, by finite–gap theory [6], it is known that regular bounded $(M-N, N)$-line solitons may be obtained from regular real quasi–periodic KP solutions when some cycles of the underlying algebraic curves degenerate to double points.

In [1], we have succeeded in connecting such two areas of mathematics - the theory of totally positive Grassmannians and the rational degenerations of the $\mathtt{M}$-curves - using the finite–gap theory for solitons of the KP equation. More precisely, for any fixed set of phases $k_1 < \cdots < k_M$ and for any fixed $\xi >> 1$, to any point of the real totally positive Grassmannian $Gr^{\mathrm{TP}}(N, M)$ we associate a rational curve $\Gamma = \Gamma(\xi)$, which is the rational degeneration of a regular $\mathtt{M}$–curve of minimal genus $g = N(M-N)$, and the Krichever divisor $\mathcal{D} = \mathcal{D}(\xi)$ of the underlying soliton solution. Moreover, the curves $\Gamma(\xi)$ are of the same topological type if $\xi >> 1$.

In [1], we have remarked that in the cases $Gr^{\mathrm{TP}}(1, M)$ - and, by duality, also $Gr^{\mathrm{TP}}(M-1, M)$ - the construction may be modified in such a way to associate the rational degeneration of a hyperelliptic $\mathtt{M}$-curve of minimal genus $g = M-1$ to the given soliton solution.

In this paper we present such modified construction for any M and we explain its relation with the duality of $Gr^{\rm TP}(1,M)$ and $Gr^{\rm TP}(M-1,M)$, using the space–time inversion transformation of soliton solutions.

2. $(M-N,N)$-line solitons via Darboux transformation, in the Sato Grassmannian and in finite–gap theory

From now on, let $N<M$. In this section, we identify the real bounded regular $(M-N,N)$-line soliton solutions in the general class of KP–soliton solutions using three techniques: Darboux transformations, Sato's dressing transformations and finite gap–theory.

First of all, we recall some useful definitions and we refer to Postnikov [23] and references therein for more details. An $N\times M$ real matrix $A\in Mat^{\rm TNN}_{N,M}$ if all the maximal $(N\times N)$ minors of A are non–negative and at least one of them is non trivial. Then the totally non–negative Grassmannian is $Gr^{\rm TNN}(N,M) = GL_N^+\backslash Mat^{\rm TNN}_{N,M}$, where GL_N^+ are the $N\times N$ real matrices with positive determinant. The totally positive Grassmannian is $Gr^{\rm TP}(N,M) = \mathcal{S}\cap Gr^{\rm TNN}(N,M)$, where $\mathcal{S}$ is the top cell in the Gelfand–Serganova decomposition of $Gr(N,M)$, *i.e.* $[A]\in Gr^{\rm TP}(N,M)$ if and only if all maximal $(N\times N)$ minors of A are positive.

The simplest way to construct KP solitons is via the Wronskian method [20]: suppose that $f^{(1)}(x,y,t),\dots,f^{(N)}(x,y,t)$ satisfy the heat hierarchy

$$\partial_y f^{(r)} = \partial_x^2 f^{(r)}, \qquad \partial_t f^{(r)} = \partial_x^3 f^{(r)}, \qquad r=1,\dots,N,$$

and let

$$\tau(x,y,t) = \mathrm{Wr_x}(f^{(1)},\dots,f^{(N)}) \equiv \left| \begin{array}{cccc} f^{(1)} & f^{(2)} & \cdots & f^{(N)} \\ \partial_x f^{(1)} & \partial_x f^{(2)} & \cdots & \partial_x f^{(N)} \\ \vdots & \vdots & \vdots & \vdots \\ \partial_x^{N-1} f^{(1)} & \partial_x^{N-1} f^{(2)} & \cdots & \partial_x^{N-1} f^{(N)} \end{array} \right|.$$

Then $u(x,y,t) = 2\partial_x^2\log(\tau(x,y,t))$, is a solution to KP-II.

Let $k_1 < k_2 < \cdots < k_M$. The $(M-N,N)$ - line soliton solutions $u(x,y,t)$ are obtained choosing

$$f^{(r)}(x,y,t) = \sum_{j=1}^{M} A_j^r E_j(x,y,t), \qquad r = 1,\dots,N, \tag{2}$$

where $A = (A_j^r)$ is a real $N \times M$ matrix and $E_j(x,y,t) = e^{\theta(k_j;x,y,t)}$ with $\theta(\lambda;x,y,t) = \lambda x + \lambda^2 y + \lambda^3 t$. In such a case

$$\tau(x,y,t) = \sum_{1\le i_1<\cdots<i_N\le M} \Delta(i_1,\dots,i_N) E_{[i_1,\dots,i_N]}(x,y,t) \tag{3}$$

where $\Delta(i_1,\dots,i_N)$ are the Plücker coordinates of the corresponding point in the real Grassmannian, $[A] \in Gr(N,M)$, and $E_{[i_1,\dots,i_N]}(x,y,t) = \mathrm{Wr}_x(E_{i_1},\dots,E_{i_N})$. Then, following [15], the $(M-N,N)$ - line soliton $u(x,y,t) = 2\partial_x^2 \log(\tau)$ is regular and bounded for all $(x,y,t) \in \mathbb{R}^3$ if and only if $[A] \in Gr^{\mathrm{TNN}}(N,M)$, *i.e.* all $N \times N$ minors $\Delta(i_1,\dots,i_N) \ge 0$.

The KP solitons are also realized as special solutions in the Sato theory of the KP hierarchy [24, 21] using the dressing transformation. Indeed let the vacuum hierarchy be

$$\begin{cases} \partial_x \Psi^{(0)} = \lambda \Psi^{(0)}, \\ \partial_{t_n} \Psi^{(0)} = \partial_x^n \Psi^{(0)} = \lambda^n \Psi^{(0)}, \quad n \ge 1, \end{cases}$$

and suppose that the dressing operator $W = 1 - w_1\partial_x^{-1} - w_2\partial_x^{-2} - \cdots$ satisfies the Sato equations $\partial_{t_n} W = (W\partial_x^n W^{-1})_+ W - W\partial_x^n$, $n \ge 1$, where the symbol $(\cdot)_+$ denotes the differential part of the given operator. Then the KP hierarchy is generated by the inverse gauge (dressing) transformation $L = W\partial_x W^{-1}$

$$\begin{cases} L\tilde{\Psi}^{(0)} = \lambda \tilde{\Psi}^{(0)}, \\ \partial_{t_n} \tilde{\Psi}^{(0)} = B_n \tilde{\Psi}^{(0)}, \quad n \ge 1; \quad B_n = (W\partial_x^n W^{-1})_+, \end{cases}$$

with $\tilde{\Psi}^{(0)} = W\Psi^{(0)}$ and $x = t_1$, $y = t_2$, $t = t_3$. In such a case the Lax operator takes the form $L = \partial_x + u_2\partial_x^{-1} + u_3\partial_x^{-2} + \cdots$, and $u_2 = \partial_x w_1$ satisfies the KP equation.

