GROUP ACTIONS ON THE SPHERE AND MULTIPLICITY RESULTS FOR THE CR-YAMABE EQUATION
AZIONI DI GRUPPO SULLA SFERA E RISULTATI DI MOLTEPLICITÀ PER L'EQUAZIONE DI YAMABE CR

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Abstract. We will show that the CR-Yamabe equation has several families of infinitely many changing sign solutions, each of them having different symmetries. The problem is variational but it is not Palais-Smale: using different complex group actions on the sphere, we will find many closed subspaces on which we can apply the minmax argument.

Sunto. Proveremo che l'equazione di Yamabe CR ammette diverse famiglie di (infinite) soluzioni a segno non costante, ognuna di esse con una distinta simmetria. Il problema è variazionale, ma non Palais-Smale: usando distinte azioni di gruppo sulla sfera, troveremo diversi sottospazi chiusi su cui poter applicare l'argomento di minmax.

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1. Introduction

The results in this note are obtained in collaboration with Ali Maalaoui and Giulio Tralli (see [16], [21]).

We will show the existence of several families of infinitely many changing sign solutions of the following sub-Riemannian Yamabe equation on the standard CR-sphere (S^{2n+1}, θ)

$$-\Delta_\theta v + c_n v = |v|^{\frac{4}{Q-2}} v, \qquad v \in \mathcal{S}^1(S^{2n+1}), \tag{1}$$

where θ is the standard Liouville (contact) form, Δ_θ denotes the related sub-Laplacian, $\mathcal{S}^1(S^{2n+1})$ is the Folland-Stein Sobolev type space on the sphere, $Q = 2n + 2$ and c_n is a

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suitable dimensional positive constant related to the (constant) Webster curvature of the sphere (see [11] for a full detailed exposition).

The problem is variational but, as in the Riemannian case, the functional associated with the equation (1) is not Palais-Smale.

For the classical Yamabe equation on $\mathbb{R}^n$, after the classification of the positive solutions (*bubbles*) in [4], the first result about changing sign solutions was proved by Ding in [8]. Following the analysis by Ambrosetti and Rabinowitz [1], Ding found a suitable subspace X of the space of the variations on which he performed the minmax argument to the restricted functional.

Later on, many authors proved the existence of changing sign solutions using other kinds of variational methods (see [2, 3] and the references therein). Finally in a couple of recent works [6, 7], del Pino, Musso, Pacard, and Pistoia found changing sign solutions, different from those of Ding, by using a superposition of positive and negative bubbles arranged on some special sets.

In the CR case, the (variational) positive solutions to the equation (1) were completely classified by Jerison and Lee in [12]. In [16], we proved that there exist changing sign solutions to (1) using a very particular group of isometries, namely the one generated by the Reeb vector field of the contact form θ on S^{2n+1}. Using the standard Hopf fibration, we showed that the restricted functional satisfies the Palais-Smale condition by showing that the new space of variation is identified with a Sobolev space on a complex projective space: in particular, due to the very special symmetry, we were able to switch from a *critical subelliptic* problem to a *subcritical elliptic* one.

Here we will show that there exist many complex group actions that lead to changing sign solutions, each of them having different symmetries. Moreover in these general cases one cannot use any analogue of the Hopf fibration, therefore we will prove the compactness condition by using a general bubbling behavior of the Palais-Smale sequences, that in our situation will lead to a contradiction on the boundedness of the energy.

Finally, we recall that in literature there are many other existence and multiplicity results about Yamabe type equations in different settings: we address the reader for instance to the papers [13], [23], [18], [17], [19], [20], [14] [15] and the references therein.

2. Structure of the equation and group actions

Let us now set

$$q^* = \frac{2Q}{Q-2}$$

and let us consider the variational problem on the sphere associated to the following functional

$$I : \mathcal{S}^1(S^{2n+1}) \to \mathbb{R}, \qquad I(v) = \frac{1}{2}\int_{S^{2n+1}} \left(|D_\theta v|^2 + c_n v^2\right) - \frac{1}{q^*}\int_{S^{2n+1}} |v|^{q^*}.$$

Here $|D_\theta v|$ stands for the Webster norm of the contact gradient $D_\theta v$, where $D_\theta = \{X_1, Y_1, \ldots, X_n, Y_n\}$ is an orthonormal basis of $ker(\theta)$; for any $j = 1, \ldots, n$ we set $Y_j = JX_j$ with J the standard complex structure on $\mathbb{C}^{n+1}$: if we identify $\mathbb{C}^{n+1} \simeq \mathbb{R}^{2n+2}$ with

$$z = (z_1, \ldots, z_{n+1}) \simeq (x_1, y_1, \ldots, x_{n+1}, y_{n+1}),$$

then J is the block matrix

$$J = \begin{pmatrix} \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & 0_{2\times 2} & \cdots & 0_{2\times 2} \\ 0_{2\times 2} & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & \ddots & 0_{2\times 2} \\ \vdots & 0_{2\times 2} & \cdots & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \end{pmatrix}.$$

We are then looking for critical points of I, knowing that any variational solution of (1) is actually a classical solution ([9, 10]).

