SOME ZAREMBA-HOPF-OLEINIK BOUNDARY COMPARISON PRINCIPLES AT CHARACTERISTIC POINTS
SUL PRINCIPIO DEL CONFRONTO DI ZAREMBA-HOPF-OLEINIK NEI PUNTI CARATTERISTICI

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ABSTRACT. We investigate the so-called *Hopf lemma* for certain degenerate-elliptic equations at characteristic boundary points of bounded open sets. For such equations, the validity of the Hopf lemma is related to the fact that the boundary of the open set reflects the underlying geometry of the specific operator. We present here some recent results obtained in [21] in collaboration with V. Martino. Our main focus is on conditions on the boundary which are stable by changing our operators in some particular classes, for example in the class of horizontally elliptic operators in non-divergence form. We also study what happens to these conditions for degenerate operators with first order terms.

SUNTO. Si desidera investigare il cosiddetto *lemma di Hopf* per alcune equazioni ellittico-degeneri nei punti del bordo di un aperto limitato che siano caratteristici per l'operatore. Per tali equazioni, la validità del lemma di Hopf è legata al fatto che il bordo dell'aperto rifletta in qualche modo la geometria che soggiace l'operatore in questione. Vengono qui presentati alcuni recenti risultati contenuti in [21], ottenuti in collaborazione con V. Martino. Si vuole prestare particolare attenzione a condizioni sul bordo che siano stabili al variare dell'operatore in particolari classi, per esempio nella classe degli operatori orizzontalmente ellittici in forma di non-divergenza. Si studia anche come cambiano queste condizioni sul bordo nel caso di operatori degeneri che ammettano termini del primo ordine.

2010 MSC. Primary 35J70; Secondary 35B51.

KEYWORDS. Hopf lemma, degenerate elliptic operators.

Bruno Pini Mathematical Analysis Seminar, Vol. 1 (2015) pp. 54–68

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ISSN 2240-2829.

1. Introduction

Let Ω be a bounded open set in $\mathbb{R}^N$ with smooth (at least of class C^2) boundary, $y \in \partial\Omega$, and ν the inner unit normal to $\partial\Omega$ at y. We say that a second-order linear partial differential operator $\mathcal{L}$ satisfies the Hopf lemma in Ω at $y \in \partial\Omega$ if, for any $u \in C^2(\Omega \cap W) \cap C^1((\Omega \cap W) \cup \{y\})$, we have

$$\begin{cases} \mathcal{L}u \leq 0 \text{ in } \Omega \cap W, \\ u > 0 \text{ in } \Omega \cap W, \\ u(y) = 0 \end{cases} \Rightarrow \quad \frac{\partial u}{\partial \nu}(y) > 0, \tag{1}$$

where W is an open neighborhood of y. For us $\mathcal{L}$ will denote an operator in the form $\mathcal{L} = \sum_{i,j=1}^N a_{ij}(p)\partial^2_{ij} + \sum_{k=1}^N b_k(p)\partial_k$: we can think the coefficients a_{ij} and b_k to be continuous functions in some open set $O \supset \overline{\Omega}$, and $A(p) = (a_{ij}(p))_{i,j=1}^N$ a symmetric nonnegative definite $N \times N$ matrix never identically vanishing. These last conditions ensure in particular the validity of a weak maximum principle for $\mathcal{L}$ (see, e.g, [12]). On the other hand all the operators will be truly degenerate, in the sense that the matrix A will have a non-trivial kernel: in this regard it is crucial the following definition.

Definition 1.1. *We say that $y \in \partial\Omega$ is characteristic for $(\mathcal{L}, \Omega)$ if $A(y)\nu = 0$.*

To the best of our knowledge, the first *Hopf lemma* is due to Zaremba [32] who recognized that, for the case of the Laplace operator $\mathcal{L} = \Delta$, an interior ball condition for Ω ensure the validity of (1). Then, the celebrated and independent papers by Hopf [9] and Oleinik [28] proved that the interior ball condition, which is suitable for Δ, is in fact suitable with respect to the Hopf lemma for every uniformly elliptic operator in non-divergence form with measurable coefficients. Besides these classical works, *Hopf lemmas* have been extensively studied under several points of view (see e.g. [8, 11, 16, 2, 26]). Since the proofs in [9, 28], there is a strict relationship with the concept of barrier functions.

