GEVREY-TYPE RESOLVENT ESTIMATES AT THE THRESHOLD FOR A CLASS OF NON-SELFADJOINT SCHRÖDINGER OPERATORS

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Abstract. In this article, we show that under some coercive assumption on the complex-valued potential $V(x)$, the derivatives of the resolvent of the non-selfadjint Schröinger operator $H = -\Delta + V(x)$ satisfy some Gevrey estimates at the threshold zero. As applications, we establish subexponential time-decay estimates of local energies for the semigroup e^{-tH}, $t > 0$. We also show that for a class of Witten Laplacians for which zero is an eigenvalue embedded in the continuous spectrum, the solutions to the heat equation converges subexponentially to the steady solution.

1. Introduction

This work is concerned with the time-decay of semigroup e^{-tH}, $t \geq 0$, where $H = -\Delta + V(x)$ is a Schrödinger operator on $\mathbb{R}^n$ with a complex-valued potential $V(x) = V_1(x) - iV_2(x)$, where either $V_1(x)$ or $V_2(x)$ are slowly decreasing like $\frac{1}{\langle x\rangle^{2\mu}}$ for some $0 < \mu < 1$. Here $\langle x\rangle = (1 + |x|^2)^{\frac{1}{2}}$.

The study of this problem is in part motivated by the large-time behaviours of solutions to the Kramers-Fokker-Planck equation with a slowly increasing potential. After a change of unknowns and for appropriate values of physical constants, the Kramers-Fokker-Planck

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equation can be written into the form

$$\partial_t u(t;x,v) + Pu(t;x,v) = 0,\ (x,v) \in \mathbb{R}^n \times \mathbb{R}^n,\ n \geq 1,\ t > 0, \tag{1.1}$$

with some initial data

$$u(0;x,v) = u_0(x,v). \tag{1.2}$$

Here P is the Kramers-Fokker-Planck operator:

$$P = -\Delta_v + \frac{1}{4}|v|^2 - \frac{n}{2} + v \cdot \nabla_x - (\nabla U(x)) \cdot \nabla_v, \tag{1.3}$$

where the potential $U(x)$ is supposed to be a real-valued C^1 function. Define $\mathfrak{M}$ by

$$\mathfrak{M}(x,v) = \frac{1}{(2\pi)^{\frac{n}{4}}} e^{-\frac{1}{2}(\frac{v^2}{2}+U(x))}. \tag{1.4}$$

Then one has $P\mathfrak{M} = 0$. If $|\nabla U(x)| \geq C > 0$ and $U(x) > 0$ outside some compact, $U(x)$ increases at least linearly and $\mathfrak{M} \in L^2$. Then 0 is in the discrete spectra of P and after suitable normalization, one has in $L^2(\mathbb{R}^{2n})$

$$e^{-tP}u_0 = \langle \mathfrak{M}, u_0 \rangle \mathfrak{M} + O(e^{-\sigma t}), \quad t \to +\infty. \tag{1.5}$$

for some $\sigma > 0$ depending on the gap between 0 and the real part of other eigenvalues of P (see for example [2, 4, 5, 8]). This result mathematically justifies the physical phenomena of trend to equilibrium. If $\nabla U(x)$ tends to zeros as $|x| \to +\infty$, 0 is the bottom of the essential spectrum of P. It may be an eigenvalue or a resonance of P and there is no gap between point 0 and the remaining part of the spectrum of P. A natural question is to study the large-time behavior of $e^{-tP}u_0$ when $\nabla U(x)$ tends to zeros as $|x| \to +\infty$. If $U(x)$ decreases sufficiently rapidly (the short-range case), one can regard the term $(\nabla U(x)) \cdot \nabla_v$ as a perturbation of a model operator P_0 which is the free Kramers-Fokker-Planck operator (without the potential). It is proven in [12] by scattering method that for short-range potentials in dimension three one has

$$e^{-tP}u_0 = \frac{1}{(4\pi t)^{\frac{3}{2}}} \langle \mathfrak{M}, u_0 \rangle \mathfrak{M} + O(t^{-\frac{3}{2}-\epsilon}), \quad t \to +\infty, \tag{1.6}$$

in appropriately weighted L^2-spaces. For long-range or slowly increasing potentials, one may need to use other models such as the Witten Laplacian

$$-\Delta_U = -(\nabla_x - \nabla U(x)) \cdot (\nabla_x + \nabla U(x)), \tag{1.7}$$

which is a self-adjoint Schrödinger operator with a slowly decreasing potential. From this point of views, the class of operators studied in this work may be considered as some model for the Kramers-Fokker-Planck equation with a long-range or a slowly increasing potential.

