GLOBAL ATTRACTORS FOR SEMILINEAR PARABOLIC PROBLEMS INVOLVING X-ELLIPTIC OPERATORS

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Abstract. We consider semilinear parabolic equations involving an operator that is X-elliptic with respect to a family of vector fields X with suitable properties. The vector fields determine the natural functional setting associated to the problem and the admissible growth of the non-linearity. We prove the global existence of solutions and characterize their longtime behavior. In particular, we show the existence and finite fractal dimension of the global attractor of the generated semigroup and the convergence of solutions to an equilibrium solution when time tends to infinity.

Sunto. Consideriamo equazioni paraboliche semilineari che coinvolgono un operatore che è X-ellittico rispetto ad una famiglia di campi vettoriali X con proprietà opportune. I campi vettoriali determinano il naturale spazio funzionale associato al problema e la crescita ammissibile di non linearità. Dimostriamo l'esistenza globale di soluzioni e caratterizziamo il loro comportamento per tempi lunghi. In particolare, dimostriamo l'esistenza e la dimensione frattale finita dell'attrattore globale del semigruppo generato e la convergenza delle soluzioni ad una soluzione di equilibrio quando il tempo tende all'infinito.

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1. Introduction

Let $\mathcal{L}$ be a second order partial differential operator in divergence form that is X-*elliptic* with respect to a family of vector fields $X = \{X_1, \ldots, X_m\}$ with certain properties. We analyze semilinear degenerate parabolic problems of the form

$$u_t = \mathcal{L}u + f(u),$$
$$u|_{\partial\Omega} = 0, \qquad u|_{t=0} = u_0, \tag{1}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$. The notion of X-elliptic operators provides a unifying framework for various types of degenerate elliptic operators. It was explicitly introduced in 2000 in [10], based on ideas applied in [3, 4, 5], however, operators that fall into this class had already been present in the literature. Since then, X-elliptic operators have been widely studied. For an overview and recent results we refer to [11] and the references therein.

The main examples of operators we consider are:

- *Sub-Laplacians on homogeneous Carnot groups* (see [2]), e.g., the Kohn Laplacian on the Heisenberg group

$$\Delta_{\mathbb{H}} = \partial^2_{xx} + \partial^2_{yy} + 4y\partial_x\partial_z - 4x\partial_y\partial_z, \qquad (x, y, z) \in \mathbb{R}^3,$$

- Δ_λ-*Laplacians* (see [7]), e.g., operators of Grushin type

$$\Delta_\alpha u = \partial^2_{xx} + |x|^{2\alpha}\partial^2_{yy}, \qquad\qquad (x, y) \in \mathbb{R}^2,\ \alpha \geq 0.$$

In these notes we show the well-posedness of (1) and characterize the longtime behavior of solutions using tools from the theory of infinite dimensional dynamical systems. Different from ordinary differential equations, for evolutionary partial differential equations (PDEs) the appropriate functional setting has to be determined and the global well-posedness to be established in each particular situation. The family of vector fields X determines the functional setting naturally associated to problem (1) and the appropriate growth conditions on the nonlinearity that imply the local well-posedness of solutions. If the nonlinearity additionally satisfies certain sign conditions, the global existence of solutions follows. Moreover, we can show the existence and finite fractal dimension of the global attractor for the generated semigroup and the convergence of solutions to an equilibrium solution as time tends to infinity.

The outline of the paper is as follows. In Section 2 we explain basic concepts from the theory of infinite dimensional dynamical systems. X-elliptic operators are introduced in Section 3, and our hypotheses and main result are stated. Examples of operators that satisfy our assumptions are given in Section 4, and in Section 5 we formulate a sketch of the proof.

2. Global attractors for the semilinear heat equation

The time evolution of many dissipative PDEs can be described in terms of semigroups in infinite dimensional function spaces, and analyzing its longtime dynamics can often be reduced to the study of the dynamics on the *global attractor*. The global attractor is a compact, strictly invariant subset of the phase space that attracts all bounded subsets as time tends to infinity. If it exists, it is unique and, in most cases, of finite fractal dimension.

