INVERSE PROBLEMS FOR PARABOLIC DIFFERENTIAL EQUATIONS FROM CONTROL THEORY

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Abstract. We are concerned with some inverse problems related to parabolic first-order and second order linear differential equations in Hilbert spaces. All the results apply well to inverse problems for partial differential equations from mathematical physics and optimal control theory. Various concrete examples are described.

Sunto. In questo articolo vengono considerati alcuni problemi inversi relativi ad equazioni differenziali paraboliche in spazi di Hilbert, sia del primo che del secondo ordine. Tutti i risultati astratti si applicano a problemi inversi per equazioni alle derivate parziali di tipo parabolico di interesse nella fisica matematica e in teoria del controllo ottimo. In effetti, sono descritti vari esempi concreti ai quali la nostra teoria si applica.

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1. Introduction

Given a Hilbert space Y, many abstract differential problems of great interest in control theory take the form

$$\begin{aligned} \frac{dy}{dt} &= Ay(t) + f(t)B\,u\,, \qquad 0 \le t \le \tau\,,\\ y(0) \,&=\, y_0 \in D(A)\,, \end{aligned}$$

where A is a generator of an analytic semigroup in Y and the operator B acts from another Hilbert space U into $[D(A^*)]'$, the dual space of $D(A^*)$ with respect to the $Y-$inner

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product, i.e.,

$$\|y\|_{[D(A^*)]'} = \|A^{-1}y\|_Y,$$

where, without loss of generality, we assume that $A^{-1} \in \mathcal{L}(Y)$. It happens, in many important situations, for some $\gamma \in (0,1)$, that $(-A)^{-\gamma}B \in \mathcal{L}(U,Y)$, see [1]-[2].

One of the main goals of this paper is to show that under suitable regularity assumptions on $g \in C([0,\tau];\mathbb{C})$, $y_0 \in D(A)$, there exists a unique strict solution $(y,f) \in C([0,\tau];Y) \times C([0,\tau];\mathbb{C})$ such that

$$A^{-1}\frac{dy}{dt} = y(t) + f(t)A^{-1}Bu, \quad 0 \le t \le \tau, \tag{1}$$

$$y(0) = y_0 \tag{2}$$

$$\Phi[A^{-1}y(t)] = g(t), \quad 0 \le t \le \tau, \tag{3}$$

where Φ is an element of Y^*. In Section 2 we recall some preliminaries. In Section 3 we handle various identification problems related to the equation $\dfrac{dy}{dt} = Ay(t) + f(t)B\,u$. In Section 4 we face the case of the second-order equations. In Section 5 we furnish some applications to partial differential equations of interest in control theory. For more details on the applications and examples we refer to [9], particularly, for second-order equations in time. see also [3], [4] and [5].

2. Some Lemmas

Let L, M be two closed linear operators in the complex Banach space X, $0 \in \rho(L)$, $D(L) \subseteq D(M)$. If $\mathcal{a}$ denotes the linear relation

$$\mathcal{a} = M^{-1}L - \gamma,$$

then X_a^θ, $0 < \theta < 1$ denotes the Banach space

$$X_a^\theta = \left\{ u \in X; \ \sup_{\xi > 0} \xi^\theta \left\| \left\{ 1 - \xi\big((\xi+\gamma)M - L\big)^{-1}M \right\} u \right\|_X < \infty \right\}. \tag{4}$$

If $\mathcal{a} = LM^{-1} - \gamma$, then

$$X_a^\theta = \left\{ u \in X; \ \sup_{\xi > 0} \xi^\theta \left\| (L-\gamma M)\big((\xi+\gamma)M - L\big)^{-1} u \right\|_X < \infty \right\}. \tag{5}$$

Concerning (5), it is also assumed that

$$\|M(\lambda M - L)^{-1}\|_{\mathcal{L}(X)} \leq c(1+|\lambda|)^{-\beta}, \tag{6}$$

for all λ with $\mathrm{Re}(\lambda-\gamma) \geq -c(1+|\mathrm{Im}\lambda|)^{\alpha}$, $\quad c>0,\ \ 0<\beta\leq\alpha\leq 1$.
Concerning (4), it is assumed that the M resolvent satisfies

$$\|(\lambda M - L)^{-1}M\|_{\mathcal{L}(X)} \leq \frac{c}{(1+|\lambda-\gamma|)^{\beta}}, \tag{7}$$

for all

$$\lambda \in \Sigma = \{\lambda \in \mathbb{C} : \ \ \mathrm{Re}(\lambda-\gamma) \geq -c(1+|\mathrm{Im}\,\lambda|)^{\alpha}\}\,, \quad \text{for } \gamma \in \mathbb{R},$$

for example, see [8]. Notice that in (4) necessarily M is bounded from X into itself.