Recall that selfadjoint Schrödinger operators with globally positive and slowly decreasing potentials have been studied in [7, 14, 15]. Assume that for some constants $\mu \in]0,1[$ and $c_1, c_2 > 0$

$$c_1\langle x\rangle^{-2\mu} \le V(x) \le c_2\langle x\rangle^{-2\mu}, \quad x \in \mathbb{R}^n. \tag{1.8}$$

Under some additional conditions, it is known ([7, 15]) that the spectral measure $E'(\lambda)$ of $-\Delta + V(x)$ is smooth at $\lambda = 0$ and satisfies for any $N \ge 0$

$$\|E'(\lambda)\|_{L^2_{\rm comp} \to L^2_{\rm loc}} = O(|\lambda|^N), \quad \lambda \to 0. \tag{1.9}$$

If $n = 1$ and if $V(x)$ is analytic, D. Yafaev ([14]) proves that

$$\|e^{-itH}\|_{L^2_{\rm comp} \to L^2_{\rm loc}} = O(e^{-ct^\beta}), \quad t \to +\infty, \tag{1.10}$$

where $\beta = \frac{1-\mu}{1+\mu}$ and c is some positive constant. The proof of [14] is based on the explicit construction of scattering solutions in one dimensional case which is not available in high dimensional case and the similar result in higher dimensions has been unknown. When specified to the selfadjoint case ($V_2 = 0$), the Gevrey estimates of the resolvent at threshold obtained in this work allow to show that under some conditions one has some subexponential estimates for the spectral measure $E'(\lambda)$ near zero and that (1.10) holds in dimensions $n \ge 2$. See [13] for more precisions.

Let $V(x) = V_1(x) - iV_2(x)$ be $-\Delta$-bounded with relative bound zero and $H = -\Delta + V(x)$. Assume that $\Re H \ge 0$ and that the following coercive estimate holds: $\exists c_0 > 0, \mu \in]0,1[$ such that

$$|\langle u, Hu\rangle| \ge c_0(\|\nabla u\|^2 + \|\langle x\rangle^{-\mu}u\|^2), \quad \forall u \in H^2(\mathbb{R}^n). \tag{1.11}$$

Then one can show the limit $G_0 = \lim_{\lambda\to 0, \lambda<0}(H-\lambda)^{-1}$ exists in some topology. For any integer $N \geq 1$, G_0^N is densely defined and one has the following Gevrey estimates of the resolvent at $z = 0$: there exists some constant $C > 0$ such that for any $\chi \in C_0^\infty(\mathbb{R}^n)$ one has

$$\|\chi G_0^N\| + \|G_0^N \chi\| \leq C_\chi C^N N^{\gamma N}, \quad \forall N \in \mathbb{N}, \tag{1.12}$$

where

$$\gamma = \frac{2\mu}{1-\mu}. \tag{1.13}$$

See Theorem 2.1. Notice that we do not need any decay condition on $V_j(x)$, therefore the above result holds for some many-body Schrödinger operators with positive interactions. If the imaginary part $-V_2(x)$ of the potential $V(x)$ satisfies some decay condition: $\exists \mu' > 0$:

$$|V_2(x)| \leq C\langle x\rangle^{-\mu'}, \quad x \in \mathbb{R}^n, \tag{1.14}$$

then one has the following subexponential time-decay estimates for the local energies of the heat equation (Theorem 3): $\exists c > 0$ such that

$$\|\chi e^{-tH}\| + \|e^{-tH}\chi\| \leq C_\chi e^{-ct^\beta}, \quad t > 0, \tag{1.15}$$

for any $\chi \in C_0^\infty(\mathbb{R}^n)$, where

$$\beta = \frac{1-\mu}{1+\mu}. \tag{1.16}$$

If $U(x)$ increases sublinearly (see conditions (4.58) for more precisions), the Witten Laplacian $-\Delta_U$ defined by (1.7) can be regarded as a compactly supported perturbation of the operator H studied in this work. The Gevrey estimates for the resolvent of H can be used to prove the trend to equilibrium with a subexponential convergence rate of local energies for the heat equation associated with $-\Delta_U$: $\exists c > 0$ such that for any $R > 0$, there exists some $C_R > 0$ such that

$$\|e^{t\Delta_U} f - \langle \varphi_0, f\rangle \varphi_0\| \leq C_R \|f\| e^{-ct^{\frac{\sigma}{2-\sigma}}}, \quad t > 0, \tag{1.17}$$

for any $f \in L^2(\mathbb{R}^n)$ with supp $f \subset \{x; |x| \leq R\}$, where φ_0 is the normalized eigenfunction of $-\Delta_U$ associated with 0 (which is also the bottom of its essential spectrum). See Corollary 4.1.

To treat more general non-selfadjoint operators for which the bottom, 0, of the essential spectrum is an eigenvalue, the main difficulty is the spectral analysis at the threshold. This subtle problem is closely related to the perturbation of arbitrary Jordan structure in infinite dimensional case and is still largely open.

The ideas of this work can also be applied to the Schrödinger semigroup e^{-itH}, $t > 0$. Under some conditions, using both the techniques of analytical dilation and analytical distorsion (which define the same set of resonances of H according to [3]), one can show that the local energies of the solutions to the Schrödinger equation also decay subexponentially with the same exponent given by (1.16). See [13].