Following [3], $u(x,y,t)$ is the $(M-N,N)$ - line soliton associated to the τ–function (3) if and only if the dressing operator takes the form $W = 1 - w_1\partial_x^{-1} - w_2\partial_x^{-2} - \cdots - w_N\partial_x^{-N}$, and $Df^{(r)} = 0$, $r = 1,\dots,N$, where

$$D \equiv W\partial_x^N = \partial_x^N - w_1(x,y,t)\partial_x^{N-1} - \ldots - w_N(x,y,t). \tag{4}$$

Regular finite–gap solutions are the complex periodic or quasi–periodic meromorphic solutions to the KP equation (1). Krichever [17, 18] has classified this class of solutions: for any non-singular genus g complex algebraic curve Γ with a marked point P_0 and a local parameter λ such that $\lambda^{-1}(P_0) = 0$, there exists a family of regular complex finite–gap solutions $u(x, y, t,)$ to (1) parametrized by non special divisors $\mathcal{D} = (P_1, \dots, P_g)$. More precisely, the Baker–Akhiezer function $\tilde{\Psi}(P; x, y, t)$ meromorphic on $\Gamma \backslash \{P_0\}$ with poles on $\mathcal{D}$ and an essential singularity at P_0 with the following asymptotics

$$\tilde{\Psi}(\lambda; x, y, t) = \big(1 + \frac{\chi_1(x, y, t)}{\lambda} + O(\lambda^{-2})\big) e^{\lambda x + \lambda^2 y + \lambda^3 t + \cdots} \quad (\lambda \to \infty),$$

is a solution to

$$\frac{\partial \tilde{\Psi}}{\partial y} = B_2 \tilde{\Psi}, \qquad \frac{\partial \tilde{\Psi}}{\partial t} = B_3 \tilde{\Psi},$$

where $B_2 \equiv (L^2)_+ = \partial_x^2 + u_2$, $B_3 = (L^3)_+ = \partial_x^3 + \frac{3}{4}(u_2 \partial_x + \partial_x u_2) + u_3$ satisfy the compatibility conditions $[-\partial_y + B_2, -\partial_t + B_3] = 0$. If the divisor $\mathcal{D}$ is non–special, then $\tilde{\Psi}$ is uniquely identified by its normalization for $P \to P_0$ [5]. Finally, $\partial_x u_3 = \frac{3}{4} \partial_y u_2$, and the KP regular finite–gap solution is

$$u_2(x, y, t) = 2\partial_x \chi_1(x, y, t) = 2\partial_x^2 \log(\Theta(Ux + Vy + Zt + z_0)) + c,$$

where $c \in \mathbb{C}$, $\Theta(z)$, $z \in \mathbb{C}^g$, is the Riemann theta–function associated to Γ, $z_0 \in \mathbb{C}^g$ is a constant vector which depends on the divisor $\mathcal{D}$, and U, V, $Z \in \mathbb{C}^g$ are the periods of certain normalized meromorphic differentials on Γ (see for instance [5] for necessary definitions and explicit formulas).

By a theorem of Dubrovin and Natanzon [7], a regular finite–gap KP–solution $u(x, y, t)$ is real (quasi)–periodic if and only if it corresponds to Krichever data on a regular $\mathtt{M}$–curve Γ. More precisely Γ must possess an anti–holomorphic involution which fixes the maximum number of ovals, $\Omega_0, \dots, \Omega_g$ such that $P_0 \in \Omega_0$ and $P_j \in \Omega_j$, $j = 1, ..., g$.

We recall that the ovals are topologically circles and, by a theorem of Harnack [12], the maximal number of components (ovals) of a real algebraic curve in the projective plane is equal to $(n-1)(n-2)/2 + 1$, where n denotes the order of the curve. The investigation of the relative positions of the branches of real algebraic curves of degree n (and similarly for algebraic surfaces) is the first part of the Hilbert's 16th problem (see [11] for a review).

According to finite–gap theory [4, 6], soliton solutions are obtained from finite–gap regular solutions in the limit in which some of the cycles of Γ become singular. In particular, the real smooth bounded $(M-N,N)$-line solitons may be obtained from regular real quasi–periodic solutions in the rational limit of $\mathtt{M}$–curves where some cycles shrink to double points. We remark that the same soliton solution may be associated in principle to topologically inequivalent rational curves (for an example see the last section).

3. Total positivity and rational $\mathtt{M}$–curves

A $N\times M$ matrix A is totally positive (respectively strictly totally positive) if all of its minors of any order are non–negative (respectively positive).

Totally positive matrices were first introduced in 1930 by Schöneberg in [25] in connection with the problem of estimating the number of real zeroes of a polynomial, and in 1935 they also arose in statistical problems in the paper by Gantmacher and M. Krein [10]. Later positive matrices arose in connection with problems from different areas of pure and applied mathematics, including small vibrations of mechanical systems, approximation theory, combinatorics, graph theory (for more details see [19, 22]). Important recent applications of total positivity are associated with the cluster algebras of Fomin and Zelevinsky [8, 9].

In a recent paper [1] we have started to investigate the relations between the realization of $(M-N,N)$–line regular bounded solitons in the Sato Grassmannian and in finite–gap theory. More precisely, to any soliton solution $([A],\mathcal{K})$, with $\mathcal{K}=\{k_1<\cdots<k_M\}$ and $[A]\in Gr^{\mathrm{TP}}(N,M)$, we associate a triple $(\Gamma,P_0,\mathcal{D})$ in agreement with the theorem of Dubrovin and Natanzon [7] and such that Γ is the rational degeneration of an $\mathtt{M}$–curve of genus $g=N(M-N)$. The arithmetic genus of Γ, g, is minimal for generic soliton data $([A],\mathcal{K})$ since it is equal to the dimension of the corresponding Grassmann cell. The general construction proposed in [1] is the following (see Figure 3):

(1) We glue $N+1$ copies of $\mathbb{C}P^1$, $\Gamma=\Gamma_0\sqcup\Gamma_1\sqcup\ldots\sqcup\Gamma_N$, at a convenient set of real ordered marked points creating double points and $N(M-N)+1$ ovals Ω_0, $\Omega_{r,j}$, $r=1,\ldots,N$, $j=1,\ldots,M-N$;

(2) We construct a vacuum wave–function $\Psi(P,\vec{t})$ meromorphic for $P \in \Gamma\backslash\{P_0\}$, where $P_0 \in \Omega_0 \cap \Gamma_0$, with the following properties:

(a) Ψ restricted to Γ_0 is the Sato vacuum wavefunction

$$\Psi^{(0)}(\lambda; x, y, t) = e^{\theta(\lambda; x,y,t)}; \tag{5}$$

(b) Ψ possesses $N(M-N)$ simple poles such that no pole is in Γ_0 and $M-N$ poles are in each Γ_r, $r = 1, \ldots, N$;

(c) Ψ possesses exactly one pole in each finite oval $\Omega_{r,j}$, $r = 1, \ldots, N$, $j = 1, \ldots, M-N$.

(3) We apply the dressing transformation (4) and we show that the normalized KP-wavefunction $\tilde{\Psi}(P,\vec{t}) = \dfrac{D\Psi(P,\vec{t})}{D\Psi(P,\vec{0})}$ is the Baker–Akhiezer function on Γ. Moreover, $\tilde{\Psi}$ has the following properties:

(a) it possesses $N(M-N)$ poles with the following rules: N poles on Γ_0 and $M-N-1$ poles on each copy of Γ_r;

(b) it possesses exactly one pole in each finite oval $\Omega_{r,j}$, $r = 1, \ldots, N$, $j = 1, \ldots, M-N$.