Definition 1.2. *Let $y \in \partial\Omega$. We say that a function h is an $\mathcal{L}$-barrier function for Ω at y if*

- *h is a C^2 function defined on an open bounded neighborhood U of y,*

- $h(y) = 0$,
- $\{p \in U \ : \ h(p) \geq 0\} \setminus \{y\} \subseteq \Omega$,
- $\mathcal{L}h \geq 0$ *in* $\{p \in U \ : \ h(p) > 0\}$,
- $\nabla h(y) \neq 0$.

It is in fact known the following

Proposition 1.1. *Fix $y \in \partial\Omega$. The existence of an interior $\mathcal{L}$-barrier function for Ω at y implies the validity of the Hopf lemma for $\mathcal{L}$ in Ω at y.*

Proof. Let u as in (1) and consider an $\mathcal{L}$-barrier function h for Ω at y, defined on U. Let $\rho > 0$ such that $\overline{B_\rho(y)} \subset U \cap W$. We set $V = \{p \in B_\rho(y) \ : \ h(p) > 0\}$, which is contained in Ω. We write $\partial V = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \{p \in B_\rho(y) \ : \ h(p) = 0\}$ and $\Gamma_2 = \partial V \setminus \Gamma_1$. Since $\Gamma_2 \subset \partial B_\rho(y) \cap \Omega$, we have $m = \min_{\overline{\Gamma}_2} u$ is strictly positive. Let us also put $M = \max_{\overline{V}} h > 0$. For $0 < \epsilon < \frac{m}{M}$, we consider $u - \epsilon h$. By construction we get $u - \epsilon h \geq 0$ on ∂V and $\mathcal{L}(u - \epsilon h) \leq 0$ in V. By the Weak Maximum Principle for $\mathcal{L}$, $u \geq \epsilon h$ in V. Since the inner unit normal to $\partial\Omega$ at y is given by $\nu = \frac{\nabla h(y)}{\|\nabla h(y)\|}$ and $y + t\nu \in V$ for small positive t, we obtain $\frac{\partial u}{\partial \nu}(y) \geq \epsilon \frac{\partial h}{\partial \nu}(y) = \epsilon \, \|\nabla h(y)\| > 0$. $\square$

Smooth domains have the interior ball property at any point $y \in \partial\Omega$, i.e. there exists a ball $B_{r_0}(p_0)$ such that $\overline{B_{r_0}(p_0)} \setminus \{y\} \subset \Omega$. This is the reason why it is easy to find an $\mathcal{L}$-barrier function h for Ω at y in the case y is non-characteristic for $(\mathcal{L}, \Omega)$. As a matter of fact, for $\alpha > 0$ big enough, the function $h_\alpha(p) = e^{-\alpha\|p-p_0\|^2} - e^{-\alpha r_0^2}$ is a barrier in a neighborhood of the non-characteristic point y (see e.g. [3]).

Therefore the real issue for the validity of the Hopf lemma is at the characteristic points. In the literature there are some positive and negative results for specific degenerate-elliptic operators. The references [4, 27, 23, 25] deal respectively with the case of the Kohn Laplacian in the Heisenberg group, generalized Greiner operators, and some Grushin-type operators. They pointed out that the boundary of the domain has somehow to reflect the geometry of the operator under consideration if one wants that the Hopf lemma holds true. The Zaremba's interior ball condition is thus replaced with an analogous condition regarding the level sets of the fundamental solution, which allows to find suitable barriers.

In Section 2, we follow at first these research lines by considering sub-Laplacians $\Delta_{\mathbb{G}}$ in general homogeneous Carnot groups: we thus show the validity of a Zaremba-type result under the condition regarding the level sets of the fundamental solution of $\Delta_{\mathbb{G}}$. Then, we focus on Hopf/Oleinik-type results. We consider the vector fields $X_1, \ldots, X_m$ generating a step-two Carnot algebra and we look at the class of operators in the form $\sum_{i,j=1}^{m} q_{ij} X_i X_j$, with (q_{ij}) uniformly positive definite. We show that an interior homogeneous $\mathbb{G}$-ball condition is suitable for every operator in this class. In Section 3 we analyze the case of operators with the presence of first-order terms. We want to deal with the model cases of $\partial^2_{xx} + x^2\partial^2_{tt} \pm \partial_t$, $\partial^2_{xx} + x^2\partial^2_{tt} \pm x\partial_t$ in order to understand the right conditions on Ω to have the Hopf property. We close the section by considering the same problem in the case of the sum of the squares of two vector fields in $\mathbb{R}^3$ satisfying the Hörmander condition at a characteristic point. In Section 4, we sum up the previous conditions and we exhibit a *natural* bounded open set where the Hopf lemma is satisfied at any point for all the non-divergence form operators uniformly elliptic along the vector fields $\partial_x, x\partial_t$ in $\mathbb{R}^2$.