Global attractors for evolutionary PDEs have been studied for several decades (e.g., see [1, 6]). A classical example is the semilinear heat equation,

$$\begin{aligned} &u_t = \mathit{\Delta} u + f(u), \\ &u|_{\partial\Omega} = 0, \qquad u|_{t=0} = u_0 \in H_0^1(\Omega), \end{aligned} \tag{2}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$. The operator $-\mathit{\Delta}$ with homogeneous Dirichlet boundary conditions generates an analytic semigroup in $L^2(\Omega)$, and the semilinear problem (2) is locally well-posed in $H_0^1(\Omega)$, if the nonlinearity f satisfies certain growth assumptions. These are determined by the critical exponent $q^* = \frac{2N}{N-2}$ in the Sobolev embedding

$$H_0^1(\Omega) \hookrightarrow L^q(\Omega), \qquad 1 \leq q \leq q^*.$$

The unique local solution exists globally in $H_0^1(\Omega)$, or blows up in finite time.

Sufficient to prevent finite time blow-up is the following sign condition on the nonlinearity,

$$\limsup_{|u|\to\infty} \frac{f(u)}{u} < \mu_1, \qquad u \in \mathbb{R},$$

where $\mu_1 > 0$ denotes the first eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary conditions. Under these assumptions solutions exist globally, and the time evolution of the initial value problem can be described in terms of a semigroup in $H_0^1(\Omega)$, given by the family of operators $S(t) : H_0^1(\Omega) \to H_0^1(\Omega)$, $t \geq 0$, where

$$S(t)u_0 := u(\cdot, t; u_0), \qquad t \geq 0,$$

and $u(\cdot, \cdot; u_0)$ denotes the solution of (2) corresponding to initial data u_0. Moreover, using tools from the theory of infinite dimensional dynamical systems allows to characterize the longtime dynamics of solutions:

The semigroup $S(t), t \geq 0$, in $H_0^1(\Omega)$ possesses a global attractor, which is connected and of finite fractal dimension. Furthermore, for any initial data $u_0 \in H_0^1(\Omega)$ the corresponding solution tends to an equilibrium solution $u^* \in \mathcal{E} = \{u \in H_0^1(\Omega) \mid \Delta u + f(u) = 0\}$ as time tends to infinity.

In these notes we extend this classical result for degenerate parabolic problems of the form (1):

- We define the adequate functional setting and show that $-\mathcal{L}$ generates an analytic semigroup in $L^2(\Omega)$.
- We formulate Sobolev type embedding properties for different classes of X-elliptic operators, that determine the appropriate growth restrictions on the nonlinearity and imply the local well-posedness.
- If the nonlinearity, in addition, satisfies certain sign conditions, solutions exist globally, and we can prove the existence and some properties of the global attractor.

3. Hypotheses and main result

Let $\mathcal{L}$ be the operator

$$\mathcal{L}u := \sum_{i,j=1}^{N} \partial_{x_i}(a_{ij}\partial_{x_j}u),$$

where the functions a_{ij} are measurable in $\mathbb{R}^N$ and $a_{ij} = a_{ji}$. We assume that there exists a family $X := \{X_1, \dots, X_m\}$ of vector fields in $\mathbb{R}^N$, $X_j = (\alpha_{j1}, \dots, \alpha_{jN})$, $j = 1, \dots, m$, such that the functions α_{jk} are locally Lipschitz continuous in $\mathbb{R}^N$. As usual, we identify the vector-valued function X_j with the linear first order partial differential operator

$$X_j = \sum_{k=1}^{N} \alpha_{jk} \partial_{x_k}, \qquad j = 1, \dots, m.$$

We suppose, moreover, that there exists a group of dilations $\delta_r : \mathbb{R}^N \longrightarrow \mathbb{R}^N$,

$$\delta_r(x) = \delta_r(x_1, \dots, x_N) = (r^{\sigma_1} x_1, \dots, r^{\sigma_N} x_N),$$

$1 = \sigma_1 \le \sigma_2 \le \cdots \le \sigma_N$, such that the vector fields X_j are δ_r-homogeneous of degree one, i.e.,

$$X_j(u(\delta_r(x))) = r(X_j u)(\delta_r(x)) \qquad \forall u \in C^\infty(\mathbb{R}^N).$$

The integer

$$Q := \sigma_1 + \cdots + \sigma_N$$

is the *homogeneous dimension* of $\mathbb{R}^N$ and plays an important role in the geometry and functional setting naturally associated to $\mathcal{L}$.