The organisation of this paper is as follows. In Section 2, under some coercive condition, we begin with an uniform energy estimate for a class of second order differential operators. The uniformity in parameters of the energy estimates allow to control the growth of powers of the resolvent at threshold in weighted spaces. In Section 3, we evaluate the numerical range of H and obtain resolvent estimates on a curve in the right half complex-plane. (1.15) is deduced from Gevrey estimates of the resolvent at threshold. In Section 4, we consider compactly supported perturbations of the operator H and prove (1.17).

2. Gevrey estimates of the resolvent at threshold

A basic ingredient of this work is a uniform a priori energy estimate. Let H is a general second order elliptic differential operator of the form

$$H = -\sum_{i,j=1}^{n} \partial_{x_i} a^{ij}(x)\partial_{x_j} + \sum_{j=1}^{n} b_j(x)\partial_{x_j} + V(x), \tag{2.18}$$

where $a^{ij}(x)$, $b_j(x)$ and $V(x)$ are complex-valued functions. Suppose that $a^{ij}(x)$'s are bounded and of class C^1 on $\mathbb{R}^n$ with bounded derivatives and that there exists $c > 0$ such that the real part of the matrix $a(x) = (a^{ij}(x))$ is positive definite satisfying

$$\Re(a^{ij}(x)) \geq cI_n, \quad \forall x \in \mathbb{R}^n. \tag{2.19}$$

Assume also that $b_j(x)$'s are bounded and that the multiplication by $V(x)$ is relatively bounded with respect to $-\Delta$ with relative bound zero. Then H defined on $D(H) =$

$H^2(\mathbb{R}^n)$ is closed. In the following we always assume without mention that the above conditions are satisfied.

Assume that $\Re H \geq 0$ and that there exists some constants $0 < \mu < 1$ and $c_0 > 0$ such that

$$|\langle Hu, u\rangle| \geq c_0(\|\nabla u\|^2 + \|\langle x\rangle^{-\mu} u\|^2), \quad \text{for all } u \in H^2, \tag{2.20}$$

$$\sup_x |\langle x\rangle^\mu b_j(x)| < \infty, \quad j = 1, \cdots, n. \tag{2.21}$$

Clearly, this assumption is satisfied if $H = -\Delta + V_1(x) - iV_2(x)$ with $V_j(x)$ real-valued and either

$$V_1(x) \geq c\langle x\rangle^{-2\mu} \tag{2.22}$$

on $\mathbb{R}^n$ (with arbitrary V_2) or $V_1 = 0$ and $V_2(x) \geq c\langle x\rangle^{-2\mu}$ on $\mathbb{R}^n$ for some $c > 0$ (strong dissipation).

Denote $b = (b_1, \cdots, b_n)$ and

$$|a|_\infty = \max_{1\leq i,j\leq n} \sup_{x\in\mathbb{R}^n} |a^{ij}(x)|, \quad |b|_{\mu,\infty} = \max_{1\leq j\leq n} \sup_{x\in\mathbb{R}^n} |\langle x\rangle^\mu b_j(x)|. \tag{2.23}$$

For $s \in \mathbb{R}$, denote

$$\varphi_s(x) = (1 + \frac{|x|^2}{R^2})^s, \tag{2.24}$$

where $R \equiv M\langle s\rangle^{\frac{1}{1-\mu}}$ with $M = M(c_0, |a|_\infty, |b|_\infty) > 1$ large enough, but independent of $s \in \mathbb{R}$. The uniformity in $s \in \mathbb{R}$ in the following Lemma is important for Gevrey estimates of the resolvent at threshold.

Lemma 2.1. *Let H be given by (2.18). Under the conditions (2.20) and (2.21) with $0 < \mu < 1$, there exists some constants $C, M > 0$ depending only on $|a|_\infty$, $|b|_{\mu,\infty}$ and c_0 given in (2.20) such that*

$$\|\langle x\rangle^{-\mu}\varphi_s(x)u\| + \|\nabla(\varphi_s(x)u)\| \leq C\|\langle x\rangle^\mu \varphi_s(x)Hu\| \tag{2.25}$$

for any $s \in \mathbb{R}$ and $u \in H^2(\mathbb{R}^n)$ with $\langle x\rangle^{|s|+\mu}Hu \in L^2$.

Lemma 2.2. *(a). G_0 is a densely defined closed operator. If H is selfadjoint (resp., maximally dissipative), then $-G_0$ is also selfadjoint (resp., maximally dissipative).*

(b). There exists some C such that

$$\|\nabla(\varphi_s G_0 \varphi_{-s} \langle x\rangle^{-\mu} w)\| + \|\langle x\rangle^{-\mu} \varphi_s G_0 \varphi_{-s} \langle x\rangle^{-\mu} w)\| \le C\|w\| \tag{2.26}$$

for all $u \in \mathcal{D}$ and $s \in \mathbb{R}$.