All the complete proofs of the announced results are contained in [21]. In [21] it is also discussed a nonlinear degenerate-elliptic case, namely the operator describing the Levi curvature for a real hypersurface in $\mathbb{C}^2$ (see [7]). Here we are not going to present this case, but we do want to mention that this is actually one of the first motivations for our investigations. As a matter of fact, the Hopf lemma for elliptic operators has been historically a crucial tool to get symmetry results via the moving planes technique (see e.g. [1, 29, 14, 15]). Symmetry results for the Levi operator have been proved in the literature [22, 10, 24, 18, 19, 20]. Nevertheless an Alexandrov-type result for the Levi curvature in its generality is still an open problem. That is why in this work we have decided to study some classes of linear operators which are strictly related with the linearization of the Levi operator.

2. Zaremba-type and Hopf/Oleinik-type results

As we mentioned, at characteristic points the Hopf property may not hold true. Let us consider for example, in $\mathbb{R}^3$, the Kohn Laplacian on the Heisenberg group

$$\Delta_{\mathbb{H}} u(x_1,x_2,t) = \partial^2_{x_1x_1} + \partial^2_{x_2x_2} - x_2\partial^2_{x_1t} + x_1\partial^2_{x_2t} + \frac{1}{4}(x_1^2+x_2^2)\partial^2_{tt} = (X_1^2+X_2^2)u,$$

where $X_1 = \partial_{x_1} - \frac{1}{2}x_2\partial_t$, $X_2 = \partial_{x_2} + \frac{1}{2}x_1\partial_t$. If $y = 0 \in \partial\Omega$ and the inner unit normal is $(0,0,1)$ at 0, then 0 is characteristic for $(\Delta_{\mathbb{H}},\Omega)$. We have the following

Counterexample 2.1. *Suppose Ω locally around 0 is described by $\{(x,t)\in\mathbb{R}^3 : t > \frac{1}{4}(x_1^2+x_2^2)\}$. Let us consider $u(x,t) = t^2 - \frac{1}{16}(x_1^2+x_2^2)^2$. Of course, $u(0) = 0$ and $u > 0$ in Ω (we can assume $\Omega \subseteq \{t > \frac{1}{4}(x_1^2+x_2^2)\}$). Moreover $\Delta_{\mathbb{H}}u(x,t) = -\frac{1}{2}(x_1^2+x_2^2) \le 0$. But $\frac{\partial u}{\partial\nu}(0) = \partial_t u(0) = 0$, thus $\Delta_{\mathbb{H}}$ does not satisfy the Hopf lemma in Ω at 0.*

Despite this counterexample, it is possible to put some natural conditions on Ω to ensure the validity of the Hopf lemma for $\Delta_{\mathbb{H}}$. This was done by Birindelli and Cutrí in [4, Lemma 2.1]: as far as we know, this was the first example in literature of Hopf lemma for a degenerate-elliptic operator at a characteristic point. They proved that an interior Koranyi-ball condition for Ω allows to find a barrier. The first thing we want to do is to prove such result in generic homogeneous Carnot groups. To this purpose let us recall some notions (more details can be found in [5]).

Let $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ be a homogeneous Carnot group, with homogeneous dimension $Q \ge 3$. Let us fix $X_1,\dots,X_m$ left-invariant vector fields δ_λ-homogeneous of degree 1, which generate the first layer of the Lie algebra of $\mathbb{G}$ $(2 \le m < N)$. We want to consider the degenerate-elliptic operator $\Delta_{\mathbb{G}} = \sum_{j=1}^m X_j^2$. We put $\Gamma(\cdot\,;p_0)$ its fundamental solution with pole at p_0. Let us denote the $\mathbb{G}$-gauge balls centered at $p_0\in\mathbb{G}$ with radius r by

$$B_r^{\mathbb{G}}(p_0) = \left\{ p\in\mathbb{R}^N \,:\, \Gamma(p;p_0) > \frac{1}{r^{Q-2}} \right\}, \tag{2}$$

i.e. the superlevel sets of the fundamental solution. We call them balls since $\Gamma(p;p_0)^{\frac{1}{2-Q}}$ defines a homogeneous symmetric norm, satisfying a pseudo-triangle inequality. By exploiting the fundamental solution as $\Delta_{\mathbb{G}}$-barrier function, we can prove the following

Proposition 2.1. *Let us assume there exist p_0 and r_0 such that*

$$y \in \partial B^{\mathbb{G}}_{r_0}(p_0), \quad \overline{B^{\mathbb{G}}_{r_0}(p_0)} \setminus \{y\} \subset \Omega.$$

Then $\Delta_{\mathbb{G}}$ satisfies the Hopf lemma in Ω at y.

