THE QUATERNIONIC HARDY SPACE AND THE GEOMETRY OF THE UNIT BALL

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ABSTRACT. The quaternionic Hardy space of slice regular functions $H^2(\mathbb{B})$ is a reproducing kernel Hilbert space. In this note we see how this property can be exploited to construct a Riemannian metric on the quaternionic unit ball $\mathbb{B}$ and we study the geometry arising from this construction. We also show that, in contrast with the example of the Poincaré metric on the complex unit disc, no Riemannian metric on $\mathbb{B}$ is invariant with respect to all slice regular bijective self maps of $\mathbb{B}$.

SUNTO. Lo spazio di Hardy di funzioni slice regolari sui quaternioni $H^2(\mathbb{B})$ è uno spazio di Hilbert con nucleo riproducente. In questa nota vediamo come questa proprietà possa essere utilizzata per costruire una metrica Riemanniana sulla palla unitaria quaternionica $\mathbb{B}$ e studiamo la geometria derivante da questa costruzione. Mostriamo inoltre che, in contrasto con l'esempio della metrica di Poincaré sul disco unitario complesso, non esiste una metrica Riemanniana su $\mathbb{B}$ che sia invariante rispetto a tutte le trasformazioni slice regolari biettive della palla in sé.

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1. INTRODUCTION

The study of function spaces on given domains often reveals geometric aspects of the domains themselves. In fact there is a rich interplay between reproducing kernel Hilbert spaces and distance functions. See [1] for an overview and several examples from one-variable holomorphic function space theory. In this note we introduce the Hardy space of slice regular functions over the quaternions and we study its relation with the geometry of the quaternionic unit ball $\mathbb{B}$. Slice regularity is, among other possible definitions, a notion of *hyper-holomorphy* for quaternionic

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functions. It was introduced in 2006 by Gentili and Struppa (see [11, 12]) and it is nowadays a well established theory, in continuous development. When compared to the well known example of quaternionic Fueter regular functions [8, 9], it has the advantage of including the identity map, the natural polynomials and converging power series of the form $\sum_{n\geq 0} q^n a_n$, where q is a quaternionic variable and the coefficients a_n are quaternions as well. The Hardy space of slice regular functions on $\mathbb{B}$ is in fact defined by

$$H^2(\mathbb{B}) := \Big\{ \sum_{n\geq 0} q^n a_n : \Big\| \sum_{n\geq 0} q^n a_n \Big\|_{H^2(\mathbb{B})} := \sqrt{\sum_{n\geq 0} |a_n|^2} < +\infty \Big\}.$$

It is possible to show that $H^2(\mathbb{B})$ is a quaternionic reproducing kernel Hilbert space, with respect to the inner product

$$\Big\langle \sum_{n\geq 0} q^n a_n, \sum_{n\geq 0} q^n b_n \Big\rangle_{H^2(\mathbb{B})} := \sum_{n\geq 0} \overline{b_n} a_n$$

and the reproducing kernel is

$$k(q,w) = k_w(q) = \sum_{n\geq 0} q^n \bar{w}^n, \quad \text{for } q, w \in \mathbb{B}.$$

For the definition and all basic results concerning quaternionic Hilbert spaces see, e.g., [13] and references therein. For the properties we are interested in, the same results hold in quaternion valued Hilbert spaces and complex valued Hilbert spaces, and the proofs are very similar.

By measuring distances between the projections of kernel functions on the unit sphere of $H^2(\mathbb{B})$ it is possible to define a metric on $\mathbb{B}$:

$$\delta_{\mathbb{B}}(p,q) := \sqrt{1 - \left| \Big\langle \frac{k_q}{\|k_q\|_{H^2(\mathbb{B})}}, \frac{k_p}{\|k_p\|_{H^2(\mathbb{B})}} \Big\rangle_{H^2(\mathbb{B})} \right|^2}.$$

The same construction in the complex case gives the pseudo-hyperbolic metric in the complex unit disc. Here we are interested in the infinitesimal version of $\delta_{\mathbb{B}}$, that is in the Riemmannian length metric associated with $\delta_{\mathbb{B}}$, and we obtain the following result.