The exponent q^* is called critical since the embedding

$$\mathcal{S}^1(S^{2n+1}) \hookrightarrow L^{q^*}(S^{2n+1})$$

is continuous but not compact: due to this lack of compactness, I does not satisfy the Palais-Smale condition.

However, we will prove the following

Theorem 2.1. *There exists a sequence of solutions $\{v_k\}$ of (1), with*

$$\int_{S^{2n+1}} |v_k|^{q^*} \longrightarrow \infty, \qquad as \quad k \to \infty.$$

Theorem 2.1 will imply that equation (1) has infinitely many changing sign solutions: in fact, by the classification result by Jerison and Lee [12], all the positive solutions of the equation (1) have the same energy.

Now, let us denote

$$\mathbb{U}(n+1) = \{g \in \mathbb{O}(2n+2),\ gJ = Jg\},$$

where $\mathbb{O}(2n+2)$ is the group of real valued $(2n+2) \times (2n+2)$ orthogonal matrices.

We explicitly note that the functional I is invariant under the action of the group $\mathbb{U}(n+1)$, i.e.

$$I(v) = I(v \circ g), \qquad \forall\, g \in \mathbb{U}(n+1).$$

If $\mathbb{G}$ is a subgroup of $\mathbb{U}(n+1)$, we define

$$X_{\mathbb{G}} = \{v \in \mathcal{S}^1(S^{2n+1}) \,:\, v \circ g = v,\ \ \forall\, g \in \mathbb{G}\}.$$

We are going to make the following assumptions on $\mathbb{G}$:

(H1) $X_{\mathbb{G}}$ is an infinite dimensional real vector space;

(H2) for any $z_0 \in S^{2n+1}$, the $\mathbb{G}$-orbit of z_0 has at least one accumulation point.

Example 2.2. *As in Ding* [8] *we can consider, for any $k \in \{1, \ldots, n\}$, the subgroups $\mathbb{G}_k = \mathbb{U}(k) \times \mathbb{U}(n+1-k)$ formed by the matrices*

$$\begin{pmatrix} g_1 & 0_{2k \times 2(n+1-k)} \\ 0_{2(n+1-k) \times 2k} & g_2 \end{pmatrix}$$

with $g_1 \in \mathbb{U}(k)$ and $g_2 \in \mathbb{U}(n+1-k)$.

The functions in $\mathcal{S}^1(S^{2n+1})$ depending only on $|z_1|, |z_2|$ (with $z = (z_1, z_2)$, $z_1 \in \mathbb{C}^k$, $z_2 \in \mathbb{C}^{n+1-k}$) belong to $X_{\mathbb{G}_k}$. Thus we immediately get that $X_{\mathbb{G}_k}$ is infinite dimensional.

Moreover, the $\mathbb{G}_k$-orbits of any point contain at least a circle. Therefore $\mathbb{G}_k$ satisfies $(H1)$ and $(H2)$.

We explicitly observe that, differently from Ding, we allow the case $k = 1$: basically this is related to the fact that the orbit of any point under the action of $\mathbb{U}(1)$ is the circle S^1, instead the orbits related to $\mathbb{O}(1)$ are $\mathbb{Z}_2$.
The following is a more general situation than can happen in this regard.

Counterexample 2.3. *For any $m \in \mathbb{N}$, let us consider the subgroups $\mathbb{G}_m = \mathbb{Z}_m \times \mathbb{U}(n)$ formed by the matrices*

$$\begin{pmatrix} \begin{pmatrix} \cos(\frac{2\pi j}{m}) & \sin(-\frac{2\pi j}{m}) \\ \sin(\frac{2\pi j}{m}) & \cos(\frac{2\pi j}{m}) \end{pmatrix} & 0_{2\times 2n} \\ 0_{2n\times 2} & g \end{pmatrix},$$

with $j \in \{0, \ldots, m-1\}$ and $g \in \mathbb{U}(n)$. These are subgroups of the group $\mathbb{G}_1$ defined in the previous example. Thus, $X_{\mathbb{G}_m}$ are infinite dimensional. On the other hand, if we fix a point $z_0 = (e^{it_0}, 0) \in \mathbb{C}^{n+1}$, its $\mathbb{G}_m$-orbit contains exactly m points. Therefore, the groups $\mathbb{G}_m$ don't satisfy our main assumption $(H2)$.