In the following we denote $\Psi^{(r)}$ (respectively $\tilde{\Psi}^{(r)}$) the restriction of Ψ (respectively of $\tilde{\Psi}$) to Γ_r, $r = 0, \ldots, N$. On Γ_r, the vacuum wave–function necessarily takes the form

$$\Psi^{(r)}(\lambda; x, y, t) = \sum_{j=1}^{M-N+1} B_j^{(r)} \frac{\prod_{s\neq j}(\lambda - \lambda_s^{(r)})}{\prod_{k=1}^{M-N}(\lambda - b_k^{(r)})} V_j^{(r)}(x, y, t), \tag{6}$$

where, for any fixed $r = 1, \ldots, N$, $\lambda_j^{(r)} \in \Gamma_r$, $j = 1, \ldots, M-N+1$, are real and ordered and

$$V_j^{(r)}(x,y,t) = \begin{cases} \Psi^{(0)}(k_{N+l-1}; x, y, t), & l = 1, \ldots, M-N+1, \quad r = 1 \\ \Psi^{(0)}(k_{N-r+1}; x, y, t), & l = 1, \quad r = 2, \ldots, N, \\ \Psi^{(r-1)}(\alpha_l^{(r-1)}; x, y, t), & l = 2, \ldots, M-N+1, \quad r = 2, \ldots, N, \end{cases}$$

with $\alpha_l^{(r-1)} \in \Gamma_{r-1}$, $l = 2, \ldots, M-N+1$, real and ordered. For any fixed $r \in \{1, \ldots, N\}$, the coefficients $B_j^{(r)}$, $j = 1, \ldots, M-N+1$, are determined imposing that

$$\lim_{P\to Q_r} \Psi^{(r)}(P, \vec{t}) = f^{(r)}(x, y, t) \tag{7}$$

where $Q_r \in \Gamma_r$ is such that $\lambda^{-1}(Q_r) = 0$, and the $\{f^{(r)}(x,y,t)\ ,\ r = 1,\dots,N\}$ form a basis of heat hierarchy solutions for the given soliton data $([A],\mathcal{K})$.

For any fixed $r \in \{1,\dots,N\}$, the poles $b_k^{(r)}$ are computed imposing the gluing conditions at the double points for all x, y, t:

(8)
$$\Psi^{(r)}(\lambda_j^{(r)};x,y,t) = \begin{cases} \Psi^{(0)}(k_{N+j-1};x,y,t), & r=1,\ \ j=1,\dots,M-N+1, \\ \Psi^{(0)}(k_{N-r+1};x,y,t), & r=2,\dots,N,\ \ j=1, \\ \Psi^{(r-1)}(\alpha_j^{(r-1)};x,y,t), & r=2,\dots,N,\ \ j=2,\dots,M-N+1. \end{cases}$$

The principal technical problem in the construction is to control the compatibility of the linear systems of equations associated to conditions (7) and (8) and to control the sign of the $B_j^{(r)}$ so to get the divisor $b_k^{(r)}$ in the prescribed position ($M-N$ poles in each copy of Γ_r, $r = 1,\dots,N$, and exactly one pole in each finite oval). Then the action of the Darboux transformation (4) is just to move the poles inside the finite ovals.

The strategy we adopt in [1] is the following:

(1) We impose that the representative matrix associated to the behaviour of the vacuum wave–function Ψ at Q_rs be upper triangular;
(2) We do the construction recursively starting from the last row of the matrix;
(3) We introduce a scaling parameter ξ to rule the position of the double points and to control the dominant phases k_l in Ψ at the double points and at Q_r, $r = 1,\dots,N$, at leading order in ξ, when $\xi >> 1$;
(4) We check that (7) and (8) give compatible linear systems when $\xi \to \infty$ and compute the matrix A associated to the leading asymptotics;
(5) We check that at leading order in ξ, the coefficients $B_j^{(r)}$ and the poles $b_k^{(r)}$ satisfy the desired requirements;
(6) We check that (7) and (8) give compatible linear systems and their solutions have the desired properties for any fixed $\xi >> 1$.

We prove that, for any soliton data $([A],\mathcal{K})$, with $[A] \in Gr^{\rm TP}(N,M)$, the construction goes through with the following choice of the marked points:

(9) $$\lambda_1^{(r)} = 0,\ \ \lambda_j^{(r)} = -\xi^{2(j-1)},\ \ \alpha_j^{(r)} = \xi^{2j-5}, \qquad j = 2,\dots,M-N+1,\ \ r = 1,\dots,N.$$

To check that (7) and (8) produce compatible systems of conditions, it is necessary to study the properties of the upper–triangular matrices A and $A(\xi)$

$$A(\xi) = A + O(\xi^{-1})$$

which rule the asymptotics of $\Psi(P;x,y,t)$ when $P \in \Gamma_r \cap \Omega_0$, $r = 1,\dots,N$. The matrix A governs the leading order asymptotics in the parameter as $\xi \to \infty$, where just the dominant phases count at the marked points. We prove that A is in banded form, that is

$$A = \begin{bmatrix} 1 & A^1_2 & A^1_3 & \dots & A^1_{M-N+1} & 0 & \dots & 0 & 0 \\ 0 & 1 & A^2_3 & \dots & A^2_{M-N+1} & A^2_{M-N+2} & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \\ 0 & \dots & 0 & 1 & \dots & \dots & \dots & A^{N-1}_{M-1} & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & \dots & A^N_{M-1} & A^N_M \end{bmatrix}, \tag{10}$$

where $A^r_j > 0$ if and only if $r \le j \le M - N + r$, $r = 1,\dots,N$. Since $[A] \in Gr^{\mathrm{TP}}(N,M)$, then A is also totally positive in classical sense with all non–trivial minors strictly positive (a minor is trivial if it is the determinant of a submatrix containing either a row or a column of zeroes).

Let us denote $A^{[r]}$ (respectively $A(\xi)^{[r]}$) the r–th row of A (respectively of $A(\xi)$). Then, by the recursive construction,

$$A(\xi)^{[r]} = A^{[r]} + \sum_{j=1}^{N-r} \epsilon^{(r)}_j(\xi) A^{[r+j]}, \tag{11}$$

where $\epsilon^{(r)}_j(\xi) \in \mathbb{R}$.

We use the following notation: $\Delta_{[i_1,\dots,i_l]}$ denotes the determinant of the submatrix of A formed by the last l rows and the columns $1 \le i_1,\dots,i_l \le M$.

The key lemma necessary to prove the compatibility of (7) and (8) in the limit $\xi \to \infty$ is the following one.

Lemma 3.1. *(Principal Algebraic Lemma* [1]*) Assume that A is in banded form as in (10) and that, after removing the first row and the first column from A, we obtain a matrix in $Gr^{TP}(N-1,M-1)$. Then $[A] \in Gr^{TP}(N,M)$ if and only if there exist $\hat{B}_n > 0$,*

$n = 1, \dots, M-N+1$, *such that*

$$A^{[1]} \equiv [1, A^1_2, \dots, A^1_{M-N+1}, 0, \dots, 0] = \sum_{n=1}^{M-N+1} \hat B_n \mathcal{E}^{[n]},$$

where $\mathcal{E}^{[1]} = [1, 0, 0, \dots, 0]$, *and, for* $n = 2, \dots, M-N+1$,

$$\mathcal{E}^{[n]} = [0, \Delta_{[2,n+1,\dots,n+N-2]}, \Delta_{[3,n+1,\dots,n+N-2]}, \dots, \Delta_{[n,n+1,\dots,n+N-2]}, 0, \dots, 0].$$