Theorem. *For any $w \in \mathbb{B}$, let us identify the tangent space $T_w\mathbb{B}$ with $\mathbb{H}$. For any vector $d \in T_w\mathbb{B}$, if $w = u + yI_w$ lies in $L_{I_w} := \mathbb{R} + \mathbb{R}I_w$ and we decompose $d = d_1 + d_2$ with d_1 in L_{I_w} and d_2 in $L_{I_w}^{\perp}$, then the length of d with respect to $g_{\mathbb{B}}$ is given by*

$$|d|^2_{g_{\mathbb{B}}(w)} = \frac{1}{(1-|w|^2)^2}|d_1|^2 + \frac{1}{|1-w^2|^2}|d_2|^2. \tag{1}$$

To understand the geometry arising from this construction, in Section 5 we study isometries of $(\mathbb{B}, g_{\mathbb{B}})$, and we sketch the proof of a characterization theorem.

Theorem. *The group of isometries of* $(\mathbb{B}, g)$ *is generated by*

(a) *regular Möbius transformation of the form*

$$q \mapsto M_\lambda(q) = (1 - q\lambda)^{-*} * (q - \lambda) = \frac{q - \lambda}{1 - q\lambda},$$

with λ *in* $(-1, 1)$*;*

(b) *isometries of the sphere of imaginary units, which in polar coordinates* $r \geq 0, t \in [0, \pi], I \in \mathbb{S}$ *read as*

$$q = re^{tI} \mapsto T_A(q) = re^{tA(I)},$$

where $A : \mathbb{S} \to \mathbb{S}$ *is an isometry of* $\mathbb{S}$*;*

(c) *the reflection in the imaginary hyperplane,*

$$q \mapsto R(q) = -\overline{q}.$$

In the previous statement the symbol $*$ denotes an appropriate multiplication operation between slice regular functions (see Section 2).
We conclude this note with a discussion on the problem of finding a Poincaré-type metric on $\mathbb{B}$, namely a Riemannian metric preserved by all slice regular self map of the unit ball.
The results presented in this note are obtained in collaboration with Nicola Arcozzi and we refer to [2] for the details of the proofs.

2. Some background on slice regular functions

Let $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ denote the non-commutative four dimensional real algebra of quaternions, where i, j, k are imaginary units $i^2 = j^2 = k^2 = -1$ subject to the rules $ij = k = -ji$. If $\mathbb{S} = \{q \in \mathbb{H} : q^2 = -1\}$ denotes the two dimensional sphere of imaginary units of $\mathbb{H}$, then we can slice the space of quaternions in copies of the complex plane intersecting along the real axis

$$\mathbb{H} = \bigcup_{I \in \mathbb{S}} (\mathbb{R} + \mathbb{R}I), \quad \mathbb{R} = \bigcap_{I \in \mathbb{S}} (\mathbb{R} + \mathbb{R}I),$$

where $L_I := \mathbb{R} + \mathbb{R}I \cong \mathbb{C}$ for any $I \in \mathbb{S}$. Each quaternion can be expressed as $q = x + yI_q$ where $x, y \in \mathbb{R}$ and $I_q \in \mathbb{S}$. To have uniqueness outside the real axis we chose $y \geq 0$. The real part of q is $\mathrm{Re}(q) = x \in \mathbb{R}$, the imaginary part is $\mathrm{Im}(q) = yI_q \in \mathbb{I} = \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$. The conjugate of q is $\bar{q} = x - yI_q$ and its modulus is given by $|q|^2 = q\bar{q}$.