Example 2.4. *In* [16] *it has been considered the case of the one-parameter group $\mathbb{G}_T$ generated by the flow of the Reeb vector field T of θ. In our notations, $\mathbb{G}_T$ is formed by the matrices $\exp(tJ)$, $t \in \mathbb{R}$, and it is a sub-group of any $\mathbb{G}_k$. The orbits are always great circles and our assumptions $(H1)$ and $(H2)$ are thus satisfied for $\mathbb{G}_T$: in particular considering the following Hopf fibration*

$$S^1 \hookrightarrow S^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n$$

where the fibers are exactly the orbits of T, we have the identification $X_{\mathbb{G}_T} \simeq \mathcal{S}^1(\mathbb{C}P^n)$.

We can provide also examples in which the groups are not in block diagonal matrices.

Example 2.5. *Let us consider the case $n = 1$, i.e. the case of S^3, and let us define the vector fields*

$$\tilde{X} = x_2\partial x_1 + y_2\partial y_1 - x_1\partial x_2 - y_1\partial y_2$$

and

$$\tilde{Y} = -y_2\partial x_1 + x_2\partial y_1 - y_1\partial x_2 + x_1\partial y_2.$$

Now we consider the one-parameter groups ($\mathbb{G}_{\tilde{X}}$ and $\mathbb{G}_{\tilde{Y}}$, respectively) generated by $\tilde{X}$ and $\tilde{Y}$: in other words,

$$\mathbb{G}_{\tilde{X}} = \{\exp(t\tilde{X}) \,:\, t \in \mathbb{R}\}, \qquad \mathbb{G}_{\tilde{Y}} = \{\exp(t\tilde{Y}) \,:\, t \in \mathbb{R}\}$$

where with some abuse of notation we can identify the vector fields with the matrices

$$\tilde{X} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \textit{and} \quad \tilde{Y} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Such groups are contained in $\mathbb{U}(2)$ since $\tilde{X}$ and $\tilde{Y}$ are skew-symmetric and they commute with J. Moreover, the vector fields are well-defined and non-vanishing everywhere in S^3, and their integral curves are always great circles. This proves in particular that $\mathbb{G}_{\tilde{X}}$ and $\mathbb{G}_{\tilde{Y}}$ satisfy our hypotheses $(H1)$ and $(H2)$.

Let us observe that also the group $\mathbb{G}_{\tilde{T}}$ generated by the vector field identified with the matrix

$$\tilde{J} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

satisfies our assumptions. Indeed, any matrix A in the Lie algebra gives rise to a linear isometry ϕ_t^A of S^3 in the following sense

$$\begin{cases} \dfrac{d}{dt}\phi_t^A(z) &= A\phi_t^A(z) \\ \\ \phi_0^A(z) &= z. \end{cases}$$

Geometrically, the isometry ϕ_t^J is given by the integral curves of the Reeb vector field T and the functions which are constant along T are the ones considered in [16]; whereas the isometry $\phi_t^{\tilde{J}}$ is given by the integral curves of $\tilde{T}$ which is the Reeb vector field of the "dual" contact form $\tilde{\theta}$ (the one with $\tilde{J}$ as complex structure instead of J). Finally, the isometries $\phi_t^{\tilde{X}}$ and $\phi_t^{\tilde{Y}}$ are given by the integral curves of the vector fields $\tilde{X}$ and $\tilde{Y}$ which are right-invariant with respect to the standard group structure in S^3: in particular they commute with the left-invariant vector fields X and Y.

3. Proof of the Theorem 2.1

The proof is based on the following lemma by Ambrosetti and Rabinowitz, which gives a condition on some particular subspaces of the space of variations on which it is allowed to perform the minmax argument (see Theorems 3.13 and 3.14 in [1]).

Lemma 3.1. *Let X be a closed subspace of $\mathcal{S}^1(S^{2n+1})$. Suppose that:*

(i) *X is infinite-dimensional;*

(ii) *$I_{|X}$, the restriction of I on X, satisfies Palais-Smale on X.*

Then $I_{|X}$ has a sequence of critical points $\{v_k\}$ in X, such that

$$\int_{S^{2n+1}} |v_k|^{q^*} \longrightarrow \infty, \qquad as \quad k \to \infty.$$

Now, suppose we are given $\mathbb{G}$ such that $X_{\mathbb{G}}$ satisfies $(H1)$ and $(H2)$. In order to apply the previous Lemma we need to show that the restricted functional $I_{|X_{\mathbb{G}}}$ is Palais-Smale. We will argue by contradiction, namely: we will consider a general Palais-Smale sequence and, since there is a precise characterization for these last ones, we will see that if Palais-Smale is violated then bubbling occurs, and the concentration set is finite and discrete, therefore the hypothesis $(H2)$ and the invariance given by the group action will lead to a contradiction on the boundedness of the energy.

