A NOTE ON VISCOUS CAPILLARY FLUIDS IN FAST ROTATION
FLUIDI VISCOSI E CAPILLARI IN ROTAZIONE RAPIDA

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Abstract. The present note is devoted to the study of singular perturbation problems for a Navier-Stokes-Korteweg system with Coriolis force. Such a model describes the motion of viscous compressible capillary fluids under the action of the Earth rotation. We are interested in the asymptotic behavior of a family of weak solutions in the limit for the Mach, the Rossby and the Weber numbers going to 0.

Sunto. La presente nota è dedicata allo studio di problemi di perturbazione singolare per un sistema di Navier-Stokes-Korteweg con forza di Coriolis. Tale modello descrive il moto di fluidi compressibili, viscosi e capillari sotto l'azione della rotazione della Terra. Ci si interessa qui al comportamento asintotico di una famiglia di soluzioni deboli nel limite a basso numero di Mach, Rossby e Weber.

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1. Introduction

In this note we review some results about singular perturbation problems for the following Navier-Stokes-Korteweg system with Coriolis force:

$$(1)\quad \begin{cases} \partial_t\rho + \mathrm{div}\ (\rho u) = 0 \\ \partial_t(\rho u) + \mathrm{div}\big(\rho u\otimes u\big) + \dfrac{1}{\varepsilon^2}\nabla P(\rho) + \dfrac{1}{\varepsilon}\mathfrak{C}(\rho,u) - \nu\mathrm{div}\big(\rho Du\big) - \dfrac{1}{\varepsilon^{2(1-\alpha)}}\rho\nabla\Delta\rho = 0. \end{cases}$$

These equations can be used to describe the motion of viscous capillary fluids under the action of the rotation of the Earth. The scalar quantity $\rho\geq 0$ represents the density

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of the fluid, while $u \in \mathbb{R}^3$ its velocity field. The smooth function P, just depending on the density, represents the pressure law of the medium, while the term $\rho\nabla\Delta\rho$ takes into account the internal tension. Finally, $\mathfrak{C}(\rho, u)$ is the Coriolis operator, which we take here equal to $e^3 \times \rho u$, where $e^3 = (0, 0, 1)$ denotes the unit vector directed along the x^3-coordinate.

The scaling appearing in (1) corresponds to take the Mach and the Rossby numbers proportional to a small parameter $\varepsilon \in]0, 1]$, and the Weber number of order $\varepsilon^{2(1-\alpha)}$, for some $\alpha \in [0, 1]$ (which means that the capillarity coefficient is supposed to equal $\varepsilon^{2\alpha}$).

For any fixed value of $\varepsilon > 0$, existence of global in time "weak solutions" to system (1) can be established as in work [3] by Bresch, Desjardins and Lin. Actually, the weak formulation adopted in that paper is a bit modified (see Definition 2.1 below) with respect to the usual one, due to a degeneracy of the model in vacuum. Roughly speaking, the idea is to localize the test-functions on regions where $\rho > 0$: this is achieved by (formally) evaluating the momentum equations on functions of the form $\rho\psi$, for smooth ψ. We remark that this is possible thanks to the additional smoothness of the density function, which is provided by the capillarity term. Such a property shows up not just in the classical energy inequality, but also through the so-called *BD entropy conservation*, a second energy inequality first discovered in [2] by Bresch and Desjardins (see also [3]) for our system, and then generalized by the same authors to different models for compressible fluids with density-dependent viscosity coefficients. At this point, let us remark that, in presence of further terms in the momentum equations (e.g. some drag forces like in [1], or a "cold pressure" term), it is possible to resort to the classical weak formulation of the system. We refer to [4] and the references therein for further details about this issue.

In the sequel, we will assume the same weak formulation as in [3]: then, we are interested in studying the asymptotic behavior of weak solutions for $\varepsilon \to 0$, and in characterizing the limit equation. In particular, this means that we are performing the incompressible and high rotation limits simultaneously; on the other hand, the assumed scaling allows us to consider either a low capillarity limit (for $0 < \alpha \leq 1$), or a constant capillarity regime (when choosing $\alpha = 0$).

In works [1] and [11], an analogous investigations is performed for similar systems. There, the authors just focus on the vanishing capillarity case; also, the study is carried out in 2-D domains and for well-prepared initial data. Here, on the contrary, we restrict our attention to the case $\alpha = 0$, in order to look at strong surface tension effects in the limit; the case $\alpha = 1$ can be treated in a very similar way, while the intermediate values $\alpha \in]0, 1[$ are technically more involved, because this choice introduces an anisotropy of scaling in the model (see paper [6]). Also, we consider the three-dimensional domain $\Omega := \mathbb{R}^2 \times]0, 1[$, for which we assume complete slip boundary conditions (to avoid boundary layers phenomena). For general ill-prepared initial data, we prove the convergence of system (1) to a 2-D modified Quasi-Geostrophic equation for the limit density, which can be seen as a sort of stream-function for the limit velocity field.

The result, formulated in Section 2, strongly relies on microlocal symmetrization and spectral analysis of the singular perturbation operator, and on the study of the propagation of acoustic-Rossby waves. In the sequel (see Sections 3 and 4), we will limit ourselves to give just a sketch of the proof. We postpone to Section 5 some comments about the case of variable rotation axis.

The analysis presented in this note is contained in works [6], [7]. We refer to them for complete proofs and further results in this direction.

2. Hypotheses and main results

Defining Ω to be the infinite slab $\mathbb{R}^2 \times]0, 1[$, we consider in $\mathbb{R}_+ \times \Omega$ the scaled Navier-Stokes-Korteweg system with Coriolis force

$$(2) \quad \begin{cases} \partial_t \rho + \mathrm{div}\, (\rho u) = 0 \\ \partial_t (\rho u) + \mathrm{div}\big(\rho u \otimes u\big) + \dfrac{1}{\varepsilon^2} \nabla P(\rho) + \dfrac{e^3 \times \rho u}{\varepsilon} - \nu \mathrm{div}\big(\rho D u\big) - \dfrac{1}{\varepsilon^{2(1-\alpha)}} \rho \nabla \Delta \rho = 0 \,, \end{cases}$$

where $\nu > 0$ denotes the viscosity of the fluid, $Du := \big(\nabla u + {}^t\nabla u\big)/2$ is the viscous stress tensor, $e^3 = (0, 0, 1)$ is the unit vector directed along the x^3-coordinate, and $0 \leq \alpha \leq 1$ is a fixed parameter.