We recall the definition of slice regularity, together with some basic property of this class of functions. In the sequel, for the sake of simplicity, we will consider functions defined on the open unit ball $\mathbb{B} := \{q \in \mathbb{H} : |q| < 1\}$. We refer to the monograph [10] for the more general case of *symmetric slice domains* and for a detailed account of the theory.

Definition 2.1. *A function $f : \mathbb{B} \to \mathbb{H}$ is said to be* slice regular *if for any $I \in \mathbb{S}$ the restriction f_I of f to $\mathbb{B}_I := \mathbb{B} \cap L_I$ has continuous partial derivatives and it is such that*

$$\overline{\partial}_I f_I(x + yI) = \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + yI) = 0$$

for all $x + yI \in \mathbb{B}_I$.

A wide class of examples is given by power series with quaternionic coefficients of the form $\sum_{n\geq 0} q^n a_n$ which converge in open balls centered at the origin. Moreover, the following characterization result holds.

Theorem 2.2. *A function f is slice regular on $\mathbb{B}$ if and only if f has a power series expansion $f(q) = \sum_{n\geq 0} q^n a_n$ converging absolutely and uniformly on compact sets of $\mathbb{B}$.*

For slice regular functions, it is possible to define an appropriate notion of derivative: the *slice (or Cullen) derivative* of a slice regular function f on $\mathbb{B}$, is the slice regular function defined by

$$\partial_c f(x + yI) = \frac{1}{2}\left(\frac{\partial}{\partial x} - I\frac{\partial}{\partial y}\right) f_I(x + yI).$$

Slice regular functions defined on $\mathbb{B}$ have a peculiar property.

Theorem 2.3 (Representation Formula). *Let f be a slice regular function on $\mathbb{B}$ and let $x+y\mathbb{S} \subset \mathbb{B}$. Then, for any $I, J \in \mathbb{S}$,*

$$f(x + yJ) = \frac{1}{2}[f(x + yI) + f(x - yI)] + J\frac{I}{2}[f(x - yI) - f(x + yI)].$$

In particular, there exist $b, c \in \mathbb{H}$ such that $f(x + yJ) = b + Jc$ for any $J \in \mathbb{S}$.

In general, the pointwise product of functions does not preserve slice regularity. Hence we introduce a new multiplication operation, which, in the special case of power series, can be defined as follows.

Definition 2.4. *Let* $f(q) = \sum_{n\geq 0} q^n a_n$*, and* $g(q) = \sum_{n\geq 0} q^n b_n$ *be slice regular functions on* $\mathbb{B}$*. Their* regular product *(or* $*$-product*) is*

$$f * g(q) = \sum_{n\geq 0} q^n \sum_{k=0}^{n} a_k b_{n-k},$$

slice regular on $\mathbb{B}$ *as well.*

The $*$-product is related to the standard pointwise product by the following formula.

Proposition 2.5. *Let* f, g *be slice regular functions on*$\mathbb{B}$*. Then*

$$f * g(q) = \begin{cases} 0 & \text{if } f(q) = 0 \\ f(q)g(f(q)^{-1}qf(q)) & \text{if } f(q) \neq 0 \end{cases}$$

The reciprocal f^{-*} of a slice regular function f with respect to the $*$-product can be defined.

Definition 2.6. *Let* $f(q) = \sum_{n\geq 0} q^n a_n$ *be a slice regular function on* $\mathbb{B}$*,* $f \not\equiv 0$*. Its* regular reciprocal *is*

$$f^{-*}(q) = \frac{1}{f * f^c(q)} f^c(q),$$

where $f^c(q) = \sum_{n=0}^{\infty} q^n \overline{a}_n$*. The function* f^{-*} *is slice regular on* $\mathbb{B} \setminus \{q \in \mathbb{B} \mid f * f^c(q) = 0\}$ *and* $f * f^{-*} = 1$ *there.*

Then we have a natural definition of *regular quotients* of slice regular functions, examples of which, that will appear in the sequel, are the *regular Möbius transformations*, of the form

$$M_a(q) = (1 - q\overline{a})^{-*} * (q - a),$$

where $a \in \mathbb{B}$. These, are slice regular bijective self-maps of the quaternionic unit ball $\mathbb{B}$ and, after multiplication on the right by unit-norm quaternions, they are the only self-maps of $\mathbb{B}$ which are slice regular and bijective. They were introduced by Stoppato in [16].