Definition 3.1. *The operator $\mathcal{L}$ is called **uniformly X-elliptic** in $\mathbb{R}^N$, if there exists a constant $C > 0$ such that*

$$\frac{1}{C} \sum_{j=1}^{m} \langle X_j(x), \xi \rangle^2 \le \sum_{i,j=1}^{N} a_{ij}(x) \xi_i \xi_j \le C \sum_{j=1}^{m} \langle X_j(x), \xi \rangle^2 \qquad \forall x, \xi \in \mathbb{R}^N,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^N$.

We define the Hilbert space H as the closure of $C_0^1(\Omega)$ with respect to the norm

$$\|u\|_H := \Big(\sum_{j=1}^{m} \|X_j u\|^2_{L^2(\Omega)} \Big)^{\frac{1}{2}}, \qquad u \in C_0^1(\Omega),$$

and assume that the following *Sobolev-type embedding* property holds:
There exists a continuous embedding

$$\text{(S)} \qquad H \hookrightarrow L^p(\Omega) \qquad \forall\, p \in [1, \frac{2Q}{Q-2}],$$

and the embedding is compact for every $p \in [1, \frac{2Q}{Q-2})$.

Under these assumptions the local well-posedness can be shown using classical techniques from the theory of analytic semigroups, if the nonlinearity is locally Lipschitz continuous and satisfies the growth restrictions

$$\text{(F1)} \qquad |f(u) - f(v)| \le c|u - v|(1 + |u|^\rho + |v|^\rho), \qquad u, v \in \mathbb{R}, \qquad 0 \le \rho < \frac{4}{Q-2},$$

for some constant $c \ge 0$.

Furthermore, the following sign conditions ensure the global existence of solutions and allow to characterize their longtime behavior:

$$\text{(F2)} \qquad \limsup_{|u|\to\infty} \frac{f(u)}{u} < \mu_1, \qquad u \in \mathbb{R},$$

where $\mu_1 > 0$ denotes the first eigenvalue of the operator $-\mathcal{L}$ with homogeneous Dirichlet boundary conditions.

Our main result is the following:

Theorem 3.1. *We assume the properties* (S), (F1) *and* (F2) *are satisfied. Then, for every initial data* $u_0 \in H$ *there exists a unique global solution* u, *and*

$$u \in C([0,\infty); H) \cap C^1((0,\infty); H).$$

The generated semigroup $S(t), t \ge 0$, *in* H *possesses a global attractor* $\mathcal{A}$, *which is connected and of finite fractal dimension. Moreover,* $\mathcal{A} = \mathcal{W}^u(\mathcal{E})$, *and for every initial data* $u_0 \in H$ *we have*

$$\lim_{t\to\infty} \operatorname{dist}_H(S(t)u_0, \mathcal{E}) = 0,$$

where $\mathcal{E} = \{u \in H \mid \mathcal{L}u + f(u) = 0\}$ *is the set of equilibrium points.*

Here, $\operatorname{dist}_H(\cdot,\cdot)$ denotes the *Hausdorff semi-distance* in H, i.e.,

$$\operatorname{dist}_H(B, A) := \sup_{b\in B} \inf_{a\in A} \|a - b\|_H \qquad \text{for subsets } A, B \subset H,$$

and $\mathcal{W}^u(\mathcal{E})$ the *unstable set* of $\mathcal{E}$, i.e.,

$$\mathcal{W}^u(\mathcal{E}) = \big\{v \in H \mid S(t)v \text{ is defined } \forall\, t \in \mathbb{R}, \operatorname{dist}_H(S(-t)v, \mathcal{E}) \to 0 \text{ as } t \to \infty\big\}.$$

We indicate the main ideas of the proof of Theorem 3.1 in Section 5. Comparing the results with the setting for the classical semilinear heat equation we observe that, since

$H_0^1(\Omega) \hookrightarrow H$, the phase spaces are less regular. Moreover, the homogeneous dimension Q replaces the Euclidean dimension N in the Sobolev-type embeddings and the growth restrictions (F1). For degenerate elliptic operators we have $Q > N$, i.e., the admissible growth of the nonlinearity decreases.

4. Classes of operators satisfying our hypotheses

We mention here two main, disjoint classes of X-elliptic operators satisfying our hypotheses, Sub-Laplacians on homogeneous Carnot groups and Δ_λ-Laplacians. For a more general setting and further examples we refer to [9]. While Sub-Laplacians satisfy Hoërmander's rank condition and are hypoelliptic, Δ_λ-Laplacians are hypoelliptic only in particular cases.