We supplement system (2) by complete slip boundary conditions. Namely, if we denote by n the unitary outward normal to the boundary $\partial\Omega$ of the domain Ω (simply, $\partial\Omega =$

$\{x^3 = 0\} \cup \{x^3 = 1\}$), we impose

$$(3)\quad (u\cdot n)_{|\partial\Omega} \,=\, u^3_{|\partial\Omega} \,=\, 0\,, \qquad (\nabla\rho\cdot n)_{|\partial\Omega} \,=\, \partial_3\rho_{|\partial\Omega} \,=\, 0\,, \qquad \big((Du)n\times n\big)_{|\partial\Omega} \,=\, 0\,.$$

In the previous system (2), the scalar function $\rho \geq 0$ represents the density of the fluid, $u \in \mathbb{R}^3$ its velocity field, and $P(\rho)$ its pressure, given by the γ-law

$$(4)\qquad P(\rho) \,:=\, \frac{1}{\gamma}\,\rho^\gamma\,, \qquad\qquad \text{for some} \quad 1 \,<\, \gamma \,\leq\, 2\,.$$

Remark 2.1. *Note that equations* (2)*, supplemented by boundary conditions* (3)*, can be recast as a periodic problem with respect to* x^3*, in the new domain*

$$\Omega \,=\, \mathbb{R}^2 \,\times\, \mathbb{T}^1\,, \qquad\qquad \textit{with} \qquad \mathbb{T}^1 \,:=\, [-1,1]/\sim\,,$$

where $\sim$ *denotes the equivalence relation which identifies* -1 *and* 1*. Indeed, it is enough to extend* ρ *and* u^h *as even functions with respect to* x^3*, and* u^3 *as an odd function.*

In what follows, we will always assume that such modifications have been performed on the initial data, and that the respective solutions keep the same symmetry properties.

We now define the notion of weak solution for our system: it is based on the one given in [3]. The requirements on the initial data and on integrability properties of respective solutions will be justified by energy estimates (see Section 3 below).

First of all, let us introduce the internal energy function $h = h(\rho)$, such that

$$h''(\rho) \,=\, \frac{P'(\rho)}{\rho} \,=\, \rho^{\gamma-2} \qquad\qquad \text{and} \qquad\qquad h(1) \,=\, h'(1) \,=\, 0\,,$$

and let us define the energies

$$(5)\qquad E_\varepsilon[\rho,u](t) \;:=\; \int_\Omega \left(\frac{1}{\varepsilon^2}\,h(\rho) \,+\, \frac{1}{2}\,\rho\,|u|^2 \,+\, \frac{1}{2\,\varepsilon^2}\,|\nabla\rho|^2\right) dx$$

$$(6)\qquad F_\varepsilon[\rho](t) \;:=\; \frac{\nu^2}{2}\int_\Omega \rho\,|\nabla\log\rho|^2\,dx \,=\, 2\,\nu^2\int_\Omega \left|\nabla\sqrt{\rho}\right|^2\,dx\,.$$

We will denote by $E_\varepsilon[\rho_0,u_0] \,\equiv\, E_\varepsilon[\rho,u](0)$ and by $F_\varepsilon[\rho_0] \,\equiv\, F_\varepsilon[\rho](0)$ the same quantities, when computed on the initial data $\big(\rho_0,u_0\big)$.

Definition 2.1. *Fix* (ρ_0,u_0) *such that* $\rho_0 - 1 \,\in\, H^1(\Omega)$*,* $\nabla\sqrt{\rho_0} \,\in\, L^2(\Omega)$ *and* $\sqrt{\rho_0}\,u_0 \,\in\, L^2(\Omega)$*, with* $\rho_0 \geq 0$ *almost everywhere.*

The couple $\big(\rho,u\big)$ *is a* weak solution *to system* (2)-(3) *in* $[0,T[\,\times\Omega$ *(for some* $T>0$*) with initial data* (ρ_0,u_0) *if the following conditions are fulfilled:*

(i) $\rho\geq 0$ *almost everywhere, and we have* $\rho-1\in L^\infty\big([0,T[\,;L^\gamma(\Omega)\big)$, $\nabla\rho$ *and* $\nabla\sqrt{\rho}\ \in L^\infty\big([0,T[\,;L^2(\Omega)\big)$ *and* $\nabla^2\rho\in L^2\big([0,T[\,;L^2(\Omega)\big)$*;*

(ii) $\sqrt{\rho}\,u\ \in L^\infty\big([0,T[\,;L^2(\Omega)\big)$ *and* $\sqrt{\rho}\,Du\ \in L^2\big([0,T[\,;L^2(\Omega)\big)$*;*

(iii) *the mass equation is satisfied in the weak sense: for any* $\phi\in\mathcal{D}\big([0,T[\,\times\Omega\big)$ *one has*

$$-\int_0^T\int_\Omega\Big(\rho\,\partial_t\phi\,+\,\rho\,u\,\cdot\,\nabla\phi\Big)\,dx\,dt\;=\;\int_\Omega\rho_0\,\phi(0)\,dx\,;$$

(iv) *the momentum equation is verified in the following sense: for* $\psi\in\mathcal{D}\big([0,T[\times\Omega\big)$,

$$\begin{aligned}(7)\quad \int_\Omega\rho_0^2u_0\cdot\psi(0)\,dx\;=\;&\int_0^T\!\!\int_\Omega\Big(-\rho^2u\cdot\partial_t\psi\,-\,\rho u\otimes\rho u:\nabla\psi\,+\,\rho^2\,(u\cdot\psi)\,\mathrm{div}\,u\,-\\ &-\frac{\gamma}{\varepsilon^2(\gamma+1)}\,P(\rho)\rho\,\mathrm{div}\,\psi\,+\,\frac{1}{\varepsilon}\,e^3\times\rho^2u\cdot\psi\,+\,\nu\rho Du:\rho\nabla\psi\,+\\ &+\nu\rho Du:(\psi\otimes\nabla\rho)\,+\,\frac{1}{\varepsilon^{2(1-\alpha)}}\,\rho^2\Delta\rho\,\mathrm{div}\,\psi\,+\,\frac{2}{\varepsilon^{2(1-\alpha)}}\,\rho\Delta\rho\nabla\rho\cdot\psi\Big)\,dx\,dt\,;\end{aligned}$$

(v) *for almost every* $t\in\,]0,T[\,$, *the following energy inequalities hold true:*

$$E_\varepsilon[\rho,u](t)\,+\,\nu\int_0^t\int_\Omega\rho\,|Du|^2\,dx\,d\tau\;\leq\;E_\varepsilon[\rho_0,u_0]$$

$$F_\varepsilon[\rho](t)\,+\,\frac{\nu}{\varepsilon^2}\int_0^t\int_\Omega P'(\rho)\,|\nabla\sqrt{\rho}|^2\,dx\,d\tau\,+\,\frac{\nu}{\varepsilon^{2(1-\alpha)}}\int_0^t\int_\Omega\left|\nabla^2\rho\right|^2\,dx\,d\tau\;\leq\;C\,(1+T)\,,$$

where the constant C *depends just on* $\big(E_\varepsilon[\rho_0,u_0],F_\varepsilon[\rho_0],\nu\big)$.

Here we consider the general case of *ill-prepared* initial data $\big(\rho,u\big)_{|t=0}\,=\,\big(\rho_{0,\varepsilon},u_{0,\varepsilon}\big)$. Namely, we suppose the following on the family $\big(\rho_{0,\varepsilon}\,,\,u_{0,\varepsilon}\big)_{\varepsilon>0}$:

(i) $\rho_{0,\varepsilon}\,=\,1\,+\,\varepsilon\,r_{0,\varepsilon}$, with $\big(r_{0,\varepsilon}\big)_\varepsilon\,\subset\,H^1(\Omega)\cap L^\infty(\Omega)$ bounded;

(ii) $\big(u_{0,\varepsilon}\big)_\varepsilon\,\subset\,L^2(\Omega)$ bounded.