3. The quaternionic Hardy space $H^2(\mathbb{B})$

As anticipated in the Introduction, a slice regular function $f(q) = \sum_{n\geq 0} q^n a_n$ belongs to the quaternionic Hardy space $H^2(\mathbb{B})$ if and only if

$$\|f\|_{H^2(\mathbb{B})} := \sqrt{\sum_{n\geq 0} |a_n|^2} < +\infty.$$

We recall that the H^2-norm can also be computed in an integral form as

$$\|f\|^2_{H^2(\mathbb{B})} = \int_0^{2\pi} |f(e^{It})|^2 dt$$

where $I \in \mathbb{S}$ is any imaginary unit and, with a slight abuse of notation, f denotes here the a.e. radial limit of the function f, see [6]. By polarizing the H^2-norm, we can endow $H^2(\mathbb{B})$ with a quaternionic Hermitian product, that can be computed as

$$\left\langle \sum_{n\geq 0} q^n a_n, \sum_{n\geq 0} q^n b_n \right\rangle_{H^2(\mathbb{B})} := \sum_{n\geq 0} \overline{b_n} a_n$$

for any $\sum_{n\geq 0} q^n a_n, \sum_{n\geq 0} q^n b_n$ in $H^2(\mathbb{B})$, thus equipping $H^2(\mathbb{B})$ with the structure of quaternionic Hilbert space.

The key property of $H^2(\mathbb{B})$ that allows us to construct an invariant metric on the unit ball $\mathbb{B}$ is that it is endowed with a reproducing kernel: for any w in $\mathbb{B}$ and any f in $H^2(\mathbb{B})$ we have

$$f(w) = \langle f, k_w \rangle_{H^2(\mathbb{B})}, \text{ where } k_w(q) = k(w,q) = \sum_{n\geq 0} q^n \overline{w}^n.$$

The first metric on $\mathbb{B}$ that we consider, denoted by $\delta_{\mathbb{B}}$, measures the distance between projections of kernel functions in the unit sphere of the Hilbert space $H^2(\mathbb{B})$:

$$\delta_{\mathbb{B}}(p,q) := \sqrt{1 - \left| \left\langle \frac{k_q}{\|k_q\|_{H^2(\mathbb{B})}}, \frac{k_p}{\|k_p\|_{H^2(\mathbb{B})}} \right\rangle_{H^2(\mathbb{B})} \right|^2}. \tag{2}$$

In the case of the complex Hardy space on the unit disc $H^2(\mathbb{D})$, this procedure leads to the pseudo-hyperbolic metric

$$\delta_{\mathbb{D}}(z,w) = \left| \frac{z-w}{1-\overline{w}z} \right|.$$

In our quaternionic setting we obtain a formally similar result. In fact, by direct computation it is possible to show that

Proposition 3.1. *Let $\delta_{\mathbb{B}}$ be defined as in* (2)*. For any $w, z \in \mathbb{B}$, $\delta_{\mathbb{B}}(z,w)$ coincides both with the value at z of the regular Möbius transformation M_w associated with w, and with the vaule at w of the regular Möbius transformation M_z associated with z, namely*

$$\delta_{\mathbb{B}}(w,z) = \left|(1-q\overline{z})^{-*} * (q-z)\right|_{|q=w} = \left|(1-q\overline{w})^{-*} * (q-w)\right|_{|q=z}.$$

4. AN INVARIANT METRIC ASSOCIATED WITH $H^2(\mathbb{B})$

The infinitesimal version of the pseudo-hyperbolic metric $\delta_{\mathbb{D}}$ in the complex disc, is the hyperbolic metric in the Riemann-Beltrami-Poincaré disc model:

$$ds^2 = \frac{|dz|^2}{(1-|z|^2)^2}.$$

By infinitesimal version of a distance δ, we mean the length metric associated with δ (see e.g. [14]). The infinitesimal metric associated with $\delta_{\mathbb{B}}$ is a Riemannian metric $g_{\mathbb{B}}$ on $\mathbb{B}$.