If $\mathbb{G}$ is the Heisenberg group $\mathbb{H}$, the $\mathbb{G}$-ball in (2) defines the Koranyi-ball, i.e. the metric balls with respect to the distance $d((x_1, x_2, t), 0) = ((x_1^2 + x_2^2)^2 + 16t^2)^{\frac{1}{4}}$. The assumption in Proposition 2.1 is saying that $\partial\Omega$ at the characteristic point has to be enough flat for the Hopf property to hold (so that to avoid behavior as in Counterexample 2.1). This is the same condition as in [4]. They used a different barrier, i.e. an exponential barrier of Hopf type.

In all the references [4, 27, 23, 25] and in our Proposition 2.1 the differential operator is fixed, and somehow also the related geometry. We want now to discuss the issue of the stability of the assumptions on Ω if we change the operator. Let us consider the following operators

$$L_Q = \sum_{i,j=1}^{m} q_{ij}(p) X_i X_j \tag{3}$$

where $Q(p)$ is symmetric and uniformly positive definite, i.e. $\lambda \mathbb{I}_m \leq Q(p) \leq \Lambda \mathbb{I}_m$ for some $\Lambda \geq \lambda > 0$.

Remark 2.1. *For operators as in* (3) *the condition of being characteristic is independent of the choice of the positive definite matrix Q, but it is determined just by the vector fields. A point $y \in \partial\Omega$ is in fact characteristic iff $X_j(y)$ is tangent to $\partial\Omega$ for all $j \in \{1, \dots, m\}$.*

If $X_1, \dots, X_m$ are the generators of the first (horizontal) layer of a homogeneous Carnot group, the operators in (3) are called *horizontally elliptic operators.* We are going to show an Hopf/Oleinik-type result in the case when the step of nilpotence of the Lie algebra is two. Let us fix some notations. Fix $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ such that the composition law $\circ$ is defined by

$$(x, t) \circ (\xi, \tau) = \left(x + \xi, t + \tau + \frac{1}{2} \langle Bx, \xi \rangle \right),$$

for $(x, t), (\xi, \tau) \in \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^N$. Here we have denoted by $\langle Bx, \xi \rangle$ the vector of $\mathbb{R}^n$ whose components are $\left\langle B^k x, \xi \right\rangle$ (for $k = 1, \dots, n$) and $B^1, \dots, B^n$ are $m \times m$ linearly independent skew-symmetric matrices. The group of dilations is defined as $\delta_\lambda((x,t)) = (\lambda x, \lambda^2 t)$ and the inverse of (x,t) is $(-x,-t)$. Up to fixing a stratification of the Lie algebra and applying a canonical isomorphism (see [5, Theorem 3.2.2]), a generic step-2 Carnot group is of this form. We can choose as homogeneous symmetric norm the function $d : \mathbb{R}^N \longrightarrow \mathbb{R}$ such that

$$d((x,t)) = \left(\|x\|^4 + \|t\|^2\right)^{\frac{1}{4}};$$

from here on we denote by $\|\cdot\|$ both the Euclidean norms in $\mathbb{R}^m$ and in $\mathbb{R}^n$. Hence, we have the homogeneous $\mathbb{G}$-ball $B^2_r(x_0,t_0) = (x_0,t_0) \circ B_r(0)$, where

$$B^2_r(0) = \{(x,t) \in \mathbb{R}^N \,:\, \|x\|^4 + \|t\|^2 < r^4\}.$$

Let us fix

$$X_i = \partial_{x_i} + \frac{1}{2}\sum_{k=1}^{n}(B^k x)_i \partial_{t_k} \qquad \text{for } i = 1, \ldots, m. \tag{4}$$

These m vector fields are left-invariant and δ_λ-homogeneous of degree 1: they generate the first layer of the Lie algebra of $\mathbb{G}$. We want to consider the operator L_Q as in (3) with respect to these specific vector fields. By exploiting the barriers built in [30] (and then used in [31]), we have the following

Theorem 2.1. *Let Ω be an open and bounded set in $\mathbb{R}^N$, and let $y = (\xi,\tau) \in \partial\Omega$ be a characteristic point. Let us assume there exist $(\xi_0,\tau_0) \in \Omega$ and $r_0 > 0$ such that*

$$y \in \partial B^2_{r_0}(\xi_0,\tau_0), \quad \overline{B^2_{r_0}(\xi_0,\tau_0)} \smallsetminus \{y\} \subset \Omega.$$

Then L_Q satisfies the Hopf lemma in Ω at y, for any horizontally elliptic operator in the step-2 Carnot group $\mathbb{G}$.