Remark 2.2. *Notice that, under our hypotheses, the energies of the initial data are uniformly bounded with respect to* ε*:*

$$E_\varepsilon[\rho_{0,\varepsilon},u_{0,\varepsilon}]\,+\,F_\varepsilon[\rho_{0,\varepsilon}]\;\leq\;K_0\,,$$

for some constant $K_0>0$ *independent of* ε*.*

Up to extraction of a subsequence, we can assume that

$$r_{0,\varepsilon} \rightharpoonup r_0 \quad \text{in } H^1(\Omega) \qquad \text{and} \qquad u_{0,\varepsilon} \rightharpoonup u_0 \quad \text{in } L^2(\Omega)\,, \tag{8}$$

where we denoted by $\rightharpoonup$ the weak convergence in the respective spaces.

For these data, we are interested in studying the asymptotic behaviour of the corresponding solutions $\big(\rho_\varepsilon\,,\,u_\varepsilon\big)_\varepsilon$ to system (2) for the parameter $\varepsilon \to 0$. As we will see, one of the main features is that the limit-flow will be *two-dimensional* and *horizontal* along the plane orthogonal to the rotation axis. Then, let us introduce some notations to describe better this phenomenon. We will always decompose $x \in \Omega$ into $x = (x^h, x^3)$, with $x^h \in \mathbb{R}^2$ denoting its horizontal component. Analogously, for a vector-field $v = (v^1, v^2, v^3) \in \mathbb{R}^3$ we set $v^h = (v^1, v^2)$, and we define the differential operators ∇_h and div_h as the usual operators, but acting just with respect to x^h. Finally, we define the operator $\nabla_h^\perp := \big(-\partial_2\,,\,\partial_1\big)$.

We restrict our attention to the case $\alpha = 0$, i.e. when the capillarity coefficient is taken to be constant. As a matter of fact, we want to put in evidence here the effects of surface tension in the limit.

Theorem 2.1. *Let $\alpha = 0$ in (2) and $1 < \gamma \leq 2$ in (4). Let $\big(\rho_\varepsilon\,,\,u_\varepsilon\big)_\varepsilon$ be a family of weak solutions (in the sense of Definition 2.1) to system (2)-(3) in $[0,T]\times\Omega$, related to initial data $\big(\rho_{0,\varepsilon}, u_{0,\varepsilon}\big)_\varepsilon$ satisfying the hypotheses (i) – (ii) and (8). Define $r_\varepsilon := \varepsilon^{-1}\,(\rho_\varepsilon - 1)$.*

Then, up to the extraction of a subsequence, one has the convergence properties

(a) $r_\varepsilon \rightharpoonup r$ *in* $L^\infty\big([0,T];H^1(\Omega)\big) \,\cap\, L^2\big([0,T];H^2(\Omega)\big)$;

(b) $\sqrt{\rho_\varepsilon}\,u_\varepsilon \rightharpoonup u$ *in* $L^\infty\big([0,T];L^2(\Omega)\big)$ *and* $\sqrt{\rho_\varepsilon}\,Du_\varepsilon \rightharpoonup Du$ *in* $L^2\big([0,T];L^2(\Omega)\big)$;

(c) $r_\varepsilon \to r$ *and* $\rho_\varepsilon^{3/2}\,u_\varepsilon \to u$ *(strong convergence) in* $L^2\big([0,T];L^2_{loc}(\Omega)\big)$,

where $r = r(x^h)$ and $u = \big(u^h(x^h), 0\big)$ are linked by the relation $u^h = \nabla_h^\perp\,(\mathrm{Id} - \Delta_h)\,r$. Moreover, r solves (in the weak sense) the modified Quasi-Geostrophic equation

$$\partial_t\Big(\big(\mathrm{Id} - \Delta_h + \Delta_h^2\big)r\Big) + \nabla_h^\perp\big(\mathrm{Id} - \Delta_h\big)r \cdot \nabla_h\Delta_h^2 r + \frac{\nu}{2}\,\Delta_h^2\big(\mathrm{Id} - \Delta_h\big)r = 0 \tag{9}$$

supplemented with the initial condition $r_{|t=0} = \widetilde{r}_0$, where $\widetilde{r}_0 \in H^3(\mathbb{R}^2)$ is the unique solution of

$$\big(\mathrm{Id} - \Delta_h + \Delta_h^2\big)\,\widetilde{r}_0 = \int_0^1 \big(\omega_0^3 + r_0\big)\,dx^3\,,$$

with r_0 and u_0 defined in (8) and $\omega_0 = \nabla\times u_0$ the vorticity of u_0.

3. Uniform bounds

The present section is devoted to show uniform bounds for the family $\big(\rho_\varepsilon, u_\varepsilon\big)_\varepsilon$.

3.1. **Energy and BD entropy estimates.** First of all, we establish energy and BD entropy estimates. The first inequality, concerning the classical energy E_ε, is obtained in a standard way, testing the momentum equation on u and integrating by parts.

Proposition 3.1. *Let (ρ, u) be a smooth solution to system* (2) *in $[0,T[\,\times\Omega$, with initial datum $\big(\rho_0, u_0\big)$, for some positive time $T>0$. Then, for all $\varepsilon>0$ and all $t\in[0,T[\,$,*

$$\frac{d}{dt}E_\varepsilon[\rho,u]\,+\,\nu\int_\Omega \rho\,|Du|^2\,dx\;=\;0\,.$$

Let us now consider the function F_ε: we have the following estimate.

Lemma 3.1. *Let (ρ, u) be a smooth solution to system* (2) *in $[0,T[\,\times\Omega$, with initial datum $\big(\rho_0, u_0\big)$, for some positive time $T>0$.*

Then there exists a "universal" constant $C>0$ such that, for all $t\in[0,T[\,$, one has

$$\begin{aligned}(10)\quad &\frac{1}{2}\int_\Omega \rho(t)\;|u(t)\,+\,\nu\,\nabla\log\rho(t)|^2\;dx\;+\\ &\qquad+\frac{\nu}{\varepsilon^2}\int_0^t\!\!\int_\Omega \left|\nabla^2\rho\right|^2\;dx\,d\tau\;+\frac{4\nu}{\varepsilon^2}\int_0^t\!\!\int_\Omega P'(\rho)\;|\nabla\sqrt{\rho}|^2\;dx\,d\tau\;\leq\\ &\qquad\qquad\leq\;C\big(F_\varepsilon[\rho_0]+E_\varepsilon[\rho_0,u_0]\big)\;+\;\frac{\nu}{\varepsilon}\left|\int_0^t\!\!\int_\Omega e^3\times u\cdot\nabla\rho\,dx\,d\tau\right|.\end{aligned}$$

The previous result is the first step in order to get BD entropy estimates. The problem is to control the Coriolis term uniformly in ε, the difficulty relying on the lack of informations on the velocity fields and their gradients. The next lemma gives us the suitable bounds.