Theorem 4.1. *Let $g_{\mathbb{B}}$ denote the length metric associated with $\delta_{\mathbb{B}}$. For any $w \in \mathbb{B}$, let us identify the tangent space $T_w\mathbb{B}$ with $\mathbb{H}$. For any vector $d \in T_w\mathbb{B}$, if w lies in L_{I_w} and we decompose $d = d_1 + d_2$ with d_1 in L_{I_w} and d_2 in $L_{I_w}^{\perp}$, then the length of d with respect to $g_{\mathbb{B}}$ is given by*

$$|d|^2_{g_{\mathbb{B}}(w)} = \frac{1}{(1-|w|^2)^2}|d_1|^2 + \frac{1}{|1-w^2|^2}|d_2|^2. \tag{3}$$

In the metric (3), the first summand is the hyperbolic metric on a slice, while the second "smaller" summand is specifically quaternionic: it measures the variation of a quaternionic Hardy function in the direction orthogonal to the slices. Its small size reflects in geometric terms the property of slice regular functions of being affine in the $\mathbb{S}$ variable, see Theorem 2.3.

The proof of this theorem follows from the application of a more general result concerning a large class of quaternionic reproducing kernel Hilbert spaces, to the case of $H^2(\mathbb{B})$.

The fact that $g_{\mathbb{B}}(w)$ measures vectors in L_{I_w} by multiplying their Euclidean length by $\frac{1}{1-|w|^2}$ means that the restriction of $g_{\mathbb{B}}$ to a slice L_I is the classical Poincaré metric in the unit disc $\mathbb{B}_I$. Using spherical coordinates, $\mathbb{B} = \{re^{tI} \mid r \in [0,1),\ t \in [0,\pi],\ I \in \mathbb{S}\}$, if $q = re^{tI}$ and we decompose the lenght element $dq = dq_1 + dq_2 \in L_{I_w} + L_{I_w}^{\perp}$, then, since dI is orthogonal to I (because I is unitary) we have $|d_1|^2 = dr^2 + r^2dt^2$ and $|d_2|^2 = r^2\sin^2 t|dI|^2$ where $|dI|$ denotes the usual two-dimensional sphere round metric on $\mathbb{S} \cong \mathbb{S}^2$. Therefore we get the expression of the metric tensor $ds^2_{g_{\mathbb{B}}}$ associated with $g_{\mathbb{B}}$ in spherical coordinates:

$$ds^2_{g_{\mathbb{B}}} = \frac{dr^2 + r^2dt^2}{(1-r^2)^2} + \frac{r^2\sin^2 t|dI|^2}{(1-r^2)^2 + 4r^2\sin^2 t}. \tag{4}$$

That is, $g_{\mathbb{B}}$ is a *warped product* of the hyperbolic metric g_{hyp} on the complex unit disc with the standard round metric $g_{\mathbb{S}}$ on the two-dimensional sphere [15].

5. Isometries and geodesics of $(\mathbb{B}, g_{\mathbb{B}})$

To uderstand the geometry of $(\mathbb{B}, g_{\mathbb{B}})$ we study its group of isometries.