Proof. We firstly show that $D(G_0)$ is dense. Remark that $\Re H \ge 0$. Let $f \in \mathcal{D}$ and $u_\epsilon = (H+\epsilon)^{-1} f$, $\epsilon > 0$. Following the proof of Lemma 2.1 with H replaced by $H+\epsilon$, one has that for any $s > 0$

$$\|\langle x\rangle^{s-\mu} \nabla u_\epsilon\| + \|\langle x\rangle^{s-2\mu} u_\epsilon\| \le C_s \|\langle x\rangle^s f\|$$

uniformly in $\epsilon > 0$. For $s > 2\mu$, this estimate implies that the sequence $\{u_\epsilon; \epsilon \in]0,1]\}$ is relatively compact in L^2. Therefore there exists a subsequence $\{u_{\epsilon_k}; k \in \mathbb{N}\}$ and $u \in L^2$ such that $\epsilon_k \to 0$ and $u_{\epsilon_k} \to u$ in L^2 as $k \to +\infty$. It follows that $Hu = f$ in the sense of distributions. The ellipticity of H implies that $u \in H^2(\mathbb{R}^n)$. Therefore $f \in R(H) = D(G_0)$. This shows that $D(G_0)$ is dense in L^2. The closeness of G_0 follows from that of H. The other assertions can be easily checked.

The argument above shows that for any $w \in \mathcal{S}$, one can find $u \in D(H)$ such that $Hu = \varphi_{-s}\langle x\rangle^{-\mu} w$. (2.26) follows from (2.18). □

Lemma 2.2 shows that for any s, $\langle x\rangle^{-\mu}\varphi_s G_0 \varphi_{-s}\langle x\rangle^{-\mu}$ defined on $\mathcal{D} = \cap_s L^{2,s}$ can be uniquely extended to a bounded operator in L^2, or in other words, for any $s \in \mathbb{R}$, G_0 is bounded from $L^{2,s}$ to $L^{2,s-2\mu}$. In addition, one has $G_0\mathcal{D} \subset \mathcal{D}$. Thus G_0^N is well defined on $\mathcal{D}$ for any $N \in \mathbb{N}^*$. By an induction, one can check that G_0^N extends to a bounded operator from $L^{2,s}$ to $L^{2,s-2N\mu}$ for any $s \in \mathbb{R}$. To simplify notation, we still denote G_0 (resp., G_0^N) its continuous extension as operator from $L^{2,s}$ to $L^{2,s-2\mu}$ (resp., from $L^{2,s}$ to $L^{2,s-2N\mu}$.

Theorem 2.1. *Let $M > 1$ be given in Lemma 2.1. Denote*

$$x_{N,r} = \frac{x}{R_{N,r}} \text{ with } R_{N,r} = M\langle (2N-1+r)\mu\rangle^{\frac{1}{1-\mu}}$$

where $N \in \mathbb{N}$ and $r \in \mathbb{R}_+$. Set $\langle x_{N,r}\rangle = (1+|x_{N,r}|^2)^{\frac{1}{2}}$. Then there exists some constant $C > 0$ such that

$$\|\langle x_{N,r}\rangle^{-(2N+r)\mu} G_0^N \langle x_{N,r}\rangle^{r\mu}\| \leq C^N \langle (2N-1+r)\mu\rangle^{\gamma N}, \tag{2.27}$$

for any integer $N \geq 1$ and any $r \geq 0$. Here

$$\gamma = \frac{2\mu}{1-\mu}. \tag{2.28}$$

The proof of Theorem 2.1 is omitted here. See [13] for the details.

Let $R(z)$ denote the resolvent of H and $\Omega(\epsilon) = \{z \in \mathbb{C}^*; \frac{\pi}{2}+\epsilon < \arg z < \frac{3\pi}{2} - \epsilon\}$, $\epsilon > 0$. Since $\Re H \geq 0$, there exists some $C_1 > 0$ such that

$$\|R(z)\| \leq \frac{C_1}{|z|}, \quad z \in \Omega.$$

As a consequence of Theorem 2.1, one obtains the following

Corollary 2.1. *Let $\epsilon > 0$ be fixed. Then there exists some constant $C > 0$ such that*

$$\|\chi \frac{d^{N-1}}{dz^{N-1}} R(z)\| \leq C_\chi C^N N^{(1+\gamma)N} \tag{2.29}$$

for any $\chi \in C_0^\infty(\mathbb{R}^n)$, $N \geq 1$ and $z \in \Omega(\epsilon)$.