Lemma 3.2. *There exists a positive constant C, just depending on K_0 (defined in Remark 2.2), such that, for any $1<\gamma\leq 2$,*

$$\frac{\nu}{\varepsilon}\left|\int_0^t\!\!\int_\Omega e^3\times u\cdot\nabla\rho\,dx\,d\tau\right|\;\leq\;C\nu(1+t)+\frac{\nu}{4\varepsilon^2}\left\|\nabla^2\rho\right\|^2_{L^2_t(L^2)}+\frac{\nu}{2\varepsilon^2}\left\|\rho^{(\gamma-1)/2}\nabla\sqrt{\rho}\right\|^2_{L^2_t(L^2)}\,.$$

From the previous inequality we deduce the BD entropy estimates for our system.

Proposition 3.2. *Let $\big(\rho_{0,\varepsilon},u_{0,\varepsilon}\big)_\varepsilon$ be a family of initial data satisfying the assumptions (i)-(ii) of Section 2, and let $\big(\rho_\varepsilon,u_\varepsilon\big)_\varepsilon$ be a family of corresponding smooth solutions.*

Then there exists a constant $C>0$ (depending just on the constant K_0 of Remark 2.2 and on ν) such that the following inequality holds true for any $\varepsilon\in\,]0,1]$:

$$F_\varepsilon[\rho_\varepsilon](t)\,+\,\frac{\nu}{\varepsilon^2}\int_0^t\int_\Omega P'(\rho_\varepsilon)\,|\nabla\sqrt{\rho_\varepsilon}|^2\,dx\,d\tau\,+\,\frac{\nu}{\varepsilon^2}\int_0^t\int_\Omega\left|\nabla^2\rho_\varepsilon\right|^2\,dx\,d\tau\,\leq\,C\,(1\,+\,t)\,.$$

3.2. **Bounds for the family of weak solutions.** From the previous energy estimates, we easily deduce the following bounds for the family $\big(\rho_\varepsilon,u_\varepsilon\big)_\varepsilon$ of weak solutions.

Proposition 3.3. *Let $\big(\rho_\varepsilon,u_\varepsilon\big)_\varepsilon$ be the family of weak solutions to system* (2) *considered in Theorem 2.1. Then it satisfies the following bounds, uniformly in ε:*

$$\sqrt{\rho_\varepsilon}\,u_\varepsilon\,\in\,L^\infty\big(\mathbb{R}_+;L^2(\Omega)\big)\qquad\text{and}\qquad\sqrt{\rho_\varepsilon}\,Du_\varepsilon\,\in\,L^2\big(\mathbb{R}_+;L^2(\Omega)\big)$$

for the velocity fields, and for the densities

$$\frac{1}{\varepsilon}\,(\rho_\varepsilon\,-\,1)\,\in\,L^\infty\big(\mathbb{R}_+;L^\gamma(\Omega)\big)\qquad\text{and}\qquad\frac{1}{\varepsilon}\,\nabla\rho_\varepsilon\,\in\,L^\infty\big(\mathbb{R}_+;L^2(\Omega)\big)\,.$$

Remark 3.1. *In particular, by interpolation we infer $\|\rho_\varepsilon-1\|_{L^\infty(\mathbb{R}_+;L^2(\Omega))}\,\leq\,C\varepsilon$.*

BD entropy estimates of Proposition 3.2 also implies the following uniform controls.

Proposition 3.4. *Let $\big(\rho_\varepsilon,u_\varepsilon\big)_\varepsilon$ be the family of weak solutions to system* (2) *considered in Theorem 2.1. Then one has the following bounds, uniformly for $\varepsilon>0$:*

$$\begin{cases}\nabla\sqrt{\rho_\varepsilon}\,\in\,L^\infty_{loc}\big(\mathbb{R}_+;L^2(\Omega)\big)\\[1ex] \dfrac{1}{\varepsilon}\,\nabla^2\rho_\varepsilon\;,\quad\dfrac{1}{\varepsilon}\,\nabla\Big(\rho_\varepsilon^{\gamma/2}\Big)\qquad\in\,L^2_{loc}\big(\mathbb{R}_+;L^2(\Omega)\big)\,.\end{cases}$$

In particular, the family $\big(\varepsilon^{-1}\,(\rho_\varepsilon-1)\big)_\varepsilon$ is bounded in $L^p_{loc}\big(\mathbb{R}_+;L^\infty(\Omega)\big)$ for any $2\leq p<4$.

Finally, let us state an important property on the quantity $D\big(\rho_\varepsilon^{3/2}\,u_\varepsilon\big)$: by writing

$$\begin{aligned}D\big(\rho_\varepsilon^{3/2}\,u_\varepsilon\big)\;&=\;\rho_\varepsilon\,\sqrt{\rho_\varepsilon}\,Du_\varepsilon\,+\,\frac{3}{2}\,\sqrt{\rho_\varepsilon}\,u_\varepsilon\,D\rho_\varepsilon\\ &=\;\sqrt{\rho_\varepsilon}\,Du_\varepsilon\,+\,(\rho_\varepsilon\,-\,1)\,\sqrt{\rho_\varepsilon}\,Du_\varepsilon\,+\,\frac{3}{2}\,\sqrt{\rho_\varepsilon}\,u_\varepsilon\,D\rho_\varepsilon\,.\end{aligned}$$

and the uniform bounds, we infer that $\Big(D\big(\rho_\varepsilon^{3/2}\,u_\varepsilon\big)\Big)_\varepsilon$ is a bounded family in $L^2_T(L^2+L^{3/2})\,\hookrightarrow\,L^2_T(L^{3/2}_{loc})$.

4. Strategy of the proof

We outline here the proof of Theorem 2.1. First of all, we study the singular perturbation operator. Then, we focus on the propagation of acoustic-Rossby waves: a direct application of RAGE Theorem to the wave system will enable us to prove suitable strong convergence properties, and then to pass to the limit in the non-linear terms. Finally, we study the limit equation.

4.1. The singular perturbation operator.

4.1. The singular perturbation operator. By uniform bounds, seeing L^∞ as the dual of L^1 and denoting by $\stackrel{*}{\rightharpoonup}$ the weak-$*$ convergence in $L^\infty\big(\mathbb{R}_+;L^2(\Omega)\big)$, we infer, up to extraction of subsequences, the following properties:

$$\sqrt{\rho_\varepsilon}\,u_\varepsilon \stackrel{*}{\rightharpoonup} u \qquad \text{in} \quad L^\infty\big(\mathbb{R}_+;L^2(\Omega)\big)\,, \qquad \sqrt{\rho_\varepsilon}\,Du_\varepsilon \rightharpoonup U \qquad \text{in} \quad L^2\big(\mathbb{R}_+;L^2(\Omega)\big)\,.$$

Working on the quantity $D\big(\rho_\varepsilon^{3/2}\,u_\varepsilon\big)$, it is possible to see that $U = Du$, as expected, and then $u\in L^2\big(\mathbb{R}_+;H^1(\Omega)\big)$.