Theorem 5.1. *The group of isometries of* $(\mathbb{B}, g_{\mathbb{B}})$ *is generated by maps of type*

(a) *regular Möbius transformation of the form*

$$q \mapsto M_\lambda(q) = (1 - q\lambda)^{-*} * (q - \lambda) = \frac{q - \lambda}{1 - q\lambda},$$

with λ *in* $(-1, 1)$*;*

(b) *isometries of the sphere of imaginary units, which in polar coordinates* $r \geq 0, t \in [0, \pi], I \in \mathbb{S}$ *read as*

$$q = re^{tI} \mapsto T_A(q) = re^{tA(I)},$$

where $A : \mathbb{S} \to \mathbb{S}$ *is an isometry of* $\mathbb{S}$*;*

(c) *the reflection in the imaginary hyperplane,*

$$q \mapsto R(q) = -\overline{q}.$$

From the expression (3) of $g_{\mathbb{B}}$, it is not difficult to see that the three families of functions (a), (b) and (c) act isometrically on $(\mathbb{B}, g_{\mathbb{B}})$.

To show that (a), (b) and (c) actually generate the group of isometries of $(\mathbb{B}, g_{\mathbb{B}})$, the main ingredients of the proof are the following:

1. *Identify two families of* 2*-dimensional totally geodesic submanifolds.*

The first family is associated with isometries of type (a) and it consists of all *slices* $\mathbb{B}_I = \mathbb{B} \cap L_I$, which are hyperbolic discs for any $I \in \mathbb{S}$. To prove that they are totally gedosic we use the fact that the restriction of $g_{\mathbb{B}}$ to $\mathbb{B}_I$ is the classical hyperbolic metric and the second component in $g_{\mathbb{B}}$ is orthogonal to $\mathbb{B}_I$.

The second family is associated with isometries of type (b) and (c) and it consists of the discs

$$D(\pi/2, C(J)) = \{re^{I\pi/2} \in \mathbb{B} : I \in \mathbb{S}, \text{ orthogonal to } J\},$$

obtained intersecting two totally geodesic 3-dimensional submanifolds: the imaginary hyperplane $\mathbb{B}(\pi/2) := \{re^{I\pi/2} \in \mathbb{B} : I \in \mathbb{S}\}$ (associated with the map R) and

$\mathbb{B}(C(J)) := \{re^{It} \in \mathbb{B} : I \in \mathbb{S},$ orthogonal to $J\}$ (associated with maps of type (b): $C(J)$ is the great circle of $\mathbb{S}$ orthogonal to J).

Notice that the two families are "orthogonal" to each other in the following sense:

$$D\left(\pi/2, \mathcal{C}(J)\right) \cap \mathbb{B}_J = \{0\} \text{ and } T_0 D\left(\pi/2, \mathcal{C}(J)\right) = T_0 \mathbb{B}_J^{\perp}.$$

Moreover, applying Möbius maps of the form M_λ to $D\left(\pi/2, \mathcal{C}(J)\right)$, we can extend the orthogonality relation from the origin to all points in $\mathbb{B} \cap \mathbb{R}$. In this way we obtain a family of totally geodesic submanifolds

$$D\left(t, \mathcal{C}(J)\right) = M_{\lambda(t)}\left(D\left(\pi/2, \mathcal{C}(J)\right)\right)$$

that, for $t \in [0, \pi]$ and $J \in \mathbb{S}/\{\pm 1\}$, defines a foliation of the manifold $\mathbb{B}$.

2. *Prove that each isometry preserves the real axis.*

To do this second step we investigate some metric properties of the discs $D\left(\frac{\pi}{2}, \mathcal{C}(J)\right)$. Since the imaginary units taken into account belong to $\mathcal{C}(J) \cong \mathbb{S}^1$, we can change coordinates, setting $I = e^{i\theta}$ and $|dI| = d\theta$, so that the metric g, on $D\left(\frac{\pi}{2}, \mathcal{C}(J)\right)$, reduces to $ds_D^2 = \frac{dr^2}{(1-r^2)^2} + \frac{r^2 d\theta^2}{(1+r^2)^2}$. It is actually convenient to parametrize $D\left(\frac{\pi}{2}, \mathcal{C}(J)\right) \subset \mathbb{I} \cong \mathbb{R}^3$ as a surface of revolution of the form $(\Phi(\rho), \Psi(\rho)\cos\theta, \Psi(\rho)\sin\theta)$, with