4.1. **Sub-Laplacians on homogeneous Carnot groups.** Let $(\mathbb{R}^N, \circ)$ be a Lie group in $\mathbb{R}^N$. We assume that $\mathbb{R}^N$ can be split as follows

$$\mathbb{R}^N = \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_n},$$

and that there exists a group of dilations $\delta_r : \mathbb{R}^N \longrightarrow \mathbb{R}^N$,

$$\delta_r(x) = \delta_r(x^{(N_1)}, \ldots, x^{(N_n)}) := (rx^{(N_1)}, \ldots, r^n x^{(N_n)}), \quad r > 0,$$

$x^{(N_i)} \in \mathbb{R}^{N_i}$, $i = 1, \ldots, n$, which are automorphisms of $(\mathbb{R}^N, \circ)$.

Let $X = \{X_1, \ldots, X_m\}$ be a family of smooth vector fields satisfying the Hörmander rank condition

$$\text{rank } \left(\text{Lie}\{X_1, \ldots, X_{N_1}\}\right)(x) = N \qquad \forall x \in \mathbb{R}^N,$$

where $\text{Lie}\{X_1, \ldots, X_{N_1}\}$ denotes the Lie-algebra generated by the family of vector fields. Moreover, we assume the vector fields X_j are left invariant on $(\mathbb{R}^N, \circ)$ and

$$X_j(0) = \frac{\partial}{\partial x_j^{(N_1)}}, \qquad j = 1, \ldots, N_1.$$

Then, $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ is a Carnot group, and the homogeneous dimension is

$$Q = N_1 + 2N_2 + \cdots + nN_n.$$

Our results apply to the Sub-Laplacian

$$\mathcal{L} = \Delta_{\mathbb{G}} = \sum_{j=1}^{N_1} X_j^2,$$

where the vector fields $X_1, \dots, X_{N_1}$ are the *generators* of $\mathbb{G}$. We remark that every Sub-Laplacian can be written in divergence form (see p. 64 in [2]). Combining previous results, the Sobolev type embeddings (S) for Sub-Laplacians can be deduced from geometric properties of the *control (or Carnot-Caratheodory) distance* related to the family of vector fields X (see [9], p. 413).

Example 4.1. *The Kohn-Laplacian on the Heisenberg group.*

The Heisenberg group $\mathbb{H}^N$, whose elements we denote by $\zeta = (x, y, z)$, is the Lie group $(\mathbb{R}^{2N+1}, \circ)$ with composition law

$$\zeta \circ \zeta' = (x + x', y + y', z + z' + 2(\langle x', y\rangle - \langle x, y'\rangle)),$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^N$. The Kohn Laplacian is the operator $\Delta_{\mathbb{H}^N} = \sum_{j=1}^N (X_j^2 + Y_j^2)$, where

$$X_j = \partial_{x_j} + 2y_j \partial_z, \quad Y_j = \partial_{y_j} - 2x_j \partial_z.$$

A natural group of dilations is given by

$$\delta_r(\zeta) = \delta_r(x, y, z) = (rx, ry, r^2 z), \qquad r > 0,$$

and the homogeneous dimension is $Q = 2N + 2$.

For further examples we refer to [2].

4.2. **The Δ_λ-Laplacian.** As in [7], we consider operators of the form

$$\Delta_\lambda := \sum_{i=1}^N \partial_{x_i}(\lambda_i^2 \partial_{x_i}),$$

where $\partial_{x_i} = \frac{\partial}{\partial_{x_i}}, i = 1, \dots, N$. The functions $\lambda_i : \mathbb{R}^N \to \mathbb{R}$ are continuous, strictly positive and of class C^1 outside the coordinate hyperplanes[1] and satisfy the following properties:

(*i*) $\lambda_1(x) \equiv 1,\ \lambda_i(x) = \lambda_i(x_1, \dots, x_{i-1}),\ i = 2, \dots, N.$

[1] $\lambda_i > 0$ in $\mathbb{R}^N \backslash \Pi$, where $\Pi = \left\{(x_1, \dots, x_N) \in \mathbb{R}^N \mid \prod_{i=1}^N x_i = 0\right\}$

(*ii*) For every $x \in \mathbb{R}^N$ the function $\lambda_i(x) = \lambda_i(x^*)$, $i = 1, \dots, N$, where

$$x^* = (|x_1|, \dots, |x_N|) \quad \text{if } x = (x_1, \dots, x_N).$$

(*iii*) There exists a constant $\hat{c} \geq 0$ such that

$$0 \leq x_k \partial_{x_k} \lambda_i(x) \leq \hat{c} \lambda_i(x) \qquad \forall k \in \{1, \dots, i-1\},\ i = 2, \dots, N,$$

for every $x \in \mathbb{R}^N_+ := \left\{(x_1, \dots, x_N) \in \mathbb{R}^N \mid x_i \geq 0\ \forall i = 1, \dots, N\right\}$.