On the other hand, thanks to the estimates for the density, we deduce that $\rho_\varepsilon \to 1$ (strong convergence) in $L^\infty\big(\mathbb{R}_+;H^1(\Omega)\big)\,\cap\,L^2_{loc}\big(\mathbb{R}_+;H^2(\Omega)\big)$, with convergence rate of order ε. So, we can write $\rho_\varepsilon = 1+\varepsilon\,r_\varepsilon$, with $\big(r_\varepsilon\big)_\varepsilon$ bounded in the previous space, and then (up to an extraction)

$$r_\varepsilon \rightharpoonup r \qquad \text{in} \quad L^\infty\big(\mathbb{R}_+;H^1(\Omega)\big)\,\cap\,L^2_{loc}\big(\mathbb{R}_+;H^2(\Omega)\big)\,. \tag{11}$$

It is also easy to get the convergences $\rho_\varepsilon u_\varepsilon \rightharpoonup u$ in $L^2\big([0,T];L^2(\Omega)\big)$ and $\rho_\varepsilon Du_\varepsilon \rightharpoonup Du$ in $L^1\big([0,T];L^2(\Omega)\big)\cap L^2\big([0,T];L^1(\Omega)\cap L^{3/2}(\Omega)\big)$, for any fixed $T>0$.

The next statement is usually referred to as the Taylor-Proudman theorem.

Proposition 4.1. *Let $\big(\rho_\varepsilon,u_\varepsilon\big)_\varepsilon$ be a family of weak solutions (in the sense of Definition 2.1 above) to system* (2)-(3)*, with data $\big(\rho_{0,\varepsilon},u_{0,\varepsilon}\big)$ satisfying the hypotheses of Section 2. Let us define $r_\varepsilon := \varepsilon^{-1}\,(\rho_\varepsilon-1)$, and let (r,u) be a limit point of the sequence $\big(r_\varepsilon,u_\varepsilon\big)_\varepsilon$.*

Then $r = r(x^h)$ and $u = \big(u^h(x^h),0\big)$, with $\mathrm{div}_h u^h = 0$; moreover, they satisfy the relation $u^h = \nabla_h^\perp\big(\mathrm{Id}-\Delta_h\big)r$.

Thanks to the previous proposition, we can define the singular perturbation operator

$$\begin{array}{rcccc} \mathcal{A}_0\,: & L^2(\Omega)\,\times\,L^2(\Omega) & \longrightarrow & H^{-1}(\Omega)\,\times\,H^{-3}(\Omega) \\ & \big(r\,,\,V\big) & \mapsto & \Big(\mathrm{div}\,V\,,\,e^3\times V\,+\nabla\big(\mathrm{Id}\,-\Delta\big)r\Big)\,. \end{array} \tag{12}$$

Direct computations immediately yield the following property on the spectrum of $\mathcal{A}_0$.

Proposition 4.2. *Let us denote by $\sigma_p(\mathcal{A}_0)$ the point spectrum of $\mathcal{A}_0$. Then $\sigma_p(\mathcal{A}_0)\,=\,\{0\}$.*

In particular, if we define by $\mathrm{Eigen}\,\mathcal{A}_0$ *the space spanned by the eigenvectors of $\mathcal{A}_0$, we have* $\mathrm{Eigen}\,\mathcal{A}_0\,\equiv\,\mathrm{Ker}\,\mathcal{A}_0$.

4.2. **Propagation of acoustic-Rossby waves.** The present paragraph is devoted to the analysis of the acoustic waves. We start by rewriting system (2) in the form

$$\begin{cases} \varepsilon\,\partial_t r_\varepsilon\,+\,\mathrm{div}\,V_\varepsilon\,=\,0 \\ \varepsilon\,\partial_t V_\varepsilon\,+\,\Big(e^3\times V_\varepsilon\,+\,\nabla\big(\mathrm{Id}\,-\Delta\big)r_\varepsilon\Big)\,=\,\varepsilon\,f_\varepsilon\,, \end{cases} \tag{13}$$

where we have set $V_\varepsilon\,:=\,\rho_\varepsilon\,u_\varepsilon$ and

$$\begin{aligned} f_\varepsilon\,\,:=\,\,&-\mathrm{div}\,\left(\rho_\varepsilon u_\varepsilon\otimes u_\varepsilon\right)\,+\,\nu\,\mathrm{div}\,\left(\rho_\varepsilon D u_\varepsilon\right)\,- \\ &-\frac{1}{\varepsilon^2}\nabla\Big(P(\rho_\varepsilon)-P(1)-P'(1)\,(\rho_\varepsilon-1)\Big)\,+\,\frac{1}{\varepsilon^2}\,\big(\rho_\varepsilon-1\big)\,\nabla\Delta\rho_\varepsilon\,. \end{aligned} \tag{14}$$

System (13) has to be read in the weak sense specified by Definition 2.1: in particular, for any $\psi\in\mathcal{D}\big([0,T[\,\times\Omega;\mathbb{R}^3\big)$, we have to test the momentum equation on $\rho_\varepsilon\,\psi$. Keeping in mind the formula

$$\begin{aligned} \langle f_\varepsilon,\phi\rangle\,\,:=\,\,&\int_\Omega\bigg(\rho_\varepsilon u_\varepsilon\otimes u_\varepsilon:\nabla\phi\,-\,\nu\,\rho_\varepsilon D u_\varepsilon:\nabla\phi\,-\,\frac{1}{\varepsilon^2}\,\Delta\rho_\varepsilon\,\nabla\rho_\varepsilon\cdot\phi\,- \\ &-\frac{1}{\varepsilon^2}\,\big(\rho_\varepsilon-1\big)\,\Delta\rho_\varepsilon\,\mathrm{div}\,\phi\,+\,\frac{1}{\varepsilon^2}\Big(P(\rho_\varepsilon)-P(1)-P'(1)\,(\rho_\varepsilon-1)\Big)\mathrm{div}\,\phi\bigg)dx\,, \end{aligned}$$

a systematic use of uniform bounds gives $\big(f_\varepsilon\big)_\varepsilon\,\subset\,L^2_T\big(W^{-1,2}(\Omega)+W^{-1,1}(\Omega)\big)$.

The main goal, now, is to apply the RAGE Theorem (see e.g. [5]) to prove dispersion of the components of the solutions which are orthogonal to $\mathrm{Ker}\,\mathcal{A}_0$. Such a kind of arguments were used in [9], in dealing with the compressible barotropic Navier-Stokes equations with Coriolis force.