$$\rho = \rho(r) = \frac{1}{2}\log\frac{1+r}{1-r}$$

the arc length of the generating curve. In the new coordinates (ρ, θ), the metric tensor is

$$ds_D^2 = d\rho^2 + \frac{1}{4}\tanh^2(2\rho)d\theta^2 = d\rho^2 + \Psi^2(\rho)d\theta^2.$$

It is possible to study geodesics of $D\left(\frac{\pi}{2}, \mathcal{C}(J)\right)$ by means of Euler-Lagrange equations and thus to prove

Lemma 5.2. *Let $J \in \mathbb{S}$. For any $q \in D\left(\frac{\pi}{2}, \mathcal{C}(J)\right)$ such that $q \neq 0$, the injectivity radius at q is finite. On the other hand, the injectivity radius at $q = 0$ is infinite.*

This important metric property allows us to conclude that all isometries map the real diameter of $\mathbb{B}$ to itself.

Notice that the Gaussian curvature K of the two-dimensional submanifold $D\left(\frac{\pi}{2}, \mathcal{C}(J)\right)$ is positive. In fact, see e.g. [7], with respect to coordinates (ρ, θ) it can be computed as

$$K = \frac{-\Psi''(\rho)}{\Psi(\rho)}$$

which is a non-negative quantity since $\Psi(\rho) = \frac{1}{2}\tanh(2\rho) \geq 0$ and $\Psi''(\rho) \leq 0$. This in particular implies that the sectional curvature of $(\mathbb{B}, g)$ is positive on all sections $D\left(\frac{\pi}{2}, \mathcal{C}(J)\right)$, while it is negative on all slices $\mathbb{B}_I$.

3. *Prove that each isometry preserves the imaginary hyperplane.*

Let Φ be an isometry of $(\mathbb{B}, g)$. Up to composition with a regular Möbius transformation of type (a) and with the map $R : q \mapsto -\overline{q}$, we can suppose that $\Phi(0) = 0$ and that, by the preceding step, $\Phi(\mathbb{B} \cap \mathbb{R}^+) = \mathbb{B} \cap \mathbb{R}^+$. Since Φ is an isometry, $\Phi(\mathbb{B}(\pi/2))$ is a totally geodesic submanifold of $\mathbb{B}$. Moreover, since $\Phi(0) = 0$, since the geodesics starting at 0 lie on slices, and since, by the first step, the slices carry the usual hyperbolic-Poincaré metric, we are able to show that Φ maps radii $\gamma_I(r) = re^{\frac{\pi}{2}I}$ to radii of the form $\Phi(\gamma_I(r)) = re^{\theta(I)\psi(I)}$ with $\theta(I) \in [0, \pi]$, and $\psi(I) \in \mathbb{S}$. Next, we show that θ must be constant on $\mathbb{S}$ and that this constant must equal $\pi/2$, thus we conclude that Φ preserves the imaginary hyperplane $\mathbb{B}(\pi/2)$.

4. *Conclusion.*

To conclude we prove that given an isometry Φ that preserves both the real axis and the imaginary hyperplane, its restriction to $\mathbb{B}(\pi/2)$ coincide with an isometry T_A of type (b). This allows us to show that $T_A^{-1} \circ \Phi$ is the identity map, thus completing the proof.