(*iv*) There exists a group of dilations $(\delta_r)_{r>0}$,

$$\delta_r : \mathbb{R}^N \to \mathbb{R}^N, \quad \delta_r(x) = \delta_r(x_1, \dots, x_N) = (r^{\sigma_1} x_1, \dots, r^{\sigma_N} x_N),$$

where $1 \leq \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_N$, such that λ_i is δ_r-*homogeneous of degree* $\sigma_i - 1$.

The last property implies that the vector fields $\lambda_i \partial_{x_i}$ are δ_r-homogeneous of degree one and the homogeneous dimension is $Q = \sigma_1 + \cdots + \sigma_N$.

Only if all functions λ_i are C^∞-smooth, Hörmander's rank condition is satisfied and the operators are hypoelliptic. We assumed in our definition of X-elliptic operators that the coefficient functions are Lipschitz continuous. The assumptions (*i*)-(*iv*) do not necessarily imply the Lipschitz continuity of the functions λ_i, but the hypotheses we need to prove our main results can still be verified. The Sobolev embeddings (S) for Δ_λ-Laplacians were proved in [7] and [3].

Example 4.2. *Operators of Grushin type.*

Let α be a nonnegative real constant. We consider the operator

$$\Delta_x + |x|^{2\alpha} \Delta_y, \qquad (x, y) \in \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}.$$

The group of dilations is given by

$$\delta_r(x, y) = \left(rx, r^{\alpha+1} y\right), \qquad r > 0,$$

and the homogeneous dimension is $Q = N_1 + (1 + \alpha) N_2$.

Example 4.3. *We split $\mathbb{R}^N$ into $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3}$ and write*

$$x = (x^{(1)}, x^{(2)}, x^{(3)}), \qquad x^{(i)} \in \mathbb{R}^{N_i},\ i = 1, 2, 3.$$

Let α, β and γ be nonnegative real constants. For the operator

$$\Delta_\lambda = \Delta_{x^{(1)}} + |x^{(1)}|^{2\alpha} \Delta_{x^{(2)}} + |x^{(1)}|^{2\beta} |x^{(2)}|^{2\gamma} \Delta_{x^{(3)}},$$

where $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})$ with

$$\begin{aligned} &\lambda_j^{(1)}(x) \equiv 1, && j = 1, \ldots, N_1, \\ &\lambda_j^{(2)}(x) = |x^{(1)}|^{\alpha}, && j = 1, \ldots, N_2, \\ &\lambda_j^{(3)}(x) = |x^{(1)}|^{\beta} |x^{(2)}|^{\gamma}, && j = 1, \ldots, N_3, \end{aligned}$$

we find the group of dilations

$$\delta_r \left(x^{(1)}, x^{(2)}, x^{(3)} \right) = \left(r x^{(1)}, r^{\alpha+1} x^{(2)}, r^{\beta + (\alpha+1)\gamma + 1} x^{(3)} \right).$$

Similarly, for operators of the form

$$\begin{aligned} \Delta_\lambda = \Delta_{x^{(1)}} &+ |x^{(1)}|^{2\alpha_{1,1}} \Delta_{x^{(2)}} + |x^{(1)}|^{2\alpha_{2,1}} |x^{(2)}|^{2\alpha_{2,2}} \Delta_{x^{(3)}} + \cdots \\ &+ \left(\prod_{i=1}^{k-1} |x^{(i)}|^{2\alpha_{k-1,i}} \right) \Delta_{x^{(k)}}, \end{aligned}$$

where $\alpha_{i,j} \geq 0$, $i = 1, \ldots, k-1, j = 1, \ldots, i$, are real constants, the group of dilations is given by

$$\delta_r \left(x^{(1)}, \ldots, x^{(k)} \right) = \left(r^{\sigma_1} x^{(1)}, \ldots, r^{\sigma_k} x^{(k)} \right)$$

with $\sigma_1 = 1$ and $\sigma_j = 1 + \sum_{i=1}^{j-1} \alpha_{j-1,i} \sigma_i$, for $i = 2, \ldots, k$.