Moreover, in such case $\hat B_1 = A^1_1 = 1$, *and*

$$\hat B_n = \frac{\Delta_{[n,\dots,n+N-1]}}{\Delta_{[n,\dots,n+N-2]}\Delta_{[n+1,\dots,n+N-1]}}, \qquad n = 2, \dots, M-N+1.$$

Thanks to the Principal Algebraic Lemma, we may associate two collections to A : matrices $\hat E^{(r)}$ and scalars $\hat B^{(r+1)}_l > 0$, $r = 0, \dots, N-1$, $l = 1, \dots, M-N+1$, which govern the gluing rules at the double points and the asymptotics at each Q_r at leading order in ξ, when $\xi \to \infty$ (see sections 3.2 and 3.3 in ([1]):

(1) Each $\hat E^{(r)}$ is an $(M-N+1) \times M$ matrix with non-negative entries
(2) For $r = 0$ the matrix $\hat E^{(0)}$ is defined by: $(\hat E^{(0)[l]})_j = \delta_j^{N+l-1}$.
(3) For $r = 1$ and $j = 1, \dots, M-N+1$, $\hat B^{(1)}_j = A^N_{N+j-1}$
(4) For each $r \in \{1, \dots, N\}$ we have: $A^{[N-r+1]} = \sum_{j=1}^{M-N+1} \hat B^{(r)}_j \hat E^{(r-1)[j]}$,
(5) For each $r \in [1, N]$ we have: $\hat E^{(r)[2,\dots,M-N+1]} = \mathcal{B}^{(r)} \hat E^{(r-1)}$, where $\mathcal{B}^{(r)}$ is lower triangular $(M-N) \times (M-N+1)$ matrix whose entries are subtraction free rational functions in $\hat B^{(r)}_j$ (for the explicit formulas see [1]).

The relations above are invariant in $Gr^{\mathrm{TP}}(N, M)$ since the elements of each matrix $\hat E^{(r)}$ and the coefficients $\hat B^{(r)}_j$ are subtraction free rational expressions in a totally positive base in the sense of Fomin and Zelevinsky (see [1]). As a consequence, all the above identities are associated to the given point in the Grassmannian and not to the representative matrix A.

We then show that (7) and (8) form a compatible system for any $\xi >> 1$ and we prove the following Theorems.

Theorem 3.1. *(The rational curve* Γ *and the vacuum wavefunction* Ψ [1]*) Let* A *be a totally positive matrix in banded form as in (10) so that* $[A] \in Gr^{TP}(N, M)$, *and let*

$\xi >> 1$. Let $\Gamma_0, \cdots, \Gamma_N$ be $N+1$ copies of $\mathbb{C}P^1$ and $P_0 \in \Gamma_0$. Let $k_1 < k_2 < \cdots < k_M$ and P_0, such that $\lambda^{-1}(P_0) = 0$, be $M+1$ marked points in Γ_0. For $r = 1, \ldots, N$, let us fix $M-N+2$ marked points in Γ_r: $\lambda_1^{(r)} = 0$, $\lambda_j^{(r)} = -\xi^{2(j-1)}$, $\alpha_j^{(r)} = \xi^{2j-5}$, $j = 2, \ldots, M-N+1$, and Q_r such that $\lambda^{-1}(Q_r) = 0$.

Then there exists a unique totally positive matrix $A(\xi) = A + O(\xi^{-1})$ and a unique vacuum wave–function $\Psi(P; x, y, t)$, meromorphic for $P \in \Gamma$, where $\Gamma = \Gamma_0 \sqcup \Gamma_1 \sqcup \cdots \sqcup \Gamma_N$, regular in $(x, y, t) \in \mathbb{R}^3$ with the following properties:

1. *$\Psi(P; 0, 0, 0) \equiv 1$, for all $P \in \Gamma$;*
2. *It satisfies (8) and $\lim_{\lambda \to \infty} \Psi^{(r)}(\lambda; x, y, t) = \sum_{j=N-r+1}^{M} \frac{A_j^r(\xi)}{\sum_{l=N-r+1}^{M} A_l^r(\xi)} E_j(x, y, t)$, $\forall x, y, t \in \mathbb{R}$, $r = 1, \ldots, N$;*
3. *its divisor of poles is $\mathcal{B} = \{b_n^{(r)}(\xi)\ ,\ n = 1, \ldots, M-N,\ r = 1, \ldots, N\}$ and it is independent of x, y, t;*
4. *it has an essential singularity at $P_0 \in \Omega_0$ such that $\Psi(\lambda; x, y, t) = e^{\theta(\lambda; x, y, t)}$.*

Moreover, the real part of Γ, $\Gamma_{\mathbb{R}}$, possesses $1 + (M-N)N$ ovals and each oval is topologically equivalent to a circle. Each double point of Γ is a common point to exactly a pair of ovals.

Let us denote Ω_0 the oval containing the infinity point $P_0 \in \Gamma_0$ (infinite oval), and let $\Omega_{r,n}$, $r = 1, \ldots, N$, $n = 1, \ldots, M-N$, be the $(M-N) \times N$ finite ovals. Then $Q_r \in \Omega_0$, $r = 1, \ldots, N$, and the finite ovals are defined by the following properties:

1. *For $n = 1, \ldots, M-N$, and for any $j = 2, \ldots, N$: $\Omega_{1,n} \cap \Gamma_0 = [k_{N+n-1}, k_{N+n}]$, $\Omega_{1,n} \cap \Gamma_1 = [\lambda_{n+1}^{(1)}, \lambda_n^{(1)}]$, and $\Omega_{1,n} \cap \Gamma_j = \emptyset$, for $j = 2, \ldots, N$;*
2. *For $r \in \{2, \ldots, N\}$: $\Omega_{r,1} \cap \Gamma_0 = [k_{N-r+1}, k_{N-r+2}]$, $\Omega_{r,1} \cap \Gamma_{r-1} = [\lambda_1^{(r-1)}, \alpha_2^{(r-1)}]$, $\Omega_{r,1} \cap \Gamma_r = [\lambda_2^{(r)}, \lambda_1^{(r)}]$, and $\Omega_{r,1} \cap \Gamma_j = \emptyset$, $\forall j \in \{1, \ldots, N\} \backslash \{r-1, r\}$;*
3. *For $r \in \{2, \ldots, N\}$ and $n \in \{2, \ldots, M-N\}$: $\Omega_{r,n} \cap \Gamma_{r-1} = [\alpha_n^{(r-1)}, \alpha_{n+1}^{(r-1)}]$, $\Omega_{r,n} \cap \Gamma_r = [\lambda_{n+1}^{(r)}, \lambda_n^{(r)}]$, and $\Omega_{r,n} \cap \Gamma_j = \emptyset$, $\forall j \in \{1, \ldots, N\} \backslash \{r-1, r\}$.*

$\Psi(\lambda; x, y, t)$ is real for $P \in \Gamma_{\mathbb{R}}$ and, in each finite oval $\Omega_{r,n}$, it possesses exactly one simple pole $b_n^{(r)}(\xi)$, whose position is independent of x, y, t, and such that $b_n^{(r)}(\xi) \in]\lambda_{n+1}^{(r)}, \lambda_n^{(r)}[\subset \Gamma_r \cap \Omega_{r,n}$.