Consider the initial value problem

$$M\frac{du}{dt} = Lu(t) + f(t)\,, \qquad 0 \leq t \leq \tau\,, \tag{8}$$

$$u(0) \ = \ u_0. \tag{9}$$

Let us recall the following results from [8] in the form of lemmas, see [8], Theorem 3.21 and Theorem 3.22, p. 63.

Lemma 2.1. *Suppose that (7) holds with $\alpha=\beta=1$. If $f \in C^{1+\theta}([0,\tau];X)$, $0<\theta<1$ and $u_0 \in D(L)$ satisfy*

$$M\left\{(\gamma M - L)^{-1}f'(0) - g_0\right\} = Lu_0 + f(0)$$

with $g_0 \in X_a^{\theta}$, see (4), then problem (8)-(9) admits a unique solution u enjoying the regularity $\dfrac{du}{dt} \in C^{\theta}([0,\tau];X)$ and $\dfrac{du}{dt} - (\gamma M - L)^{-1}f' \in B([0,\tau];X_a^{\theta})$ where $B([0,\tau];X_a^{\theta})$ denotes the space of all bounded $X-$valued functions on $[0,\tau]$, endowed with the sup-norm.

Lemma 2.2. *Suppose that (7) holds with $\alpha=\beta=1$. Let $0<\theta<1$. Then, for any $f \in C^1([0,\tau];X) \cap B([0,\tau];Y_{M,L}^{\theta})$, with $f' \in B([0,\tau];Y_{M,L}^{\theta})$ and for any $u_0 \in D(L)$ satisfying $Lu_0 + f(0) = Mg_0$ with $g_0 \in X_a^{\theta}$ (cfr. [4]), problem (8)-(9) possesses a unique strict solution on $[0,\tau]$ which satisfies*

$$\frac{du}{dt} \in B([0,\tau];X_a^{\theta}), \quad Lu \in C^{\theta}([0,\tau];X).$$

Here

$$Y^{\theta}_{M,\,L} = \left\{ f \in X;\ \ \sup_{\xi>0} \xi^{\theta} \left\| \big((\xi+\gamma)M - L\big)^{-1} f \right\|_X < \infty \right\}, 0 < \theta < 1,$$

with

$$\|f\|_{Y^{\theta}_{M,\,L}} = \sup_{\xi>0} \xi^{\theta} \left\| \big((\xi+\gamma)M - L\big)^{-1} f \right\|_X .$$

Next we recall a general identification result described recently in the paper [7], cfr. Theorem 5.1

Lemma 2.3. *Let L, M be two closed linear operators, $0 \in \rho(L)$, $D(L) \subseteq D(M)$, such that (5) holds. Assume that $2\alpha + \beta > 2$, $3 - 2\alpha - \beta < \theta < 1$. Let $y_1 \in D(L)$ such that*

$$My_1 = Ly_0 + z\big[g'(0) - \Phi[Ly_0] - \Phi[h(0)]\big]/\Phi[z] + h(0),$$

with $y_0 \in D(L)$, $z \in (X, D(A))_{\theta,\infty}$, $A = LM^{-1}$, $\Phi \in X^*$, $\Phi[My_0] = g(0)$, $h \in C^1([0,\tau];X)$, $h \in B\big([0,\tau];(X,D(A))_{\theta,\infty}\big)$. Then the identification problem

$$My'(t) = Ly(t) + f(t)z + h(t), \quad 0 \le t \le \tau, \tag{10}$$

$$y(0) = y_0 \tag{11}$$

$$\Phi[My(t)] = g(t), \quad 0 \le t \le \tau, \tag{12}$$

admits a unique strict solution $(y,f) \in C^1([0,\tau];D(L)) \times C^1([0,\tau];\mathbb{C})$ such that $(My')' - f'(.)z - h'(.) \in C^{(2\alpha+\beta-3+\theta)/\alpha}([0,\tau];X) \cap B\left([0,\tau]; X_A^{2\alpha+\beta-3+\theta}\right)$.