Theorem 4.1 (RAGE)**.** *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B} : D(\mathcal{B}) \subset \mathcal{H} \longrightarrow \mathcal{H}$ a self-adjoint operator. Denote by $\Pi_{\rm cont}$ the orthogonal projection onto the subspace $\mathcal{H}_{\rm cont}$, where we set $\mathcal{H} = \mathcal{H}_{\rm cont} \oplus \overline{{\rm Eigen}(\mathcal{B})}$ and $\overline{\Theta}$ is the closure of a subset Θ in $\mathcal{H}$. Finally, let $\mathcal{K} : \mathcal{H} \longrightarrow \mathcal{H}$ be a compact operator. Then, in the limit for $T \to +\infty$ one has*

$$\left\| \frac{1}{T} \int_0^T e^{-it\mathcal{B}}\, \mathcal{K}\, \Pi_{\rm cont}\, e^{it\mathcal{B}}\, dt \right\|_{\mathcal{L}(\mathcal{H})} \longrightarrow 0\,.$$

The previous theorem implies the following consequences.

Corollary 4.1. *Under the hypotheses of Theorem 4.1, suppose also that $\mathcal{K}$ is self-adjoint, with $\mathcal{K} \geq 0$. Then there exists a function μ, with $\mu(\varepsilon) \to 0$ for $\varepsilon \to 0$, such that:*

1) *for any $Y \in \mathcal{H}$ and any $T > 0$, one has*

$$\frac{1}{T} \int_0^T \left\| \mathcal{K}^{1/2}\, e^{it\mathcal{B}/\varepsilon}\, \Pi_{\rm cont} Y \right\|_{\mathcal{H}}^2 dt \,\leq\, \mu(\varepsilon)\, \|Y\|_{\mathcal{H}}^2\,;$$

2) *for any $T > 0$ and any $X \in L^2\big([0,T];\mathcal{H}\big)$, one has*

$$\frac{1}{T^2} \left\| \mathcal{K}^{1/2}\, \Pi_{\rm cont} \int_0^t e^{i(t-\tau)\mathcal{B}/\varepsilon}\, X(\tau)\, d\tau \right\|_{L^2([0,T];\mathcal{H})}^2 \leq\, \mu(\varepsilon)\, \|X\|_{L^2([0,T];\mathcal{H})}^2\,.$$

We now come back to our problem. For any fixed $M > 0$, define the space H_M by

$$H_M \,:=\, \Big\{ (r,V) \in L^2(\Omega) \times L^2(\Omega) \;\big|\; \widehat{r}(\xi^h,k) \equiv 0\,,\; \widehat{V}(\xi^h,k) \equiv 0 \quad \text{for } \left|\xi^h\right| + |k| > M \Big\}\,:$$

it is a Hilbert space, endowed with the scalar product

$$\langle (r_1,V_1)\,,\,(r_2,V_2)\rangle_{H_M} \,:=\, \langle r_1\,,\,({\rm Id}-\Delta) r_2\rangle_{L^2} \,+\, \langle V_1\,,\,V_2\rangle_{L^2}\,. \tag{15}$$

In fact, it is easy to verify that the previous bilinear form is symmetric and positive definite. Moreover, we have $\|(r,V)\|_{H_M}^2 \,=\, \left\|({\rm Id}-\Delta)^{1/2} r\right\|_{L^2}^2 + \|V\|_{L^2}^2$. Straightforward computations also show that $\mathcal{A}_0$ is skew-adjoint with respect to $\langle\,\cdot\,,\,\cdot\,\rangle_{H_M}$:

$$\langle \mathcal{A}_0(r_1,V_1)\,,\,(r_2,V_2)\rangle_{H_M} \,=\, -\,\langle (r_1,V_1)\,,\,\mathcal{A}_0(r_2,V_2)\rangle_{H_M}\,.$$

Let $P_M : L^2(\Omega) \times L^2(\Omega) \longrightarrow H_M$ be the orthogonal projection onto H_M. For a fixed $\theta \in \mathcal{D}(\Omega)$ such that $0 \leq \theta \leq 1$, we define the operator

$$\mathcal{K}_{M,\theta}(r,V) \,:=\, \Big(\big({\rm Id}-\Delta\big)^{-1} P_M\big(\theta\, P_M r\big)\,,\, P_M\big(\theta\, P_M V\big) \Big)\,.$$

Note that $\mathcal{K}_{M,\theta}$ is self-adjoint and positive with respect to the scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_{H_M}$; moreover it is compact by Rellich-Kondrachov theorem.

We want to apply the Theorem 4.1 to

$$\mathcal{H} \,=\, H_M\ , \quad \mathcal{B} \,=\, i\,\mathcal{A}_0\ , \quad \mathcal{K} \,=\, \mathcal{K}_{M,\theta} \quad \text{and} \quad \Pi_{\mathrm{cont}} \,=\, Q^{\perp}\,,$$

where Q and $Q^{\perp}$ are the orthogonal projections onto respectively $\mathrm{Ker}\,\mathcal{A}_0$ and $\big(\mathrm{Ker}\,\mathcal{A}_0\big)^{\perp}$.

We set $(r_{\varepsilon,M}, V_{\varepsilon,M}) \,:=\, P_M(r_\varepsilon, V_\varepsilon)$: from system (13) we get

$$\varepsilon\,\frac{d}{dt}\big(r_{\varepsilon,M}\,,\,V_{\varepsilon,M}\big) \,+\, \mathcal{A}\big(r_{\varepsilon,M}\,,\,V_{\varepsilon,M}\big) \,=\, \varepsilon\,\big(0\,,\,f_{\varepsilon,M}\big)\,, \tag{16}$$

where $\big(0\,,\,f_{\varepsilon,M}\big) \,\in\, H_M^* \cong H_M$ acts on any $(s,W)\in H_M$ like $\langle\big(0,f_\varepsilon\big),\big(s,P_M\big(\rho_\varepsilon\,W\big)\big)\rangle_{H_M}$. By Bernstein inequalities, for any $T>0$ fixed and any $W\in H_M$ one has

$$\begin{aligned}\big\|P_M\big(\rho_\varepsilon W\big)\big\|_{L^2_T(W^{1,\infty}\cap H^1)} \;&\leq\; C(M)\;\|\rho_\varepsilon W\|_{L^2_T(L^2)}\\ &\leq\; C(M)\,\Big(\|W\|_{L^2_T(L^2)} \,+\, \|\rho_\varepsilon - 1\|_{L^\infty_T(L^2)}\,\|W\|_{L^2_T(L^\infty)}\Big)\,,\end{aligned}$$

for some constant $C(M)$ depending only on M. This fact, combined with the uniform bounds we established on f_ε, entails $\big\|\big(0,f_{\varepsilon,M}\big)\big\|_{L^2_T(H_M)} \,\leq\, C(M)$. Therefore, applying Q to (16) and using uniform bounds for $\big(\partial_t Q\big(r_{\varepsilon,M}\,,\,V_{\varepsilon,M}\big)\big)_\varepsilon$ (with respect to ε, for any $M>0$ fixed), Ascoli-Arzelà Theorem implies, for $\varepsilon\to 0$, the strong convergence

$$Q\big(r_{\varepsilon,M}\,,\,V_{\varepsilon,M}\big) \,\longrightarrow\, \big(r_M\,,\,u_M\big) \qquad \text{in} \quad L^2\big([0,T]\times K\big)\,. \tag{17}$$