Finally, the coefficients and the poles of Ψ *as in (6) satisfy* $B_j^{(r)} = \frac{\hat B_j^{(r)}}{\sum\limits_{l=1}^{M-N+1} \hat B_l^{(r)}} + O(\xi^{-1})$, $b_k^{(r)} = -\frac{\sum\limits_{l=1}^{k} \hat B_l^{(r)}}{\sum\limits_{l=1}^{k+1} \hat B_l^{(r)}} \xi^{2(l-1)}(1+O(\xi^{-1}))$, *for* $r=1,\dots,N$, $j=1,\dots,M-N+1$, $k=1,\dots,M-N$.

We remark that the topological type of Γ is the same for all values of the parameter $\xi >> 1$. Then the Darboux transformation (4) moves each divisor point $b_j^{(r)}$ inside its oval in such a way that, at the double points of Γ, the divisor points of $D\Psi$ may only occur in couples (see [1]). In the latter case we use the following counting rule: we attribute one divisor point to the first oval and the second divisor point to the other oval.

Theorem 3.2. *(The Baker-Akhiezer function on* Γ *and the Krichever divisor* $\mathcal{D}$ [1]*) Under the hypotheses of Theorem 3.1, let D be the dressing transformation associated to the soliton data* $([A],\mathcal{K})$ *and defined in (4). Then* $\tilde\Psi(P;x,y,t) = \dfrac{D\Psi(P;x,y,t)}{D\Psi(P,0,0,0)}$, $P\in\Gamma$, *is the normalized Baker-Akhiezer function on* Γ *associated to* $([A],\mathcal{K})$*, and it has the following properties:*

1. *it is meromorphic for* $P\in\Gamma\backslash\{P_0\}$ *and regular for all* x,y,t;
2. *it is real for* $P\in\Gamma_{\mathbb{R}}$ *and real* x,y,t;
3. *its divisor of poles is* $\mathcal{D} = \{\gamma_j^{(0)}\ ,\ j=1,\dots,N\}\cup\{\gamma_n^{(r)}\ ,\ n=1,\dots,M-N-1,\ r=1,\dots,N\}$ *and it is independent of* x,y,t.

Moreover, for any fixed x,y,t, $\tilde\Psi$ *has the following properties:*

1. $P_0\in\Gamma_0$, *is an essential singularity and* $\tilde\Psi^{(0)}(\lambda;x,y,t) = \prod\limits_{j=1}^{N} \frac{\lambda-\gamma_j^{(0)}(x,y,t)}{\lambda-\gamma_j^{(0)}} e^{\theta(\lambda,x,y,t)}$;
2. $\tilde\Psi^{(1)}(\lambda_j^{(1)},x,y,t) = \tilde\Psi^{(0)}(k_{N+j-1},x,y,t)$, *for all* $j\in\{1,\dots,M-N+1\}$;
3. *For* $r=2,\dots,N$, $\tilde\Psi^{(r)}(\lambda_1^{(r)},x,y,t) = \tilde\Psi^{(0)}(k_{N-r+1},x,y,t)$;
4. *For* $r=2,\dots,N$, *and* $j=2,\dots,M-N+1$, $\tilde\Psi^{(r)}(\lambda_j^{(r)},x,y,t) = \tilde\Psi^{(r-1)}(\alpha_j^{(r-1)},x,y,t)$.

The Krichever divisor $\mathcal{D}$ *has the following properties:*

1. *The component* Γ_0 *contains exactly* N *divisor points* $\gamma_1^{(0)},\dots,\gamma_N^{(0)}$;
2. *For any* $r\in\{1,\dots,N\}$, Γ_r *contains exactly* $M-N-1$ *divisor points* $\gamma_1^{(r)},\dots,\gamma_{M-N-1}^{(r)}$;

(3) *For any* $r \in \{0, \ldots, N\}$*, all divisor points lying in* Γ_r *are pairwise different;*

(4) $\mathcal{D} \cap \Omega_0 = \emptyset$*, that is no divisor point occurs in the infinite oval;*

(5) $\mathcal{D} \subset \bigcup_{r,j} \Omega_{r,j}$*, that is each divisor point is real and lies in some finite oval;*

(6) *Each finite oval* $\Omega_{r,j}$ *contains exactly one divisor point according to the counting rule.*

In [1] we also give explicit estimates of the divisor points in $\mathcal{D}$ for $\xi >> 1$.

4. Degenerations of Hyperelliptic M–curves associated to points in $Gr^{\mathrm{TP}}(1, M)$ and $Gr^{\mathrm{TP}}(M-1, M)$

It is a relevant open question to classify which $(M-N, N)$–soliton solutions may be associated to Krichever data on rational degenerations of a given class of M–curves. In this section, for any fixed soliton data $([A], \mathcal{K})$ with $[A]$ either in $Gr^{\mathrm{TP}}(1, M)$ or $Gr^{\mathrm{TP}}(M-1, M)$ and $\mathcal{K} = \{k_1 < k_2 < \cdots < k_M\}$, we show how to assign the Krichever data on $(\tilde{\Gamma}, P_-, \zeta)$, where $\tilde{\Gamma} = \Gamma_+ \sqcup \Gamma_-$ is the rational degeneration of a hyperelliptic curve of genus $g = M-1$, with affine part

$$\tilde{\Gamma} \ : \ \{(\zeta, \mu) \in \mathbb{C}^2 \ : \ \mu^2 = \prod_{j=1}^{M} (\zeta - k_j)^2\}, \tag{12}$$

$\Gamma_+ = \{(\zeta, \mu(\zeta)\,;\, \zeta \in \mathbb{C}\}$, $\Gamma_- = \{(\zeta, -\mu(\zeta)\,;\, \zeta \in \mathbb{C}\}$, and $P_\pm \in \Gamma_\pm$ are such that $\zeta^{-1}(P_\pm) = 0$. Moreover let σ be the hyperelliptic involution which exchanges Γ_+ with Γ_-, *i.e.* $\sigma(\zeta, \mu(\zeta)) = (\zeta, -\mu(\zeta))$. In the following we use the same notation for the points in $\tilde{\Gamma}$ and their ζ–coordinates.

Lemma 4.1. *Let* $[a] \in Gr^{TP}(1, M)$*, with* $a = [a_1, \ldots, a_M]$*,* $a_j > 0$*,* $j = 1, \ldots, M$*, let* $k_1 < k_2 < \cdots < k_M$*, and define*

$$\Psi_{1,M}(\zeta; x, y, t) = \begin{cases} \Psi^{(0)}(\zeta; x, y, t), & \text{if } \zeta \in \Gamma_-, \\ \Psi_+(\zeta; x, y, t) \equiv \displaystyle\sum_{j=1}^{M} \frac{a_j}{\sum\limits_{n=1}^{M} a_m} \frac{\prod_{s \neq j} (\zeta - k_s)}{\prod_{l=1}^{M-N} (\zeta - b_l^{(1)})} \Psi^{(0)}(k_j; x, y, t), & \text{if } \zeta \in \Gamma_+, \end{cases} \tag{13}$$