Here $(X, D(A))_{\theta,\infty}$ denotes the real interpolation space between X and $D(A)$(endowed with the graph norm) and

$$X^{\theta}_A = \left\{ u \in X;\ \ \sup_{t>0} t^{\theta} \left\| A^0 \big(t - A\big)^{-1} u \right\|_X < \infty \right\},$$

where

$$A^0\big(t - A\big)^{-1} = -I + t(t-A)^{-1}.$$

A weaker formulation of problem (10)-(12) reads

$$\frac{d}{dt}\big(My(t)\big) = Ly(t) + f(t)z + h(t), \quad 0 \le t \le \tau, \tag{13}$$

$$(My)(0) = My_0 \tag{14}$$

$$\Phi[My(t)] = g(t), \quad 0 \le t \le \tau, \tag{15}$$

and is described, in more generality, in [6], Theorem 6.3. We write it in the form of lemma as follows.

Lemma 2.4. *Suppose* $2\alpha+\beta+\theta>3$, $y_0 \in D(L)$, $Ly_0 \in (X, D(A))_{\theta,\infty}$, $h \in C([0,\tau];X) \cap B\big([0,\tau];(X,D(A))_{\theta,\infty}\big)$, $g \in C^1([0,\tau];\mathbb{C})$, $\Phi \in X^*$, $\Phi[My_0] = g(0)$, $\Phi[z] \neq 0$. *Then problem (13)-(15) admits a unique solution* (y,f) *such that*

$$My \in C^1([0,\tau];X), \quad f \in C([0,\tau];\mathbb{C}),$$

$$Ly \in C^{(2\alpha+\beta-3+\theta)/\alpha}\big([0,\tau];X\big) \cap B\left([0,\tau];X_A^{(2\alpha+\beta-3+\theta)/\alpha}\right)$$

3. First-Order Case

In order to face problem (1)-(3), we shall assume, following [9], that $-A$ admits bounded imaginary powers. This guarantees, in particular that

$$(Y,D(A))_{\theta,2} = [Y,D(A)]_\theta = D\big((-A)^\theta\big) \subseteq (Y,D(A))_{\theta,\infty},$$

where $[Y,D(A)]_\theta$ denotes the complex interpolation space of index θ between Y and $D(A)$, see [13].

Assume $(-A)^{\theta_0-1}B \in \mathcal{L}(U,Y)$, $0<\theta_0<1$. Then clearly, $A^{-1}B \in \mathcal{L}(U,Y)$ too.

We see that if $g \in C^1([0,\tau];\mathbb{C})$, it follows, from (3) after applying Φ to equation (1), that

$$g'(t) = \Phi[y(t)] + f(t)\Phi[A^{-1}Bu], \quad 0 \le t \le \tau,$$

and thus, if $\Phi[A^{-1}Bu] \neq 0$, then necessarily

$$f(t) = \frac{1}{\Phi[A^{-1}Bu]}\big[g'(t) - \Phi[y(t)]\big], \quad 0 \le t \le \tau. \tag{16}$$

Taking (16) into account, equation (1) becomes

$$A^{-1}y'(t) = y(t) - Cy(t) + \frac{g'(t)}{\Phi[A^{-1}Bu]}A^{-1}Bu, \quad 0 \le t \le \tau, \tag{17}$$

where C is the operator

$$C \in \mathcal{L}(Y, Y_A^{\theta_0}), \quad Cy = \frac{\Phi[y]}{\Phi[A^{-1}Bu]}A^{-1}Bu$$

provided that $A^{-1}Bu \in Y_A^{\theta_0}$ where Y_A^{θ} represents the space as X_A^{θ} with Y instead of X. Notice that $(-A)^{-1}B = (-A)^{-\theta_0}(-A)^{\theta_0-1}Bu \in D\big((-A)^{\theta_0}\big) = [Y, D(A)]_{\theta_0} \subseteq (Y, D(-A))_{\theta_0,\infty}$.