5. Sketch of the proof

We provide a sketch of the proof of Theorem 3.1 and refer for all details to Section 4 in [8] and Section 3 in [9].

5.1. **Preliminaries.** We consider $\mathcal{L}$ in $L^2(\Omega)$ with domain

$$\mathcal{D}(\mathcal{L}) = \left\{ u \in H \mid \exists\, c \geq 0 \text{ such that } |a(u,v)| \leq c \|v\|_{L^2(\Omega)} \ \forall v \in H \right\},$$

$$\langle -\mathcal{L} u, v \rangle_{L^2(\Omega)} = a(u,v) \qquad \forall u \in \mathcal{D}(\mathcal{L}),\ v \in H,$$

where the bilinear form a in H is defined as

$$a(u,v) := \int_\Omega \sum_{i,j=1}^N a_{ij}(x)\partial_{x_i}u(x)\partial_{x_j}v(x)\,dx, \qquad u,v \in H.$$

We observe that the operator $\mathcal{L}$ is densely defined and self-adjoint in $L^2(\Omega)$, and that the Sobolev embeddings (S) and the X-ellipticity of $\mathcal{L}$ imply the following *Poincaré-type inequality*,

$$\|u\|^2_{L^2(\Omega)} \le c\,a(u,u) \qquad \forall u \in H,$$

for some constant $c \ge 0$. Consequently, $A := -\mathcal{L}$ is positive sectorial and generates an analytic semigroup in $L^2(\Omega)$, which we denote by $e^{\mathcal{L}t}, t \ge 0$.

Moreover, since A is positive, self-adjoint, and has compact inverse by the Sobolev type embeddings (S), there exists an orthonormal basis of $L^2(\Omega)$ of eigenfunctions $\psi_j \in H, j \in \mathbb{N}$, of A with eigenvalues

$$0 < \mu_1 \le \mu_2 \le \dots, \qquad \mu_j \to \infty \text{ as } j \to \infty.$$

We denote the fractional power spaces associated to A by $X^\alpha = \big(\mathcal{D}(A^\alpha), \langle\cdot,\cdot\rangle_{X^\alpha}\big)$, $\alpha \in \mathbb{R}$, where $\langle u,v\rangle_{X^\alpha} = \langle A^\alpha u, A^\alpha v\rangle_{X^0}$, $u,v \in \mathcal{D}(A^\alpha)$, and

$$\mathcal{D}(A^\alpha) = \Big\{\psi = \sum_{j\in\mathbb{N}} c_j\psi_j,\ c_j \in \mathbb{R} \;\Big|\; \sum_{j\in\mathbb{N}} \mu_j^{2\alpha}c_j^2 < \infty\Big\},$$
$$A^\alpha\psi = A^\alpha \sum_{j\in\mathbb{N}} c_j\psi_j = \sum_{j\in\mathbb{N}} \mu_j^\alpha c_j\psi_j.$$

The embedding $X^\alpha \hookrightarrow X^\beta$ is compact for $\alpha > \beta$, and in this notation, we have

$$X^1 = \mathcal{D}(A), \quad X^{\frac{1}{2}} = H, \quad X^0 = L^2(\Omega), \quad X^{-\frac{1}{2}} = H',$$

where H' denotes the dual space of H.

5.2. **Local well-posedness.** Fractional power spaces are useful to show the well-posedness and deduce regularity properties of solutions of parabolic problems with sectorial operators. We recall the following classical result.

Proposition 5.1. *Let B be a positive sectorial operator in a Hilbert space X^0 and for some $\alpha \in [0,1)$ let $F : X^\alpha \to X^0$ be Lipschitz continuous on bounded subsets of X^α. Then,* *for every* $u_0 \in X^\alpha$ *there exists a unique local solution* $u \in C([0,T);X^\alpha) \cap C^1((0,T);X^0)$ *of the initial value problem*

$$u_t = Bu + F(u), \qquad u|_{t=0} = u_0,$$

which is defined on the maximal interval of existence $[0,T)$, $T > 0$.

Either $T = \infty$, *or, if* $T < \infty$, *then* $\limsup_{t\to T} \|u(t)\|_{X^\alpha} = \infty$. *Moreover, the solution satisfies* $u \in C((0,T);X^1) \cap C^1((0,T);X^\gamma)$ *for all* $\gamma \in [0,1)$.