On the other hand, by Duhamel's formula, solutions to equation (16) can be written as

$$\big(r_{\varepsilon,M}\,,\,V_{\varepsilon,M}\big)(t) \,=\, e^{i\,t\,\mathcal{B}/\varepsilon}\big(r_{\varepsilon,M}\,,\,V_{\varepsilon,M}\big)(0) \,+\, \int_0^t e^{i\,(t-\tau)\,\mathcal{B}/\varepsilon}\,\big(0\,,\,f_{\varepsilon,M}\big)\,d\tau\,. \tag{18}$$

Note that, by definition (and since $[P_M,Q]=0$),

$$\Big\|(\mathcal{K}_{M,\theta})^{1/2}\;Q^{\perp}\big(r_{\varepsilon,M}\,,\,V_{\varepsilon,M}\big)\Big\|^2_{H_M} \;=\; \int_\Omega \theta\;\big|Q^{\perp}\big(r_{\varepsilon,M}\,,\,V_{\varepsilon,M}\big)\big|^2\;dx\,.$$

Therefore, a straightforward application of Corollary 4.1 (recalling also Proposition 4.2) gives that, for $T>0$ fixed and for ε going to 0,

$$Q^{\perp}\big(r_{\varepsilon,M}\,,\,V_{\varepsilon,M}\big) \,\longrightarrow\, 0 \qquad \text{in} \quad L^2\big([0,T]\times K\big) \tag{19}$$

for any fixed $M>0$ and any compact set $K\subset\Omega$.

4.3. **Passing to the limit.** Thanks to relations (19) and (17), and to a careful analysis of the high frequencies remainders, we deduce the following proposition.

Proposition 4.3. *For any $T > 0$, for $\varepsilon \to 0$ one has, up to extraction of a subsequence, the strong convergences*

$$r_\varepsilon \longrightarrow r \qquad \text{and} \qquad \rho_\varepsilon^{3/2}\, u_\varepsilon \longrightarrow u \qquad \text{in} \quad L^2\big([0,T]; L^2_{loc}(\Omega)\big)\,.$$

As a consequence of Proposition 4.3 and uniform bounds, by interpolation we get also the strong convergence

$$\nabla r_\varepsilon \longrightarrow \nabla r \qquad \text{in} \qquad L^2\big([0,T]; L^2_{loc}(\Omega)\big)\,. \tag{20}$$

In order to compute the limit system, let us take $\phi \in \mathcal{D}\big([0,T[\,\times\Omega\big)$, with $\phi = \phi(x^h)$, and use $\psi = \big(\nabla_h^\perp\phi, 0\big)$ as a test function in equation (7). Since $\operatorname{div}\psi = 0$, we get

$$\begin{aligned}\int_0^T\!\!\int_\Omega\bigg(&-\rho_\varepsilon^2\, u_\varepsilon\cdot\partial_t\psi \,-\, \rho_\varepsilon u_\varepsilon\otimes\rho_\varepsilon u_\varepsilon : \nabla\psi \,+\\ &+\rho_\varepsilon^2\,(u_\varepsilon\cdot\psi)\operatorname{div}u_\varepsilon \,+\, \frac{1}{\varepsilon}\,e^3\times\rho_\varepsilon^2 u_\varepsilon\cdot\psi \,+\, \nu\rho_\varepsilon Du_\varepsilon : \rho_\varepsilon\nabla\psi \,+\\ &+\nu\rho_\varepsilon Du_\varepsilon : (\psi\otimes\nabla\rho_\varepsilon) \,+\, \frac{2}{\varepsilon^2}\,\rho_\varepsilon\Delta\rho_\varepsilon\nabla\rho_\varepsilon\cdot\psi\bigg)\,dx\,dt \,=\, \int_\Omega \rho_{0,\varepsilon}^2\, u_{0,\varepsilon}\cdot\psi(0)\,dx\,.\end{aligned} \tag{21}$$

Now we rewrite the rotation term by using the weak formulation of the mass equation:

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^T\!\!\int_\Omega e^3\times\rho_\varepsilon^2 u_\varepsilon\cdot\psi \;&=\; \frac{1}{\varepsilon}\int_0^T\!\!\int_\Omega \rho_\varepsilon u_\varepsilon^h\cdot\nabla_h\phi \,+\, \frac{1}{\varepsilon}\int_0^T\!\!\int_\Omega(\rho_\varepsilon-1)\,\rho_\varepsilon u_\varepsilon^h\cdot\nabla_h\phi\\ &=\; -\int_\Omega r_{0,\varepsilon}\,\phi(0) \,-\, \int_0^T\!\!\int_\Omega r_\varepsilon\,\partial_t\phi \,+\, \int_0^T\!\!\int_\Omega r_\varepsilon\,\rho_\varepsilon\,u_\varepsilon^h\cdot\nabla_h\phi\,.\end{aligned}$$

Due to the strong convergence of r_ε in $L^2_T(L^2)$, it is easy to see that the expression on the right-hand side of the previous relation converges.

Concerning the capillarity term, we can write

$$\frac{2}{\varepsilon^2}\int_0^T\!\!\int_\Omega \rho_\varepsilon\Delta\rho_\varepsilon\nabla\rho_\varepsilon\cdot\psi \,=\, \frac{2}{\varepsilon^2}\int_0^T\!\!\int_\Omega \Delta\rho_\varepsilon\,\nabla\rho_\varepsilon\cdot\psi \,+\, \frac{2}{\varepsilon^2}\int_0^T\!\!\int_\Omega(\rho_\varepsilon-1)\;\Delta\rho_\varepsilon\,\nabla\rho_\varepsilon\cdot\psi\,.$$

By uniform bounds, we gather that the second term goes to 0; on the other hand, combining (20) with the weak convergence of Δr_ε in $L^2_T(L^2)$ implies that also the first term converges for $\varepsilon \to 0$.

Putting these last two relations into (21) and using convergence properties established above in order to pass to the limit, we arrive at the equation

$$\int_0^T \int_\Omega \Big(- u \cdot \partial_t \psi \, - \, u \otimes u : \nabla\psi \, - \, r\, \partial_t \phi \, + $$
$$ + r\, u^h \cdot \nabla_h \phi \, + \, \nu D u : \nabla \psi \, + \, 2\, \Delta r\, \nabla r \cdot \psi \Big)\, dx\, dt \; = \; \int_\Omega \big(u_0 \cdot \psi(0) \, + \, r_0\, \phi(0) \big)\, dx\,.$$

Now we use that $\psi = \big(\nabla_h^\perp \phi, 0\big)$ and that, by Proposition 4.1, $u = \big(\nabla_h^\perp \widetilde{r}, 0\big)$, where we have set $\widetilde{r} \,:=\, (\mathrm{Id}\, - \Delta)$; recall also that all these functions do not depend on x^3. Then, integrating by parts, it is easy to prove that the previous expression equals the weak formulation (in the classical sense) of the Quasi-Geostrophic type equation of Theorem 2.1, which is now completely proved.