If we think of the example of the Heisenberg group, we are saying that the Koranyi-ball condition which is natural for the sum of squares $\Delta_{\mathbb{H}}$ is appropriate also for operators as in (3) with respect to the Heisenberg vector fields.

3. Examples and counterexamples with first-order terms

If we consider vector fields different from the ones in (4), the statement analogous to Theorem 2.1 is false as we can see with the following counterexample.

Remark 3.1. *Let us consider in $\mathbb{R}^2$ the two vector fields $X_1 = \partial_x$ and $X_2 = x\partial_t$. If we look at the operator $X_1^2 + X_2^2 = \partial_{xx}^2 + x^2\partial_{tt}^2$, we can realize that an Hopf lemma at characteristic points can be proved under the assumption of the interior homogeneous ball $\{x^4 + t^2 < r^4\}$ (see [23, 25]). On the other hand, if we consider the operators as in (3) built with respect to these two vector fields, this condition is not the right one. As a matter of fact, let us pick*

$$Q = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \quad \text{and} \quad L_Q = \partial_{xx}^2 - x\partial_{xt}^2 + x^2\partial_{tt}^2 - \frac{1}{2}\partial_t.$$

Fix Ω such that it is contained in the halfspace $\{t > 0\}$, but it is flat enough to have the interior homogeneous ball property at $(0,0)$. Consider the function $u(x,t) = \left(t + \frac{1}{8}x^2\right)^\alpha$ with $0 < \alpha - 1 < \frac{1}{26}$. A straightforward calculation shows this is a counterexample to Hopf for L_Q.

The reason of the behavior described in Remark 3.1 is the presence of the first order term $-\frac{1}{2}\partial_t$. In order to understand what happens in presence of such terms, we will always denote in this section by Ω a bounded open set in some $\mathbb{R}^N$ such that $0 \in \partial\Omega$, the positive t-direction determines the inner unit normal and it is a characteristic direction for the operator at 0.

Following Birindelli and Cutrí [4, Remark 2], we can see that the Koranyi-ball condition for $\Omega \subset \mathbb{R}^3$ at 0 is enough to ensure the Hopf property also for an operator like

$$\Delta_{\mathbb{H}} + k_1(x,t)\partial_{x_1} + k_2(x,t)\partial_{x_2} + (x_1^2 + x_2^2)\gamma(x,t)\partial_t,$$

with bounded k_1, k_2, γ. The barrier can be chosen as for $\Delta_{\mathbb{H}}$. This is the exact behavior Monticelli noted in [25, Lemma 4.1] for Grushin-type equations. He proved in particular a Hopf lemma in $\Omega \subset \mathbb{R}^2$ at 0 for operators like

$$\partial_{xx}^2 + x^2\partial_{tt}^2 + k(x,t)\partial_x + x^2\gamma(x,t)\partial_t,$$

with bounded k and γ, under a homogeneous interior ball condition. Having in mind these examples and Remark 3.1, we want to understand the behavior of Ω if we want an Hopf lemma for operators as $\Delta_{\mathbb{H}} \pm \partial_t$, $\partial^2_{xx} + x^2\partial^2_{tt} \pm \partial_t$, $\partial^2_{xx} + x^2\partial^2_{tt} \pm x\partial_t$. In order to do this let us go back to the heat operator, seen as a degenerate elliptic operator in $\mathbb{R}^{N+1}$.

Remark 3.2. *Let us denote $\mathcal{H} = \mathcal{H}^- = \Delta_x - \partial_t$ in $\mathbb{R}^{N+1}$. Suppose Ω strictly contains $\{(x,t) \in \mathbb{R}^{N+1} : t + \frac{1}{2N}\|x\|^2 > 0\}$, at least locally around 0. Then $h(x,t) = t + \frac{1}{2N}\|x\|^2$ is clearly an $\mathcal{H}$-barrier function for Ω at 0, and thus $\mathcal{H}$ satisfies the Hopf lemma in Ω at 0. We cannot do much more better than this. As a matter of fact, suppose that Ω is contained in the region $\{(x,t) \in \mathbb{R}^{N+1} : t + \beta_0\|x\|^2 > 0\}$, for some $0 < \beta_0 < \frac{1}{2N}$. Then, we can choose $\beta_0 < \beta < \frac{1}{2N}$ and $\epsilon = \frac{(\beta-\beta_0)(1-2\beta N)}{4\beta^2} > 0$, and we can consider $u(x,t) = (t + \beta\|x\|^2)^{1+\epsilon}$. This function is $C^2(\Omega) \cap C^1(\Omega \cup \{0\})$, $u(0) = 0$, $u > 0$ in Ω, and $u_t(0) = 0$. Moreover*

$$\mathcal{H}u(x,t) = -(1-2\beta N)(1+\epsilon)(t+\beta\|x\|^2)^{-1+\epsilon}\left(t + \beta_0\|x\|^2\right) \le 0 \quad \text{in } \Omega.$$