To conclude the local well-posedness of (1) in $H = X^{\frac{1}{2}}$ we differentiate two cases.

- If f satisfies the growth restriction (F1) with $0 \le \rho < \frac{2}{Q-2}$, it follows from Hölder's inequality and the Sobolev embeddings (S) that f is Lipschitz continuous on bounded subsets from $X^{\frac{1}{2}}$ to X^0. For initial data $u_0 \in X^{\frac{1}{2}}$ the local well-posedness and regularity of solutions is an immediate consequence of Proposition 5.1.
- Otherwise, if $\frac{2}{Q-2} \le \rho < \frac{4}{Q-2}$, using complex interpolation and the Sobolev embeddings (S) we can show that there exists $\alpha \in (0, \frac{1}{2})$ such that the mapping $f : X^{\frac{1}{2}} \to X^{-\alpha}$ is Lipschitz continuous on bounded subsets (see Lemma 1, [8]). The operator $A = -\mathcal{L}$ can be extended to a positive sectorial operator in $X^{-\alpha}$ with domain $X^{1-\alpha}$. The local well-posedness and regularity of solutions then follows from Proposition 5.1 by considering the initial value problem in $X^{-\alpha}$,

$$u_t = -Au + f(u), \qquad u|_{t=0} = u_0 \in X^{\frac{1}{2}}.$$

Consequently, if (F1) holds, problem (1) is locally well-posed in H, and the solution $u \in C([0,T);H) \cap C^1((0,T);H)$, where $[0,T)$, $T > 0$, is the maximal interval of existence. Moreover, either u exists globally, or $\limsup_{t\to T} \|u(t)\|_H = \infty$.

5.3. **Global existence and longtime behavior of solutions.** The global existence of solutions can be shown by considering the Lyapunov functional $\Phi : H \to \mathbb{R}$,

$$\Phi(u) := \frac{1}{2} a(u,u) - \int_\Omega F(u(x))dx,$$

where $F(u) := \int_0^u f(s)ds$ denotes the primitive of f. If u is a weak solution of (1), then $u \in C([0,T);H) \cap C^1((0,T);H)$, and $\Phi(u(\cdot)) \in C\big([0,T);\mathbb{R}\big) \cap C^1\big((0,T);\mathbb{R}\big)$. Multiplying the PDE by u_t it follows that

$$\frac{d}{dt}\Phi(u(t)) = -\|u_t(t)\|^2_{L^2(\Omega)} < \infty, \qquad t \in (0,T),$$

and the growth restriction (F1), the sign condition (F2) and Young's inequality yield an estimate of the form

$$C_1\left(1+\|u(t)\|_H^2\right) \le \Phi(u(t)) \le \Phi(u_0) \le C_2\left(1+\|u_0\|_H^2+\|u_0\|^{\rho+2}_{L^{\rho+2}(\Omega)}\right),$$

for some constants $C_1, C_2 \ge 0$. The Sobolev embeddings (S) now imply that solutions are uniformly bounded in H for $t > 0$ and therefore exist globally.

We denote by $S(t), t \ge 0$, the semigroup in H generated by problem (1),

$$S(t)u_0 := u(t;u_0), \qquad t \ge 0,$$

where u denotes the global weak solution corresponding to initial data $u_0 \in H$.

The existence and structure of the global attractor can be deduced from the following well-known result.

Proposition 5.2. *Let $S(t), t \ge 0$, be an asymptotically compact semigroup in a Banach space that possesses a Lyapunov functional. If the set of equilibrium points $\mathcal{E}$ is bounded and for every bounded subset $B \subset V$ the orbit $\gamma^+(B) = \bigcup_{t\ge 0} S(t)B$ is bounded, then, the global attractor exists, is connected and $\mathcal{A} = \mathcal{W}^u(\mathcal{E})$.*

We observed that Φ is a Lyapunov functional for the semigroup $S(t), t \ge 0$, and orbits of bounded sets are bounded. The sign condition (F2) further implies that the set of equilibrium points $\mathcal{E}$ is bounded in H, and it can be shown that the semigroup $S(t), t \ge 0$, satisfies the *smoothing property* in bounded subsets $D \subset H$:

There exists $t^* > 0$ and a constant $\kappa^* > 0$ such that

$$\|S(t^*)u_0 - S(t^*)v_0\|_H \le \kappa^*\|u_0 - v_0\|_{L^2(\Omega)} \qquad \forall\, u_0, v_0 \in D. \tag{3}$$

The asymptotic compactness of the semigroup is an immediate consequence of this property, and by Proposition 5.2 the global attractor $\mathcal{A}$ exists, is connected and $\mathcal{A} = \mathcal{W}^u(\mathcal{E})$.