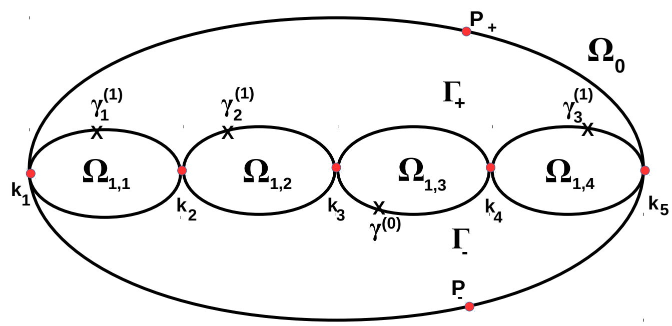

FIGURE 1. $\tilde{\Gamma}$ *for a point in* $Gr^{TP}(1,5)$*; the Krichever divisor is in generic position since* $\gamma^{(0)}$ *does non coincide with any double point.*

where the poles $b_l^{(1)}$ *are defined imposing* $\Psi_+(k_j;x,y,t) = \Psi^{(0)}(k_j;x,y,t)$*, for all* $j = ,\dots,M$ *and, for all* $x,y,t\in\mathbb{R}$*. Then* Ψ *is meromorphic on* $\tilde{\Gamma}\backslash\{P_-\}$ *and* $b_l^{(1)} \in]k_l,k_{l+1}[\cap\Gamma_+$*,* $l = 1,\dots,M-1$*.*

The proof is straightforward and we omit it. We remark that, for any $\xi >> 1$, the curves $\tilde{\Gamma}$ and $\Gamma = \Gamma(\xi)$ as in Theorem 3.1, are not topologically equivalent if $M \geq 2$. In the following Corollary we list the properties of the Krichever divisor on $\tilde{\Gamma}$.

Corollary 4.1. *Let the Darboux transformation be* $D^{(1)} = \partial_x - w_1(x,y,t)$*, where* $w_1(x,y,t) = \partial_x \log(f(x,y,t))$ *and* $f(x,y,t) = \sum_{j=1}^{M} a_j E_j(x,y,t)$*. Then* $\tilde{\Psi}_{1,M}(\zeta;x,y,t) = \dfrac{D^{(1)}\Psi_{1,M}(\zeta;x,y,t)}{D^{(1)}\Psi_{1,M}(\zeta;0,0,0)}$*, is the Baker–Akhiezer function of* $([a],\mathcal{K})$ *on* $(\tilde{\Gamma},P_-,\zeta)$*. The pole divisor of* $\tilde{\Psi}_{1,M}(\zeta;x,y,t)$ *is* $\mathcal{D}_{1,M} = \{\gamma^{(0)}\}\cup\mathcal{D}_+$*, where* $\gamma^{(0)} = \dfrac{\sum_{j=1}^{M} k_j a_j}{\sum_{j=1}^{M} a_j} \in \Gamma_-$ *and* $\mathcal{D}_+ = \{\gamma_1^{(1)},\dots\gamma_{M-2}^{(1)}\} \subset \Gamma_+$*. Moreover, there exists* $s\in\{1,\dots,M-1\}$ *such that* $\gamma^{(0)} \in [k_s,k_{s+1}[\,\cap\,\Gamma_-$*, and* $\mathcal{D}_+ \subset (\,]k_1,k_M[\,\backslash\,]k_s,k_{s+1}[\,)\,\cap\,\Gamma_+$*.*

In particular, if $\gamma^{(0)} \neq k_s$*, then* $\mathcal{D}_+ \cap (\,]k_l,k_{l+1}[\,\cap\Gamma_+\,) \neq \emptyset$*, for any* $l = 1,\dots,M-1$*,* $l\neq s$*. Otherwise, if* $\gamma^{(0)} = k_s \in \Gamma_-$*, then* $s\in\{2,\dots,M-1\}$*, and there exists* $\bar{l} \in \{1,\dots,M-2\}$ *such that* $\gamma_{\bar{l}}^{(1)} = k_s \in \Gamma_+$ *and* $\mathcal{D}_+ \cap (\,]k_{s-1},k_{s+1}[\,\backslash\,\{k_s\}) = \emptyset$*.*

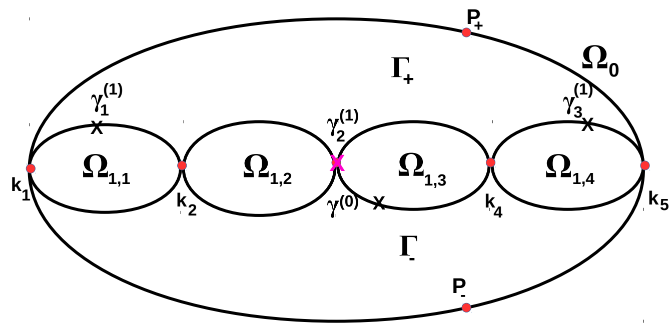

FIGURE 2. *The same as in Figure 1 with a Krichever divisor in special position: the poles $\gamma^{(0)}$ and $\gamma_2^{(1)}$ coincide with the double point k_3.*

In Figures 1 and 2 we show two possible positions of the divisor for a soliton solution $([a], \mathcal{K})$ when $[a] \in Gr^{\mathrm{TP}}(1,5)$ and $\tilde{\Gamma}$ is the rational degeneration of a genus $g = 4$ hyperelliptic curve.

Let $x_i > 0$, $i = 1, \dots, M-1$, $x_M = 1$, and define

$$C^i_j = \begin{cases} 1 & \text{if } j = i, \\ \dfrac{x_i}{x_{i+1}} & \text{if } j = i+1, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Then $[C] \in Gr^{\mathrm{TP}}(M-1, M)$. The double points (9) on Γ_r, $r = 1, \dots, M-1$, are just

$$\lambda_1^{(r)} = 0, \qquad \lambda_2^{(r)} = -1, \qquad \alpha_2^{(r)} = \xi^{-1}, \tag{15}$$

and the Darboux transformation for $([C], \mathcal{K})$ is $D^{(M-1)} = \partial_x^{M-1} - w_1(x,y,t)\partial_x^{M-2} - \cdots - w_{M-1}(x,y,t)$, with $f^{(r)}(x,y,t) = x_{M-r+1}E_{M-r}(x,y,t) + x_{M-r}E_{M-r+1}(x,y,t)$, $r = 1, \dots, M-1$. Let the rational curve Γ and the vacuum wavefunction $\Psi(P; x, y, t)$ be as in Theorem 3.1 and define $\tilde{\Psi}_{M-1,M}(P; x, y, t) := \dfrac{D^{(M-1)}\Psi(P; x, y, t)}{D^{(M-1)}\Psi(P; 0, 0, 0)}$, $P \in \Gamma \backslash \{P_0\}$. Then the following Corollary holds true.

Corollary 4.2. $\tilde{\Psi}_{M-1,M}$ *restricted to* $\Gamma_1 \sqcup \Gamma_2 \sqcup \cdots \sqcup \Gamma_{M-1}$ *is constant w.r.t. the spectral parameter* λ*, and the Krichever divisor* $\mathcal{D}_{M-1,M} = \{\gamma_1^{(0)}, \dots, \gamma_{M-1}^{(0)}\}$ *satisfies*

$$x_{i+1} \prod_{l=1}^{M} (k_i - \gamma_l^{(0)}) + x_i \prod_{l=1}^{M} (k_{i+1} - \gamma_l^{(0)}) = 0, \qquad i = 1, \dots, M-1, \tag{16}$$

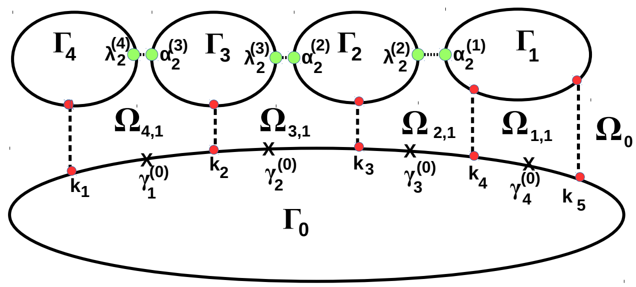

FIGURE 3. *According to Theorem 3.1, for any point in* $Gr^{TP}(4,5)$, $\Gamma = \Gamma_0 \sqcup \hat{\Gamma}_+$, *with* $\hat{\Gamma}_+ = \Gamma_1 \sqcup \cdots \sqcup \Gamma_4$, *is the rational degeneration of a hyperelliptic* M*–curve of genus* $g = 4$. *The red and green points are the marked double points.*

with $\gamma_l^{(0)} \in]k_l, k_{l+1}[\cap \Gamma_0$, *for* $l = 1, \dots, M-1$.