Therefore u is a counterexample to the Hopf property in Ω. We stress that $t \sim -\frac{1}{2N}\|x\|^2$ is the behavior of the level set of the fundamental solution for $\mathcal{H}$ up to lower order terms (fundamental solution with pole at some $(0, -|t_0|)$ and passing through 0).
The operator $\mathcal{H}^+ = \Delta_x + \partial_t$ has the same behavior. It satisfies the Hopf lemma in the sets Ω which are "flat enough" to contain strictly the paraboloid $\{(x,t) \in \mathbb{R}^{N+1} : t > \frac{1}{2N}\|x\|^2\}$. And it does not satisfies the Hopf lemma in Ω if Ω is contained in a region delimited by a steeper paraboloid $\{(x,t) \in \mathbb{R}^{N+1} : t > \beta_0\|x\|^2\}$, for some $\beta_0 > \frac{1}{2N}$.

Despite the non-parabolicity aspects, the degenerate-elliptic operators

$$\Delta_{\mathbb{H}} \pm \partial_t, \qquad \partial^2_{xx} + x^2\partial^2_{tt} \pm \partial_t$$

behave the same way as $\mathcal{H}^{\pm}$ regarding to the Hopf-property at 0. As a matter of fact, if there exists an open neighborhood U of 0 such that

$$\left\{(x_1,x_2,t) \in U \subset \mathbb{R}^3 : t \ge \pm\frac{1}{4}(x_1^2+x_2^2)\right\} \smallsetminus \{(0,0,0)\} \subset \Omega \subset \mathbb{R}^3$$

$$\text{or} \qquad \left\{(x,t) \in U \subset \mathbb{R}^2 : t \ge \pm\frac{1}{2}x^2\right\} \smallsetminus \{(0,0)\} \subset \Omega \subset \mathbb{R}^2,$$

then (respectively) $\Delta_{\mathbb{H}} \pm \partial_t$ or $\partial^2_{xx} + x^2\partial^2_{tt} \pm \partial_t$ satisfies the Hopf lemma in Ω at 0. The barriers can be easily constructed as $h(x_1, x_2, t) = t \mp \frac{1}{4}(x_1^2 + x_2^2)$ or $h(x, t) = t \mp \frac{1}{2}x^2$. On the other hand, as in Remark 3.2, the conditions on Ω cannot be improved drastically. If

$$\Omega \subseteq \{(x_1, x_2, t) \in \mathbb{R}^3 \,:\, t > \pm\beta_0(x_1^2 + x_2^2)\} \quad \text{for some positive } \beta_0 \gtrless \frac{1}{4}$$
$$\text{or} \qquad \Omega \subseteq \{(x, t) \in \mathbb{R}^2 \,:\, t > \pm\beta_0 x^2\} \quad \text{for some positive } \beta_0 \gtrless \frac{1}{2},$$

then the Hopf lemma does not hold true in Ω at 0. The functions $u(x_1, x_2, t) = (t \mp \beta(x_1^2 + x_2^2))^\alpha$ or $u(x, t) = (t \mp \beta x^2)^\alpha$ work as counterexamples for suitable choices of $\alpha > 1$ and β ($\beta_0 \gtrless \beta \gtrless \frac{1}{4}$ or $\beta_0 \gtrless \beta \gtrless \frac{1}{2}$).

Remark 3.3. *We can perform an analysis similar to Remark 3.2 for*

$$\partial^2_{xx} + x\partial_t, \qquad \text{in } \mathbb{R}^2.$$

This is well-studied in literature: it is the stationary part of the Kolmogorov operator, and it is an example of the so-called degenerate Ornstein-Uhlenbeck operators (see [13, 17, 6]*). Suppose there exists an open neighborhood U of 0 such that*

$$\left\{(x, t) \in U \subset \mathbb{R}^2 \,:\, t \geq \frac{1}{6}x^3\right\} \smallsetminus \{(0, 0)\} \subset \Omega,$$

then $\partial^2_{xx} + x\partial_t$ satisfies the Hopf lemma in Ω at 0. As before, a barrier can be easily constructed as $h(x, t) = t - \frac{1}{6}x^3$. In order to construct counterexamples analogue to the previous ones, let us define the following function

$$f_{\beta_0^\pm}(x) = \begin{cases} \beta_0^+ x^3 & \text{if } x > 0 \\ \\ \beta_0^- x^3 & \text{if } x < 0. \end{cases}$$