As an application, which also shows the relationship between the metric $g_{\mathbb{B}}$ and the Hardy space $H^2(\mathbb{B})$, we see that the restriction of the metric $g_{\mathbb{B}}$ to a three-dimensional sphere $r\partial\mathbb{B}$ of radius r, induces the volume form

$$dVol_{r\partial\mathbb{B}}(re^{tI}) = \frac{r^3 \sin^2 t}{(1-r^2)((1-r^2)^2 + 4r^2\sin^2(t))} dt dA_{\mathbb{S}}(I)$$

where $dA_{\mathbb{S}}$ denotes the area element of the two-dimensional sphere $\mathbb{S}$. After a normalization $dVol_{r\partial\mathbb{B}}$ induces a volume form on the boundary $\partial\mathbb{B}$ of the unit ball: if $u = e^{sJ} \in \partial\mathbb{B}$, we have

$$\begin{aligned} dVol_{\partial\mathbb{B}}(u) &:= \lim_{r\to 1^-}(1-r^2)dVol_{r\partial\mathbb{B}}(ru) = \lim_{r\to 1^-} \frac{(1-r^2)r^3\sin^2 s}{(1-r^2)((1-r^2)^2+4r^2\sin^2(s))}dtdA_{\mathbb{S}}(I) \\ &= \frac{1}{4}dtdA_{\mathbb{S}}(I). \end{aligned}$$

Notice that $dVol_{\partial\mathbb{B}}$ is the product of the usual spherical metric on the two-dimensional sphere $\mathbb{S}$ with the metric dt on circles $\partial\mathbb{B}_I$ which appears in the definition of Hardy spaces given in [6]. Moreover denoting (with a slight abuse of notation) the radial limit by f itself, we have

Proposition 5.3. *If $f \in H^2(\mathbb{B})$, then*

$$\frac{1}{Vol_{\partial\mathbb{B}}(\partial\mathbb{B})}\int_{\partial\mathbb{B}} |f(u)|^2 dVol_{\partial\mathbb{B}}(u) = ||f||^2_{H^2(\mathbb{B})}.$$

6. POINCARÉ TYPE METRICS ON $\mathbb{B}$

Bisi and Gentili proved in [3] that the usual Poincaré metric on $\mathbb{B}$ is invariant under classical (non-regular) Möbius maps. On the contrary, as shown by Bisi and Stoppato in [4], the same metric is not preserved by regular Möbius maps associated with a non-real point. In the subsequent paper [5] the same authors obtain the proof of an analogue of the Schwarz-Pick Lemma, that, in some sense, motivates the search of a Riemannian metric which is preserved by every slice regular bijective self map of $\mathbb{B}$. In fact one of the statements of their result read as follows.

Theorem 6.1 (Bisi, Stoppato, [5])**.** *Let $f : \mathbb{B} \to \mathbb{B}$ be a slice regular function and let $q_0 \in \mathbb{B}$. Then*

$$\left|(f(q)-f(q_0)) * \left(1-\overline{f(q_0)} * f(q)\right)^{-*}\right| \leq \left|(q-q_0)*(1-\bar{q}_0 * q)^{-*}\right|.$$

Moreover the equality holds for some $q \neq q_0$ if and only if f is a regular Möbius map.

Even if the previous result suggests that regular Möbius play an important role in the study of the intrinsic geometry of the unit ball, there is no Riemannian metric on $\mathbb{B}$ which is invariant under any regular Möbius function, unless the Möbius function is already an isometry for the metric defined in Theorem 4.1. If a geometric invariant for slice regular functions on the quaternionic ball exists, it has to be other than a Riemannian metric.

Theorem 6.2. *Let a be a point of $\mathbb{B} \setminus \mathbb{R}$. There is no Riemannian metric m on $\mathbb{B}$ having as isometry the regular Möbius map $\varphi : q \mapsto (1 - q\overline{a})^{-*} * (a - q)$.*

The proof consists in showing that the real differential of $\varphi \circ \varphi$ at the origin maps the unit disc in $T_0\mathbb{B} \cap L_{I_a}^{\perp}$ into a proper subset of itself. Hence φ cannot be isometric.

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