Identities (16) are easily deduced from the equations $D^{(M-1)} f^{(r)} = 0$, $r = 1, \dots, M$.

On each Γ_r, $r = 1, \dots, M-1$, let us perform the linear substitution

$$\zeta = c_0^{(r)} \lambda + c_1^{(r)},$$

where $c_1^{(r)} = k_{M-r}$, and $c_0^{(r)}$ are recursively defined

$$c_0^{(1)} = k_{M-1} - k_M, \qquad c_0^{(r)} = \frac{c_0^{(r-1)}}{\xi} + k_{M-r} - k_{M-r+1}, \; r = 2, \dots, M-1.$$

In the new local coordinate the double points are

$$\lambda_j^{(r)} = \begin{cases} k_{M-r}, & \text{if } j = 1, \quad r = 1, \dots, M-1, \\ k_M, & \text{if } j = 2, \quad r = 1, \\ \alpha_2^{(r-1)} = \frac{c_0^{(r-1)}}{\xi} + k_{M-r+1} \in]k_{M-r}, k_{M-r+1}], & \text{if } j = 2, \quad r = 2, \dots, M-1. \end{cases}$$

Let us notice that

$$\lim_{\xi \to \infty} c_0^{(r)} = k_{M-r} - k_{M-r+1}, \qquad \lim_{\xi \to \infty} \lambda_2^{(r)} = k_{M-r+1}, \;\; r = 2, \dots, M-1.$$

Then the following Corollary to Theorem 3.1 holds true

Corollary 4.3. *Let* $k_1 < k_2 < \cdots < k_M$, $\xi >> 1$ *and* $[C] \in Gr^{TP}(M-1, M)$. *Then the curve* Γ *constructed in Theorem 3.1 is the rational degeneration of a regular hyperelliptic*

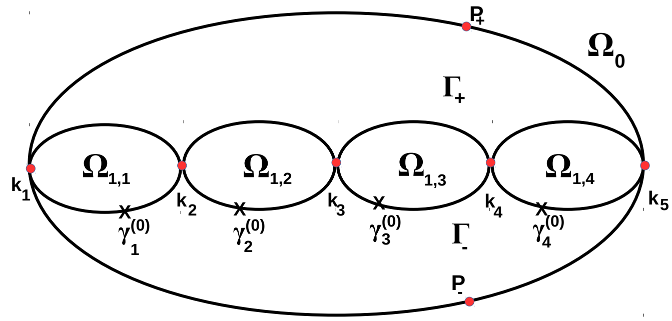

FIGURE 4. *The same point in $Gr^{TP}(4,5)$ as in Figure 3: $\Gamma_- = \Gamma_0$, while Γ_+ is the desingularization of $\Gamma_1 \sqcup \Gamma_2 \sqcup \Gamma_3 \sqcup \Gamma_4$. Each finite oval $\Omega_{1,j}$ is the desingularization of the corresponding oval $\Omega_{j,1}$, $j = 1, \dots, M-1$ of Figure 3 and coincides with the ovals defined for points in $Gr^{TP}(1,5)$ in Figures 1 and 2. The Krichever divisor is the same as in Figure 3, since $\mathcal{D} \subset \Gamma_0$ and $\Gamma_0 = \Gamma_-$.*

curve of genus $g = M-1$, $\Gamma = \Gamma_- \sqcup \hat{\Gamma}_+$ such that $\hat{\Gamma}_+ = \Gamma_1 \sqcup \ldots \sqcup \Gamma_{M-1}$ is Γ_+ with $M-2$ extra double points $\lambda_2^{(r)} = \alpha_2^{(r-1)}$, $r = 2, \dots, M-1$.

The double points $\lambda_2^{(r)}$, $\alpha_2^{(r-1)}$, $r = 2, \dots, M-1$, are due to the construction and may be eliminated since the Baker–Akhiezer function $\tilde{\Psi}_{M-1,M}$ restricted to $\hat{\Gamma}_+$ is constant with respect to the spectral parameter.

For any $[C] \in Gr^{\mathrm{TP}}(M-1, M)$, we now desingularize explicitly Γ to $\tilde{\Gamma}$ using the duality property between points in $Gr^{\mathrm{TP}}(1, M)$ and in $Gr^{\mathrm{TP}}(M-1, M)$. The duality between Grassmann cells is naturally linked to the space–time transformation of soliton solutions and it has been used to classify the asymptotic behavior of soliton solutions (see [2, 26] and references therein).

Let $f(x,y,t) = \sum_{j=1}^{M} a_j E_j(x,y,t)$ be the τ–function (3) of $([a], \mathcal{K})$, with $[a] \in Gr^{\mathrm{TP}}(1, M)$ and define $\tilde{f}(x,y,t) = \frac{f(x,y,t)}{\prod_{j=1}^{M} E_j(x,y,t)}$. Then $\tilde{f}(x,y,t)$ is equivalent to $f(x,y,t)$, since $u(x,y,t) =$

$2\partial_x^2 \log(f(x,y,t)) = 2\partial_x^2 \log(\tilde{f}(x,y,t))$, while

$$\tau(x,y,t) = \tilde{f}(-x,-y,-t) = \sum_{j=1}^{M} a_j \prod_{l\neq j} E_l(x,y,t),$$

is the τ–function of the soliton solution $([C],\mathcal{K})$, with $[C] \in Gr^{\mathrm{TP}}(M-1,M)$ defined by the dual transformation

$$a_j = \Delta_{[\hat{j}]}(C) \prod_{1\le i<l\le M; i,l\neq j} (k_l - k_i), \qquad j = 1,\dots,M. \tag{17}$$

Here $\Delta_{[\hat{j}]}(C)$ is the minor obtained eliminating the j–th column from C. Let us notice that if we apply the space–time transformation twice we go back to the initial soliton $([a],\mathcal{K})$.

The space–time transformation preserves the rational curve $\tilde{\Gamma}$ defined in (12) and associated to $[a]$, and it moves the Krichever divisor points inside its finite ovals. Indeed, let C be as in (14). Then $\Delta_{[\hat{j}]}(C) = x_j$, $j=1,\dots,M$, and (17) define a point $[a] \in Gr^{\mathrm{TP}}(1,M)$.

Theorem 4.1. *Let $([C],\mathcal{K})$, with $\mathcal{K} = \{k_1 < k_2 < \cdots < k_M\}$ and $[C] \in Gr^{TP}(M-1,M)$ with C as in (14). Let $\tilde{\Gamma}$ be as in (12) and let $\Psi_{1,M}(\zeta;x,y,t)$ be the vacuum wavefunction defined in (13) for the soliton solution $([a],\mathcal{K})$, $[a] \in Gr^{TP}(1,M)$, with a_j defined by (17).*

Then $\Psi_{1,M}(\zeta;x,y,t)$ is also the vacuum wavefunction for $([C],\mathcal{K})$on $\tilde{\Gamma}$, and

$$\tilde{\Psi}(P;x,y,t) = \frac{D^{(M-1)}\Psi_{1,M}(P;x,y,t)}{D^{(M-1)}\Psi_{1,M}(P;0,0,0)}, \qquad P \in \tilde{\Gamma}\backslash\{P_-\}, \ \ (x,y,t) \in \mathbb{C}^3,$$

is the normalized Baker–Akhiezer function on $(\tilde{\Gamma}, P_-, \zeta)$ for the soliton solution $([C],\mathcal{K})$.