The finite fractal dimension of $\mathcal{A}$ is a consequence of the smoothing property (3) and Proposition 18 in [9]. Finally, the convergence of trajectories to equilibrium solutions can be deduced from the smoothing property and the invariance principle of LaSalle (see Proposition 8 in [9]).

6. Concluding remarks

We have shown that the theory of analytic semigroups and global attractors extends to a large class of degenerate parabolic problems involving X-elliptic operators. The family of vector fields X determines the natural functional setting and the admissible growth of the nonlinearity. Comparing our results with the semilinear heat equation, the phase spaces are weaker, and the homogeneous dimension Q plays the same role as the Euclidean dimension for the classical Laplacian in the growth restrictions on the non-linearity. The critical exponent decreases with increasing Q.

To study semilinear equations with critical nonlinearities is subject of future work. Moreover, we aim to establish blow-up results for solutions of semilinear problems with X-elliptic operators, as well as of linear equations involving singular potentials.

These notes are based on the references [8] and [9]. We restricted the presentation to degenerate parabolic problems and particular examples of X-elliptic operators, for a more general setting and further results we refer to [9]:

- The vector fields X need not be homogeneous with respect to a group of dilations, and the homogeneous dimension can be defined in a broader context. Sobolev type embeddings such as (S) are still valid for X-elliptic operators that do not belong to the classes discussed in Section 4.
- Besides parabolic problems we analyzed semilinear degenerate damped hyperbolic equations of the form

$$\begin{aligned} u_{tt} + \beta u_t &= \mathcal{L}u + f(u), \\ u\big|_{\partial\Omega} &= 0, \\ u\big|_{t=0} &= u_0 \in H, \quad u_t\big|_{t=0} = u_1 \in L^2(\Omega), \end{aligned}$$

  where the constant β is positive. The global well-posedness and the existence and finite fractal dimension of the global attractor can be shown, if f satisfies property (F2) and the growth restrictions (F1) with exponent $0 \le \rho < \frac{2}{Q-2}$.

References

[1] A. V. Babin, M. I. Vishik. Attractors for Evolution Equations, North-Holland, Amsterdam (1992).

[2] A. Bonfiglioli, E. Lanconelli, F. Uguzzoni. Stratified Lie Groups and Potential Theory for their Sub-Laplacians, Springer-Verlag, Berlin, Heidelberg (2007).

[3] B. Franchi, E. Lanconelli. *An embedding theorem for Sobolev spaces related to non-smooth vector fields and Harnack inequality.* Comm. Partial Differential Equations, **9** (1984) 1237–1264.

[4] B. Franchi, E. Lanconelli. *Hölder regularity theorem for a class of linear nonuniformly elliptic operators with measurable coefficients.* Ann. Sc. Norm. Super. Pisa Cl. Sci., **10** (1983) 523–541.

[5] B. Franchi, E. Lanconelli. *Une métrique associée à une classe d'operateurs elliptiques dégénérés.* Conference on linear partial and pseudodifferential operators (Torino 1982), Rend. Sem. Mat. Univ. Politec. Torino 1983, Special Issue (1984) 105–114.

[6] J. K. Hale. Asymptotic Behavior of Dissipative Systems, American Mathematical Society, Providence, Rhode Island (1988).

[7] A. E. Kogoj, E. Lanconelli. *On semilinear Δ_λ-Laplace equation.* Nonlinear Anal., **75** (2012) 4637–4649.

[8] A. E. Kogoj, S. Sonner. *Attractors for a class of semi-linear degenerate parabolic equations.* J. Evol. Equ., **13** (2013) 675–691.

[9] A. E. Kogoj, S. Sonner. *Attractors met X-elliptic operators.* J. Math. Anal. Appl., **420** (2014) 407–434.

[10] E. Lanconelli, A. E. Kogoj. *X-elliptic operators and X-control distances.* Contributions in honor of the memory of Ennio De Giorgi, Ric. Mat., **49** (2000) suppl. 223–243.

[11] F. Uguzzoni. *Estimates of the Green function for X-elliptic operators.* Math. Ann., **361** (2015) 169–190.

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