Hence, we have the following

Lemma 3.2. *Let $\mathbb{G}$ be a subgroup of $\mathbb{U}(n+1)$ that satisfies $(H2)$. Then $I_{|X_{\mathbb{G}}}$, the restriction of I on $X_{\mathbb{G}}$, satisfies the Palais-Smale compactness condition on $X_{\mathbb{G}}$.*

Proof. Let us first recall a general bubbling behavior of the Palais-Smale sequences (P-S) of the functional I, [5]. Let $\{v_k\}$ be a (P-S)$_c$ sequence, that is

$$I(v_k) \to c, \qquad I'(v_k) \to 0$$

as $k \to \infty$. Then there exist $m \geq 0$, m sequences $z_k^j \to z_j \in S^{2n+1}$ (for $1 \leq j \leq m$), m sequences of positive numbers R_k^j converging to zero, and a solution $v_\infty \in \mathcal{S}^1(S^{2n+1})$, such that up to a subsequence

$$v_k = v_\infty + \sum_{j=1}^{m} v_{k,j} + o(1), \text{ as } k \to \infty.$$

where $v_{k,j}$ are bubbles concentrating in z_k^j with concentrations R_k^j.
Moreover,

$$I(v_k) = I(v_\infty) + \sum_{j=1}^{m} I(v_j) + o(1), \text{ as } k \to \infty. \tag{2}$$

The important claim for what we need is that the blow-up set

$$\Theta = \{z_j \in S^{2n+1}, \quad 1 \leq j \leq m\}$$

is discrete and finite. Now we are looking at the functional $I_{|X_{\mathbb{G}}}$, so we have that our (P-S) sequence is invariant under the action of $\mathbb{G}$ and this means that if $z \in \Theta$ is a concentration point, then the whole orbit of z would be, which is impossible under our assumption. In particular this would contradict the energy quantization (2).
Indeed, let us assume for the sake of simplicity that we have only one concentration point $z_0 \in \Theta$ and let $(g_i)_{1 \leq i \leq l}$ be l elements in $\mathbb{G}$: then $g_i \cdot z_0$ are also concentration points in Θ. In particular

$$c = \lim_{k \to \infty} I(v_k) = I(v_\infty) + \sum_{i=1}^{l} I(v_0) = I(v_\infty) + lI(v_0) \tag{3}$$

with v_0 the bubble concentrating at z_0. Now we observe that $I(v_0) \neq 0$, since from the equation satisfied by v_0 we have that

$$\int_{S^{2n+1}} |D_\theta v_0|^2 + c(n)v_0^2 = \int_{S^{2n+1}} |v_0|^{q^*}.$$

Therefore

$$I(v_0) = \Big(\frac{1}{2} - \frac{1}{q^*}\Big) \int_{S^{2n+1}} |v_0|^{q^*}$$

and this last quantity is different from zero if bubbling occurs.
Finally, since $\mathbb{G}$ satisfies the hypothesis $(H2)$, the orbit of z_0 has at least one accumulation

point on the sphere, therefore Θ contains infinitely many points: hence, by letting $l \to \infty$ in (3), we get a contradiction. $\square$

Now we will prove the main theorem.

Proof. (of Theorem 2.1)
Let us take any $\mathbb{G}$ subgroup of $\mathbb{U}(n+1)$ that satisfies assumptions $(H1)$ and $(H2)$: the examples in Section 2 provide the existence of a large class of such groups. By the previous lemma, we have that $I_{|X_{\mathbb{G}}}$ satisfies Palais-Smale on $X_{\mathbb{G}}$. Therefore $X_{\mathbb{G}}$ satisfies conditions (i) and (ii) in the lemma by Ambrosetti and Rabinowitz, so that $I_{|X_{\mathbb{G}}}$ has a sequence of critical points $\{v_k\}$ in $X_{\mathbb{G}}$, such that

$$\int_{S^{2n+1}} |v_k|^{q^*} \longrightarrow \infty, \qquad \text{as} \quad k \to \infty.$$

On the other hand, we have that the functional I is invariant under the action of $\mathbb{G}$. By the Principle of Symmetric Criticality [22], this implies that any critical point of the restriction $I_{|X_{\mathbb{G}}}$ is also a critical point of I on the whole space of variations $\mathcal{S}^1(S^{2n+1})$. This ends the proof. $\square$

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