Suppose that

$$\Omega \subseteq \left\{(x, t) \in \mathbb{R}^2 \,:\, t > f_{\beta_0^\pm}(x)\right\}$$

for some $\beta_0^+ > \frac{1}{6}$ and $0 < \beta_0^- < \frac{1}{6}$. Then, we can consider the function $u(x, t) = (t - f_{\beta^\pm})^\alpha$, with $\beta_0^+ > \beta^+ > \frac{1}{6}$, $\beta_0^- < \beta^- < \frac{1}{6}$, and $\alpha > 1$. The function $f_{\beta^\pm}$ is smooth enough to ensure that $u \in C^2(\Omega) \cap C^1(\Omega \cup \{0\})$. Moreover u is positive in Ω, and $u(0) = u_t(0) = 0$.

Suitable choices of β^+, β^-, α *give* $(\partial^2_{xx} + x\partial_t)\, u \le 0$ *in* Ω *and* u *is thus a counterexample to the Hopf lemma in* Ω.

Let us now turn our attention to the degenerate-elliptic operator

$$\partial^2_{xx} + x^2\partial^2_{tt} + x\partial_t.$$

Also for this one, assuming that there exists an open neighborhood U of 0 such that

$$\left\{(x,t) \in U \subset \mathbb{R}^2 \,:\, t \ge \frac{1}{6}x^3\right\} \smallsetminus \{(0,0)\} \subset \Omega,$$

$(\partial^2_{xx} + x^2\partial^2_{tt} + x\partial_t)$ satisfies the Hopf property in Ω at 0: $h(x,t) = t - \frac{1}{6}x^3$ is in fact still a barrier. However we cannot be as precise as in Remark 3.3. We are able to find a counterexample just assuming that $\Omega \subseteq \{(x,t) \in \mathbb{R}^2 \,:\, t > \beta_0 x^2\}$ for some positive β_0. In this case, by taking $0 < \beta < \beta_0$ and considering $\Omega \subset \{x^2 \le \beta^2\}$, the function $u(x,t) = (t - \beta x^2)^\alpha$ works as counterexample to Hopf in Ω for some $\alpha > 1$.

Remark 3.4. *Let us just mention that, for any* $k \in \mathbb{N}$*, the operator*

$$\partial^2_{xx} + x^k\partial_t, \qquad \text{in } \mathbb{R}^2,$$

behaves regarding the Hopf property as the ones described in Remark 3.2 (in the case of k *even) and Remark 3.3 (for* k *odd) with the natural adjustments.*

The behavior observed in the previous specific degenerate-elliptic examples occurs also in different situations. Let us consider in $\mathbb{R}^3$ the two vector fields

$$X_1 = \partial_{x_1} + b^1(x)\partial_t, \qquad X_2 = \partial_{x_2} + b^2(x)\partial_t, \tag{5}$$

where $b = (b^1, b^2) : U_0 \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is smooth, defined in an open neighborhood U_0 of $(0,0)$, and such that $b(0,0) = (0,0)$. Suppose that

$$[X_1, X_2](0) = b^2_{x_1}(0,0) - b^1_{x_2}(0,0) \ne 0. \tag{6}$$

We want to investigate the operator

$$X_1^2 + X_2^2 = \Delta_x + 2b^1(x)\partial^2_{x_1 t} + 2b^2(x)\partial^2_{x_2 t} + \left((b^1(x))^2 + (b^2(x))^2\right)\partial^2_{tt} + \operatorname{div}(b)\partial_t.$$

Since $b(0) = 0$, the positive t-direction is characteristic at 0. To study the Hopf property, let us define the following polynomial of degree 3

$$F_b(x) = \frac{1}{2}\left\langle \mathcal{J}^s b(0)x, x\right\rangle + \frac{1}{6}\left(b^1_{x_1,x_1}(0)x_1^3 + 3b^1_{x_1,x_2}(0)x_1^2x_2 + 3b^2_{x_1,x_2}(0)x_1x_2^2 + b^2_{x_2,x_2}(0)x_2^3\right),$$

where $\mathcal{J}^s b(0)$ is the symmetric part of the Jacobian matrix of b at $(0, 0)$. We can prove the following