Proof. Notice that $\|zR(z)\|$ is uniformly bounded for $z \in \Omega(\epsilon)$ and that

$$\frac{d^{N-1}}{dz^{N-1}} R(z) = (N-1)! R(z)^N = (N-1)! G_0^N (1+zR(z))^N.$$

By Theorem 2.1, there exists some constant $C_1 > 0$ such that

$$\|\chi G_0^N (1+zR(z))^N\| \leq \|\langle x_{N,r}\rangle^{2\mu N} \chi\|_{L^\infty} \times C_1^N$$

for any $\chi \in C_0^\infty(\mathbb{R}^n)$, $N \geq 1$ and $z \in \Omega(\epsilon)$. Let $R > 0$ be such that supp $\chi \subset b(0,R)$. Then One can check that

$$\|\langle x_{N,r}\rangle^{2\mu N}\chi\|_{L^\infty} \leq \|\chi\|_{L^\infty}(1+\frac{R^2}{M^2((2N-1)\mu)^{\frac{2}{1-\mu}}})^{\mu N} \leq C_\chi 2^{\mu N}$$

for some constant C_χ depending on χ and R, but independent of N. This proves (2.26) with $C = C_1 2^\mu$ independent of χ. □

3. Subexponential time-decay for the heat equation

From now on, we consider the Schrödinger operator $H = -\Delta + V(x)$ with $V(x) = V_1(x) - iV_2(x)$, $V_1(x), V_2(x)$ being real. Denote $R(z) = (H - z)^{-1}$. Theorem 2.1 can be used to prove subexponential time-decay for local energies of solutions to the heat and Schrödinger equations. To do this, we use Cauchy integral formula for semigroups and need some information of the resolvent on a contour in the right half complex plane passing through the origin.

Proposition 3.1. *Assume that $\Re H \geq -a\Delta$ for some $a > 0$ and that the imaginary part of the potential $V(x)$ verifies the estimate*

$$|V_2(x)| \leq C\langle x\rangle^{-2\mu'}, \quad \forall x \in \mathbb{R}^n, \tag{3.30}$$

where $0 < \mu' < \frac{n}{2}$ and $0 < \mu' \leq 1$. Then there exists some constant $C_0 > 0$ such that the numerical range $N(H)$ of H is contained in a region of the form $\{z; \Re z \geq 0, |\Im z| \leq C_0(\Re z)^{\mu'}\}$. Consequently, for any $A_0 > C_0$ there exists some constant M_0 such that

$$\|R(z)\| \leq \frac{M_0}{|z|^{\frac{1}{\mu'}}} \tag{3.31}$$

for $z \in \Omega := \{z \in \mathbb{C}^*; |z| \leq 1, \Re z < 0$ or $\Re z \geq 0, |\Im z| > A_0(\Re z)^{\mu'}\}$.

Proof. For $z = \langle u, Hu\rangle \in N(H)$ where $u \in D(H)$ and $\|u\| = 1$, one has

$$\begin{aligned} \Re z &= \Re\langle u, Hu\rangle \geq a\|\nabla u\|^2 \\ |\Im z| &\leq \langle u, |V_2|u\rangle \leq C\|\langle x\rangle^{-\mu'}u\|^2. \end{aligned}$$

According to the generalized Hardy inequality ([6]), one has for $0 < \mu' < \frac{n}{2}$

$$\|\langle x\rangle^{-\mu'}u\|^2 \leq \frac{\Gamma(\frac{n-2\mu'}{4})^2}{2^{2\mu'}\Gamma(\frac{n+2\mu'}{4})^2}\||\nabla|^{\mu'}u\|^2. \tag{3.32}$$

Let $\hat{u}$ denote the Fourier transform of u normalized so that $\|\hat{u}\| = \|u\|$ and $\tau = \|\nabla u\|$. Then

$$\begin{aligned} \||\nabla|^{\mu'}u\|^2 &= \||\xi|^{\mu'}\hat{u}\|^2 = \||\xi|^{\mu'}\hat{u}\|^2_{L^2(|\xi|\geq\tau)} + \||\xi|^{\mu'}\hat{u}\|^2_{L^2(|\xi|<\tau)} \\ &\leq \tau^{2(\mu'-1)}\||\xi|\hat{u}\|^2_{L^2(|\xi|\geq\tau)} + \tau^{2\mu'}\|\hat{u}\|^2_{L^2(|\xi|<\tau)} \\ &\leq 2\tau^{2\mu'} = 2\|\nabla u\|^{2\mu'}. \end{aligned}$$

In the first inequality above, the condition $0 < \mu' \leq 1$ is used. This proves that $|\Im z| \leq C_0(\Re z)^{\mu'}$ when $z \in N(H)$. The other assertions of Proposition are immediate, since $\sigma(H) \subset \overline{N(H)}$ and

$$\|R(z)\| \leq \frac{1}{\mathrm{dist}(z, N(H))}.$$

□

Notice that under the conditions of Proposition 3.1, one can not exclude possible accumulation of complex eigenvalues towards zero. Making use of Proposition 3.1, one can prove as in Corollary 2.1 the following

Corollary 3.1. *Under the conditions of proposition 3.1, let κ be an integer suhc that $\kappa + 1 \geq \frac{1}{\mu'}$. Then there exists some constant $C > 0$ such that for any $\chi \in C_0^\infty(\mathbb{R}^n)$, one has*

$$\|\chi(x)\frac{d^{N-1}}{dz^{N-1}}R(z)\| \leq C_\chi C^N N^{(1+(1+\kappa)\gamma)N}, \quad \forall N \geq 1, \tag{3.33}$$

uniformly in $z \in \Omega$. Here Ω is defined as in Proposition 3.1.