Finally the divisor $\mathcal{D}_{M-1,M}$ is obtained from the vacuum divisor $\{b_1^{(1)},\dots,b_{M-1}^{(1)}\}$ by the hyperelliptic involution σ,

$$\gamma_l^{(0)} = \sigma(b_l^{(1)}), \qquad l = 1,\dots,M-1.$$

Proof. By construction $\tilde{\Psi}(\zeta;x,y,t) = \tilde{\Psi}_{M-1,M}(\zeta;x,y,t)$, on Γ_-. On Γ_+, $\tilde{\Psi}(\zeta;x,y,t)$ is meromorphic of degree $M-1$ in ζ and, forall $j,s = 1,\dots,M$, and forall x,y,t, it satisfies

$$\tilde{\Psi}(k_j;x,y,t) = \tilde{\Psi}(k_s;x,y,t) = \frac{\left(\prod\limits_{i=1}^{M} E_i(x,y,t)\right)\left(\sum\limits_{j=1}^{M} x_j \Big[\prod\limits_{1\le n<m<M,n,m\neq j} (k_m - k_n)\Big]\right)}{\sum\limits_{j=1}^{M} x_j \left[\prod\limits_{1\le n<m<M,n,m\neq j} (k_m - k_n) \prod\limits_{i\neq j}^{M} E_i(x,y,t)\right]}.$$

Then $\tilde{\Psi}(k_j; x, y, t)$ is constant with respect to the spectral parameter on Γ_+. Finally $\Psi(k_j; x, y, t) = E_j(x, y, t)$, $j = 1, \dots, M$ is equivalent to

$$x_{i+1} \prod_{l=1}^{M} (k_i - b_l^{(1)}) + x_i \prod_{l=1}^{M} (k_{i+1} - b_l^{(1)}) = 0, \qquad i = 1, \dots, M-1.$$

The equations above are just (16). Then clearly the divisors $\{b_1^{(1)}, \dots, b_{M-1}^{(1)}\}$ and $\mathcal{D}_{M-1,M}$ are related by the hyperelliptic involution σ. □

References

[1] S. Abenda, P.G. Grinevich *Rational degenerations of M-curves, totally positive Grassmannians and KP–solitons*, arXiv:1506.00563 submitted.

[2] S. Chakravarty, Y. Kodama. *Classification of the line-soliton solutions of KPII.* J. Phys. A **41** (2008), 275209, 33 pp.

[3] S. Chakravarty, Y. Kodama. *Soliton solutions of the KP equation and application to shallow water waves.* Stud. Appl. Math. **123** (2009) 83-151.

[4] L.A, Dickey. Soliton equations and Hamiltonian systems. Second edition. Advanced Series in Mathematical Physics, 26. World Scientific Publishing Co., Inc., River Edge, NJ, 2003. xii+408 pp.

[5] B.A. Dubrovin. *Theta-functions and nonlinear equations. (Russian) With an appendix by I. M. Krichever.* Uspekhi Mat. Nauk. **36** (1981), 1180.

[6] B.A. Dubrovin, I.M. Krichever, S.P. Novikov. Integrable systems. Dynamical systems, IV, 177-332, Encyclopaedia Math. Sci., 4, Springer, Berlin, 2001.

[7] B. A.Dubrovin, S.M. Natanzon. *Real theta-function solutions of the Kadomtsev-Petviashvili equation.* Izv. Akad. Nauk SSSR Ser. Mat. **52** (1988) 267-286.

[8] S. Fomin, A. Zelevinsky. *Double Bruhat cells and total positivity.* J. Amer. Math. Soc. **12** (1999) 335-380.

[9] S. Fomin, A. Zelevinsky. *Cluster algebras I: foundations.* J. Am. Math. Soc. **15** (2002) 497-529.

[10] F.R. Gantmacher, M.G. Krein. *Sur les matrices oscillatoires.* C.R.Acad.Sci.Paris **201** (1935) 577-579.

[11] D.A. Gudkov. *The topology of real projective algebraic varieties.* Russ. Math. Surv. **29** (1974) 1-79.

[12] A. Harnack. *Über die Vieltheiligkeit der ebenen algebraischen Curven.* Math. Ann. **10** (1876) 189-199.

[13] R. Hirota. The direct method in soliton theory. Cambridge Tracts in Mathematics, 155. Cambridge University Press, Cambridge, 2004. xii+200 pp.

[14] B.B. Kadomtsev, V.I. Petviashvili. *On the stability of solitary waves in weakly dispersive media.*, Sov. Phys. Dokl. **15** (1970) 539-541.

[15] Y. Kodama,L.K. Williams. *The Deodhar decomposition of the Grassmannian and the regularity of KP solitons.* Adv. Math. **244** (2013) 979-1032.

[16] Y. Kodama,L.K. Williams. *KP solitons and total positivity for the Grassmannian.* Invent. Math. **198** (2014) 637-699.

[17] I. M. Krichever. *An algebraic-geometric construction of the Zakharov-Shabat equations and their periodic solutions.* (Russian) Dokl. Akad. Nauk SSSR **227** (1976) 291-294.

[18] I. M. Krichever. *Integration of nonlinear equations by the methods of algebraic geometry.* (Russian) Funkcional. Anal. i Priloen. **11** (1977) 15-31, 96.

[19] G. Lusztig. *A survey of total positivity.* Milan J. Math. **76** (2008) 125-134.

[20] V. B. Matveev, M.A. Salle. Darboux transformations and solitons. Springer Series in Nonlinear Dynamics. Springer-Verlag, Berlin, 1991.

[21] T. Miwa, M. Jimbo, E. Date. Solitons. Differential equations, symmetries and infinite-dimensional algebras. Cambridge Tracts in Mathematics, 135. Cambridge University Press, Cambridge, 2000. x+108 pp.

[22] A. Pinkus. Totally positive matrices. Cambridge Tracts in Mathematics, 181. Cambridge University Press, Cambridge, 2010. xii+182 pp.

[23] A. Postnikov. *Total positivity, Grassmannians, and networks.*, arXiv:math/0609764 [math.CO]

[24] M. Sato. *Soliton equations as dynamical systems on infinite-dimensional Grassmann manifold.* in: Nonlinear PDEs in Applied Sciences (US-Japan Seminar, Tokyo), P. Lax and H. Fujita eds., North-Holland, Amsterdam (1982) 259-271.

[25] J. Schoenberg. *Über variationsvermindende lineare Transformationen.* Mathematishe Zeitschrift **32** (1930) 321-328.

[26] Y. Zarmi. *Vertex dynamics in multi-soliton solutions of Kadomtsev-Petviashvili II equation.* Nonlinearity **27** (2014), 14991523.

Dipartimento di Matematica, Università di Bologna, P.zza di Porta San Donato 5, I-40126 Bologna BO, ITALY

E-mail address: simonetta.abenda@unibo.it