Theorem 3.1. *Let X_1, X_2 the two vector fields in $\mathbb{R}^3$ defined in* (5)*, satisfying* (6)*. Suppose $\Omega \subset \mathbb{R}^3$ is a bounded open set with $0 \in \partial\Omega$, and with $(0, 0, 1)$ as inner unit normal at 0. Suppose also there exist an open neighborhood U of $(0, 0, 0)$ and a positive constant γ such that*

$$\left\{(x, t) \in U \subset \mathbb{R}^3 \,:\, t \geq F_b(x) + \gamma \left\|x\right\|^4\right\} \smallsetminus \{(0, 0, 0)\} \subset \Omega.$$

Then the operator $\mathcal{L} = X_1^2 + X_2^2$ satisfies the Hopf lemma in Ω at 0.

The comparison with the degree-3 polynomial F_b is suggested by the examples in the first part of this section (and looking at the first order term $\operatorname{div}(b)\partial_t$). In the case of the Heisenberg vector fields, where $b(x) = \frac{1}{2}(-x_2, x_1)$, we have $F_b \equiv 0$ and Theorem 3.1 gives back the flatness condition of Birindelli and Cutrí.

4. All around the boundary

Let us now go back to the study of some families of operators L_Q as in (3), which started in Section 2 and was interrupted with Remark 3.1.

Throughout the paper we have considered conditions on the behavior of $\partial\Omega$ around a characteristic point. We would like to exploit here that analysis in order to construct bounded open sets Ω in which our operators satisfy the Hopf lemma at every boundary point. It is not difficult to convince ourselves that this is not possible for operators like ∂^2_{xx} in $\mathbb{R}^2$ or $\Delta_x \pm \partial_t$ in $\mathbb{R}^{N+1}$. Nonetheless it is possible for some families of operators L_Q, and it is possible in a uniform way with respect to uniformly positive definite matrices Q.

In the sets

$$B^2_{r_0}(\xi_0, \tau_0) = (\xi_0, \tau_0) \circ \left\{(x, t) \in \mathbb{R}^m \times \mathbb{R}^n \,:\, \left\|x\right\|^4 + \left\|t\right\|^2 < r^4\right\} \subset \mathbb{R}^N$$

every operator $L_Q = \sum_{i,j=1}^m q_{i,j}(x,t)X_iX_j$ (with $X_i = \partial_{x_i} + \frac{1}{2}\sum_{k=1}^n (B^k x)_i \partial_{t_k}$ as in (4)) satisfies the Hopf lemma at every boundary point of $B^2_{r_0}(\xi_0,\tau_0)$. This holds true for any symmetric uniformly positive definite matrix $Q(x,t)$. As a matter of fact, the characteristic points of $\partial B^2_{r_0}(\xi_0,\tau_0)$ are just the ones of the form (ξ_0,τ) (with $\|\tau-\tau_0\| = r_0^2$) and the barrier functions constructed in [30] are actual L_Q-barrier functions in $B^2_{r_0}(\xi_0,\tau_0)$ at those points.

On the other hand, we have seen in Remark 3.1 that, for the vector fields $X_1 = \partial_x$ and $X_2 = x\partial_t$ in $\mathbb{R}^2$, the operators L_Q may not satisfy the Hopf lemma in the homogeneous ball $\{(x,t)\in\mathbb{R}^2 \,:\, x^4+t^2<1\}$. We have to change this set accordingly to what we have showed in Section 3. To this aim, let us fix $\Lambda \geq \lambda > 0$ and define the following bounded open set
$$B_{\frac{\Lambda}{\lambda}} = \left\{(x,t)\in\mathbb{R}^2 \,:\, x^4 - \frac{1}{2}\left(\frac{\Lambda}{\lambda}-1\right)x^2 + t^2 < 1\right\} \subset \mathbb{R}^2.$$
We can prove the following

Proposition 4.1. *For any 2×2 symmetric matrix $Q(x,t)$ such that $\lambda\mathbb{I}_2 \leq Q \leq \Lambda\mathbb{I}_2$, the operator $L_Q = \sum_{i,j=1}^2 q_{i,j}(x,t)X_iX_j$ satisfies the Hopf lemma in $B_{\frac{\Lambda}{\lambda}}$ at any point of its boundary.*

We would like to stress that, in the case $\Lambda=\lambda$, L_Q is forced to be $\lambda(\partial^2_{xx} + x^2\partial^2_{tt})$: the set $B_{\frac{\Lambda}{\lambda}}$ coincides with the homogeneous ball $\{x^4+t^2<1\}$ and we recover the condition in [23, 25].

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