As another consequence of Proposition 3.1, we obtain the following estimate on the expansion of the resolvent at 0:

Corollary 3.2. *Under the conditions of Proposition 3.1, assume in addition (2.20) with $\mu \in]0,1[$. Then there exists some constant $c > 0$ such that for any $z \in \Omega$ and z near 0, one has for some N (depending on z) such that*

$$\|\langle x_{N,0}\rangle^{-2N\mu}(R(z) - \sum_{j=0}^{N} z^j G_0^{j+1})\| \leq e^{-c|z|^{-\frac{1}{\gamma}}}. \tag{3.34}$$

Here $\langle x_{N,0}\rangle$ is defined in Theorem 2.1 with $r = 0$.

Theorem 3.1. *Assume that (2.20) is satisfied for some* $\mu \in]0,1[$, *that* $\Re H \geq -a\Delta$ *for some* $a > 0$ *and that the imaginary part of the potential* $V(x)$ *verifies*

$$|V_2(x)| \leq C\langle x\rangle^{-2\mu'}, \quad \forall x \in \mathbb{R}^n, \tag{3.35}$$

for some $\mu' \in]0, \frac{1}{2}[$. *Let* $\chi \in C_0^\infty(\mathbb{R}^n)$ *be equal to* 1 *in the unit ball and be supported in the ball of the radius two. Then there exists some constants* $C, c > 0$ *such that*

$$\|\chi(Ct^{-\frac{1}{1+\mu}}x)e^{-tH}\| + \|e^{-tH}\chi(Ct^{-\frac{1}{1+\mu}}x)\| \leq Ce^{-ct^\beta}, \quad t > 0, \tag{3.36}$$

where

$$\beta = \frac{1-\mu}{1+\mu}. \tag{3.37}$$

In particular, there exists $c > 0$ *such that for any* $R > 0$ *one has*

$$\|e^{-tH}f\| \leq C_R e^{-ct^\beta}\|f\|, \quad t > 0, \tag{3.38}$$

for all $f \in L^2(\mathbb{R}^n)$ *supported in some ball* $\{|x| \leq R\}$,

Theorem is deduced from Theorem 2.1 and Proposition 3.1. See [13] for details.

4. An application to the Witten Laplacian

In this Section, we apply the Gevrey estimates obtained before to a class of operators which are compactly supported perturbation of operator H studied before. Consider operator P of the form

$$P = H + W(x). \tag{4.39}$$

where $H - \Delta + V(x)$ is a non-selfadjoint Schrödinger operator satisfying the conditions of Theorem 3 and $W \in L^\infty(\mathbb{R}^n)$ with compact support. Then the essential spectrum of P is equal to $[0, +\infty[$ and the accumulation points of the eigenvalues of P are contained in $\mathbb{R}_+$. We want to study the large-time asymptotics of the semigroup e^{-tP} as $t \to +\infty$.

From Theorem 2.1, one sees that for any $s \in \mathbb{R}$, G_0W is a compact operator in $L^{2,s}$ and -1 is an eigenvalue of G_0W in $L^{2,s}$ if and only if 0 is an eigenvalue of P in L^2 and $\ker(1 + G_0W)$ in $L^{2,s}$ coincides with the eigenspace of P with eigenvalue 0.

Proposition 4.1. *Assume that* 0 *is not an eigenvalue of* P*. Then* P *has no eigenvalue in a region* Ω *of the form*

$$\Omega_\delta = \{|z| < \delta, \Re z < 0\} \cup \{|z| < \delta, \Re z \geq 0, |\Im z| > \delta^{-1}(\Re z)^{\mu'}\} \tag{4.40}$$

for some $\delta > 0$*. There exists some constant* $c > 0$ *such that*

$$\|\chi(e^{-tP} - \sum_{\lambda \in \sigma_p P; \Re\lambda \leq 0} e^{-tP}\Pi_\lambda)\| + \|(e^{-tP} - \sum_{\lambda \in \sigma_p P; \Re\lambda \leq 0} e^{-tP}\Pi_\lambda)\chi\| \leq C_\chi e^{-ct^\beta} \tag{4.41}$$

for any $t > 0$ *and* $\chi \in C_0^\infty(\mathbb{R}^n)$*. Here* Π_λ *is the Riesz projection associated with the eigenvalue* λ *of* P*.*