5. Remarks for variable rotation axis

Let us spend here a few words on the case of variable rotation axis, namely when the Coriolis operator is given by

$$\mathfrak{C}(\rho, u) \; = \; \mathfrak{c}\, e^3 \, \times \, \rho\, u\,, \tag{22}$$

for a suitable non-constant function $\mathfrak{c}$. This is important, since considering a constant rotation axis is an approximation which is physically consistent in regions which are very far from the equatorial zone and from the poles, and which are not too extended: in general, the dependence of the Coriolis force on the latitude should be taken into account.

The case of variable axis (22) was considered first in [10] by Gallagher and Saint-Raymond for the classical incompressible Navier-Stokes equations. There, the authors assumed that $\mathfrak{c} = \mathfrak{c}(x^h)$ is a smooth function of the horizontal variables only, and that it satisfies the following non-degeneracy condition:

$$\lim_{\delta \to 0} \, \mathcal{L}\left(\left\{ x^h \, \in \, \mathbb{R}^2 \; \middle| \; \left| \nabla_h \mathfrak{c}(x^h) \right| \, \leq \, \delta \right\} \right) \; = \; 0\,, \tag{23}$$

where $\mathcal{L}(\mathcal{O})$ denotes the 2-dimensional Lebesgue measure of a set $\mathcal{O} \subset \mathbb{R}^2$. The previous techincal assumptions are motivated by the strategy of the proof.

As a matter of fact, one has to remark that, for variable rotation axis, the singular perturbation operator becomes variable coefficients, so that spectral analysis is out of use.

Then, the idea is to resort to compensated compactness arguments to prove the convergence in the non-linear terms: namely, after a regularization procedure and integration by parts, one takes advantage of the structure of the system to find special cancellations and properties which enable to pass to the limit.

Let us mention that the same technique was used also in [8] by Feireisl, Gallagher, Gérard-Varet and Novotný, in dealing with the compressible barotropic Navier-Stokes equations with Earth rotation, when centrifugal force is taken into account. Indeed, the presence of this last term allows to consider non-constant limit density profiles $\widetilde{\rho}$ in the regime of low Mach number, and therefore variable coefficients appear in the singular perturbation operator. We point out that the previous technical assumptions on $\mathfrak{c}$ are replaced in paper [8] by suitable properties for $\widetilde{\rho}$, which can be deduced by the analysis of its diagnostic equation.

Let us come back to the case of Navier-Stokes-Korteweg system (1), with $\mathfrak{C}$ given by (22) and still satisfying hypothesis (23). We focus again on the case $\alpha = 0$ (the other values of α can be treated in an analogous way), and we suppose the pressure term P to be now given by the sum of a standard barotropic law P_b and a singular law P_s, in order to recover stability of the system even on vacuum and to resort to the classical weak formulation (see Section 1 above). For simplicity of exposition, we omit here the precise assumptions on the singular pressure law and on the initial data: very few things change with respect to Section 2, and one has just to add a condition on $1/\rho_{0,\varepsilon}$ in order to exploit the presence of P_s in the energy estimates.

For notation convenience, let us also introduce the operator $\mathfrak{D}_{\mathfrak{c}}$: for any scalar function $f = f(x^h)$, we set $\mathfrak{D}_{\mathfrak{c}}(f) := D_h\big(\mathfrak{c}^{-1}\,\nabla_h^\perp f\big)$.

In [7] we proved the following convergence result, where we looked for minimal regularity assumptions for $\mathfrak{c}$.

Theorem 5.1. *Under the previous hypotheses, suppose that $\mathfrak{c} \in W^{1,\infty}(\mathbb{R}^2)$ is $\neq 0$ almost everywhere and it verifies condition* (23)*. Let us also assume that $\nabla_h \mathfrak{c} \in \mathcal{C}_\mu(\mathbb{R}^2)$, for some admissible modulus of continuity μ.*

Let $\big(\rho_\varepsilon\,,\,u_\varepsilon\big)_\varepsilon$ *be a family of weak solutions (in the classical sense) to system* (2)-(3) *in* $[0,T]\times\Omega$*, related to (suitable) initial data* $\big(\rho_{0,\varepsilon},u_{0,\varepsilon}\big)_\varepsilon$*. Define* $r_\varepsilon\,:=\,\varepsilon^{-1}\,(\rho_\varepsilon-1)$*.*

Then, up to the extraction of a subsequence, one has the same convergence properties (a)-(b) of Theorem 2.1, where, this time, $r=r(x^h)$ *and* $u=\big(u^h(x^h),0\big)$ *verify the relation* $\mathfrak{c}(x^h)\,u^h\;=\;\nabla_h^\perp\,(\mathrm{Id}\,-\,\Delta_h)\,r$*. Moreover,* r *solves (in the weak sense) the equation*

$$\partial_t\left(r\,-\,\mathrm{div}_h\!\left(\frac{1}{\mathfrak{c}^2}\,\nabla_h\big(\mathrm{Id}\,-\,\Delta_h\big)r\right)\right)\,+\,\nu\;{}^t\mathfrak{D}_{\mathfrak{c}}\,\circ\,\mathfrak{D}_{\mathfrak{c}}\big((\mathrm{Id}\,-\,\Delta_h)r\big)\;=\;0$$

supplemented with the initial condition $r_{|t=0}\;=\;\widetilde{r}_0$*, where* $\widetilde{r}_0$ *is defined by*

$$\widetilde{r}_0\,-\,\mathrm{div}_h\!\left(\frac{1}{\mathfrak{c}^2}\,\nabla_h\big(\mathrm{Id}\,-\,\Delta_h\big)\widetilde{r}_0\right)\;=\;\int_0^1\Big(\mathrm{curl}_h\big(\mathfrak{c}^{-1}\,u_0^h\big)\,+\,r_0\Big)\,dx^3\,.$$

Remark 5.1. *Notice that the limit equation is linear for variable rotation axis: indeed, the dynamics is much more constrained in this case. Let us also note the appearance of variable coefficients in the limit equation.*

The proof of Theorem 5.1 uses analogous arguments as those in [10]. The main novelty here is the presence of an additional non-linear term, due to capillarity; nonetheless, it turns out that this item exactly cancels out with another one, coming from the analysis of the convective term. In addition, the regularization process presents some complications with respect to [10], because one has less available controls for the velocity fields.

As it was already the case in [10], the compensated compactness arguments work under high regularity assumptions on the function $\mathfrak{c}$: here, we looked for minimal conditions for it in order to prove the result. Having $\mathfrak{c}\in W^{1,\infty}$ seems to be a necessary hypothesis, together with (23), for making this strategy work; on the other hand, boundedness of the second derivatives was used in [10] to control some remainders created by the regularization procedure (essentially, commutators between a smoothing operator and the variable coefficient). Theorem 5.1 shows that it is sufficient to have $\nabla_h\mathfrak{c}$ continuous, for some admissible modulus of continuity μ; in [7] we also proved that, if μ decays to 0 suitably fast (so fast to annihilate a logarithmic divergence), then it is enough to impose Zygmund type conditions and to control the second variation of $\nabla\mathfrak{c}$ by μ.

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