Proof. Let $W(z) = 1 + R(z)W$. Theorem 2.1 and Proposition 3.1 show that $W(z) = 1 + G_0W + O(|z|)$ in $\mathcal{L}(L^2)$ for $z \in \Omega$, where Ω is defined in Proposition 3.1. The condition that 0 is not an eigenvalue of P implies that there exists some $\delta > 0$ such that $1 + R(z)W$ is invertible in $\mathcal{L}(L^2)$ for $z \in \Omega_\delta$ with uniformly bounded inverse. Consequently P has no eigenvalue in Ω_δ. By Theorem 2.1, one sees that the derivatives of $W(z)$ and $R(z)\chi$ for $z \in \Omega_\delta$ admit limits at the point $z = 0$ and satisfy the Gevrey estimates in $\mathcal{L}(L^2)$:

$$\|\frac{d^N}{dz^N}R(0)\chi\| + \|\frac{d^N}{dz^N}W^{(n)}(0)\| \leq C_\chi C^N N^{(1+\gamma)N} \tag{4.42}$$

From the resolvent equation

$$(P - z)^{-1}\chi = (1 + R(z)W)^{-1}R(z)\chi,$$

we deduce that the resolvent of P satisfies the same kind of Gevrey estimates at the point $z = 0$. (4.41) is deduced from the resolvent estimates by using contour integral. The details are omitted here. □

When 0 is an eigenvalue of P, the kernel of $1 + G_0W$ becomes non-trivial. The threshold spectral analysis for non-selfadjoint operators is a subtle problem due to the possible presence of arbitrary Jordan structures. For this reason, we restrict ourselves to the selfadjoint case.

Theorem 4.1. *Assume that* 0 *is an eigenvalue of* $P = H + W(x)$*, where* H *is a selfadjoint operator satisfying the conditions of Theorem 3 and* $W(x)$ *a real, bounded and compactly* *supported function. Then one has*

$$\|\chi(e^{-tP} - \sum_{\lambda\in\sigma_p(P);\lambda\le 0} e^{-t\lambda}\Pi_\lambda)\| + \|(e^{-tP} - \sum_{\lambda\in\sigma_p(P);\lambda\le 0} e^{-t\lambda}\Pi_\lambda)\chi\| \le C_\chi e^{-ct^\beta} \tag{4.43}$$

for any $t > 0$ *and* $\chi \in C_0^\infty(\mathbb{R}^n)$. *Here* $\beta = \frac{1-\mu}{1+\mu}$ *and* Π_λ *is the spectral projection associated with the eigenvalue* λ *of* P.

Theorem 4.1 is an easy consequence of the following resolvent asymptotics with Gevrey estimates on the remainder.

Theorem 4.2. *Let* P *satisfy the conditions of Theorem 4.1. Then there exists some* $\delta > 0$ *such that*

$$(P-z)^{-1} = -\frac{\Pi_0}{z} + R_1(z) \tag{4.44}$$

for $z \in \Omega(\delta) = \{z \in \mathbb{C}; |z| < \delta, |\arg z| > \delta\}$ *where the remainder* $R_1(z)$ *satisfies the following Gevrey estimates:* $\exists C > 0$ *such that for any* $\chi \in C_0^\infty(\mathbb{R}^n)$ *one has for some constant* C_χ

$$\|\chi \frac{d^N}{dz^N} R_1(z)\| + \|\frac{d^N}{dz^N} R_1(z)\chi\| \le C_\chi C^N N^{(1+\gamma)N}, \tag{4.45}$$

for any $N \in \mathbb{N}$ *and* $z \in \Omega(\delta)$.

Proof. We use the Grushin-Feshbach method to study the low-energy asymptotics for the resolvent of P by using the equation

$$(P-z)^{-1} = (1 + R(z)W)^{-1}R(z). \tag{4.46}$$

Since the method is well-known, we only sketch some details and insist on the Gevrey estimates of the remainder. Let m be the multiplicity of the zero eigenvalue of P. Then $\ker_{L^{2,s}}(1+G_0W)$ is independent of $s \in \mathbb{R}$ and coincides with the eigenspace of P associated withe the eigenvalue 0. We need only to work in $L^2(\mathbb{R}^n)$ $(s = 0)$.

Let $\psi_1, \cdots, \psi_m$ be a basis of $\ker(1 + G_0W)$ such that

$$\langle \psi_j, -W\psi_k\rangle = \delta_{jk}, \quad j, k = 1, \cdots, m. \tag{4.47}$$

(4.47) can be realized because $\phi \to \langle \phi, -W\phi \rangle$ is positive definite on $\ker(1 + G_0 W)$ (see [9]). Define $Q : L^2 \to L^2$ by

$$Qf = \sum_{j=1}^{m} \langle -W\psi_j, f \rangle \psi_j, \quad f \in L^2. \tag{4.48}$$

Then one can show that for $\epsilon > 0$ small enough

$$E(z) = (Q'(1 + R(z)W)Q')^{-1}Q' \tag{4.49}$$

is well-defined and continuous in $z \in \Omega(\delta)$.

Define $T : \mathbb{C}^m \to D(P)$ and $S : L^2 \to \mathbb{C}^m$ by

$$\begin{aligned} Tc &= \sum_{j=1}^{m} c_j \psi_j, \quad c = (c_1, \cdots, c_m) \in \mathbb{C}^m, \\ Sf &= (\langle -W\psi_1, f \rangle, \cdots, \langle -W\psi_m, f \rangle) \in \mathbb{C}^m, \quad f \in L^2. \end{aligned}$$

Set $W(e) = (1 + R(z)W)$ and

$$E_+(z) = T - E(z)W(z)T, \tag{4.50}$$

$$E_-(z) = S - SW(z)E(z), \tag{4.51}$$

$$E_{-+}(z) = -SW(z)T + SW(z)E(z)W(z)T. \tag{4.52}$$

Then one has the formula

$$(1 + R(z)W)^{-1} = E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z) \text{ on } H^{1,-s}. \tag{4.53}$$

Since Notice that by the selfadjointness of H, $\|zR(z)\|$ is bounded for $z \in \Omega(\delta)$. The Gevrey estimates given in Corollary 3.1 hold for $R(z)$ with $\kappa = 0$. Consequently $E(z), E_\pm(z)$ and $E_{-+}(z)$ satisfy the similar Gevrey estimates for z in $\Omega(\delta)$. $E_{-+}(z)$ can be calculated. One has

$$E_{-+}(z) = -z\Psi + z^2 r_1(z) \tag{4.54}$$

where $\Psi = (\langle\psi_j, \psi_k\rangle)_{1\le j,k\le m}$ is positive definite and $r_1(z)$ satisfies the Gevrey estimates in $\Omega(\delta)$. Consequently,

$$E_{-+}(z)^{-1} = -\frac{\Psi^{-1}}{z} + \tilde{r}_1(z) \tag{4.55}$$

with $\tilde{r}_1(z)$ satisfies the Gevrey estimates in $\Omega(\epsilon)$. This proves that $(1 + R(z)W)^{-1}$ is of the form

$$(1 + R(z)W)^{-1} = \frac{A_0}{z} + B(z) \tag{4.56}$$

where A_0 is an operator of rank m and $B(z)$ satisfies the Gevrey estimates in $\Omega(\delta)$. From this we deduce that

$$(P - z)^{-1} = -\frac{\Pi_0}{z} + R_1(z) \tag{4.57}$$

where $R_1(z)$ satisfies the Gevrey estimates

$$\|\frac{d^N}{dz^N} R_1(z)\chi\| \le C_\chi C^N N^{(1+\gamma)N}$$

for z in $\Omega(\delta)$. This proves (4.44). □

Finally, consider the Witten-Laplacian $-\Delta_U$ with $U \in C^2(\mathbb{R}^n; \mathbb{R})$ such that for some $\sigma \in]0, 1[$ and $c_1, C_1 > 0$, one has

$$U(x) \ge 0, \quad |\nabla U(x)| \ge c_1\langle x\rangle^{\sigma-1}, \quad |\Delta U(x)| \le C_1\langle x\rangle^{\sigma-2} \tag{4.58}$$

for x outside some compact. Then

$$-\Delta_U = -(\nabla_x + \nabla U(x)) \cdot (\nabla_x + \nabla U(x)) = -\Delta + |\nabla U(x)|^2 - \Delta U(x)$$

Let $H = -\Delta + V(x)$ with

$$V(x) = \chi_R(x) + (1 - \chi_R(x))(|\nabla U(x)|^2 - \Delta U(x)),$$

χ_R is some cut-off function with $0 \le \chi_R(x) \le 1$ and $\chi_R(x) = 1$ for $|x| \le R$; 0 for $|x| \ge 2R$ for some $R > 1$ sufficiently large. Then one can check that H satisfies the conditions of Theorem 3 with $\mu = 1 - \sigma$ and $\mu' = 0$ and $-\Delta_U = H + W(x)$ where $W(x)$ is of compact support. The continuous spectrum of $-\Delta_U$ is equal to $[0, +\infty[$ and the only eigenvalue of $-\Delta_U$ is zero which is simple. As a consequence of Theorem 4.1, one obtains the following

Corollary 4.1. *Assume the condition (4.58). Let* $\varphi_0(x) = c_0 e^{U(x)}$ *with the normalizing constant* c_0 *chosen such that* $\|\varphi_0\| = 1$*. Then there exists some constant* $c > 0$ *such that for any* $R > 0$*, there exists some constant* $C_R > 0$ *such that*

$$\|e^{t\Delta_U} f - \langle \varphi_0, f\rangle \varphi_0\| \le C_R e^{-ct^{\frac{\sigma}{2-\sigma}}} \|f\|, \quad t > 0, \tag{4.59}$$

for any $f \in L^2(\mathbb{R}^n)$ *with supp* $f \subset \{x; |x| \le R\}$.

Corollary 4.1 shows that for the Witten Laplacian in a scattering situation, distribution functions converge subexponentially to the equilibrium. It remains to see if the similar result also holds for the Kramers-Fokker-Planck operators.

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