This step transforms the inverse problem (1)-(3) to the direct problem to find a solution $y(.)$ to (17), (2). Our aim is to apply Lemma 2.1, with $M = A^{-1}, \quad L = I - C$.

We verify that the resolvent estimate (7) holds in this case. Indeed,

$$
\begin{aligned}
(\lambda A^{-1} - 1 + C)^{-1}A^{-1} &= \left[(\lambda A^{-1} - 1)\big(1 + (\lambda A^{-1} - 1)^{-1}C\big)\right]^{-1} A^{-1} \\
&= \big(I - (\lambda A^{-1} - 1)^{-1}C\big)^{-1}(\lambda A^{-1} - 1)^{-1}A^{-1} \\
&= \big(I - A(\lambda - A)^{-1}C\big)^{-1}(\lambda - A)^{-1}.
\end{aligned}
$$

On the other hand, it is seen from [8], p. 49, that

$$
\left\|A(\lambda - A)^{-1}u\right\|_Y \le c|\lambda|^{-\theta_0}\|u\|_{Y_A^{\theta_0}}, \quad \text{for all } \lambda \in \Sigma_1.
$$

Since $C \in \mathcal{L}(Y, Y_A^{\theta_0}) = \mathcal{L}\big(Y,\, (Y, D(A))_{\theta_0,\infty}\big)$, we conclude $1 - A(\lambda - A)^{-1}C$ has a bounded inverse for all $|\lambda|$ large enough. Therefore the required estimate (7) is obtained. We write

$$
\begin{aligned}
&1 - \xi\big((\xi+\gamma)A^{-1} - I + C\big)^{-1}A^{-1} \\
&\qquad = 1 - \frac{\xi}{\xi+\gamma}\big((\xi+\gamma)A^{-1} - I + C\big)^{-1}\big((\xi+\gamma)A^{-1} - I + C + I - C\big) \\
&\qquad = 1 - \frac{\xi}{\xi+\gamma} - \frac{\xi}{\xi+\gamma}\big((\xi+\gamma)A^{-1} - I + C\big)^{-1}(I - C) \\
&\qquad = \frac{\gamma}{\xi+\gamma} - \frac{\xi}{\xi+\gamma}\big((\xi+\gamma)A^{-1} - I + C\big)^{-1}(I - C) \\
&\qquad = \frac{\gamma}{\xi+\gamma} - \frac{\xi}{\xi+\gamma}\Big(I - A\big(\xi+\gamma - A\big)^{-1}C\Big)^{-1} A(\xi+\gamma-A)^{-1}(I - C).
\end{aligned}
$$

This implies that $Y_a^{\theta_0}$, $a = A(1 - C)$ in (4), coincides

$$
\big\{y \in Y,\ (1 - C)y \in (X, D(A))_{\theta_0,\infty}\big\}.
$$

But

$$
Cy = \frac{\Phi[y]}{\Phi[A^{-1}Bu]}A^{-1}Bu = \frac{-\Phi[y]}{\Phi[A^{-1}Bu]}(-A)^{-\theta_0}(-A)^{\theta_0-1}Bu \in D\big((-A)^{-\theta_0}\big) \subseteq (Y, D(A))_{\theta_0,\infty}.
$$

As a consequence, applying Lemma 2.1, we can write the result as follows

Theorem 3.1. *Suppose* $(-A)^{\theta-1}B \in \mathcal{L}(U,Y)$, $0<\theta<1$, $g \in C^{2+\theta}([0,\tau];\mathbb{C})$, $\Phi[A^{-1}Bu] \neq 0$, $A^{-1}\left(\left(\gamma A^{-1}-1+C\right)^{-1}\dfrac{g'(0)}{\Phi[A^{-1}Bu]}A^{-1}Bu - g_0\right) = y_0 - Cy_0 + \dfrac{g'(0)}{\Phi[A^{-1}Bu]}A^{-1}Bu$ *for some* $g_0 \in (Y, D(A))_{\theta,\infty}$ $\left(\text{and hence } (1-C)y_0 + \dfrac{g'(0)}{\Phi[A^{-1}Bu]}A^{-1}Bu \in D(A)\right)$. *Then problem (1)-(3) admits a unique solution* $(y,f) \in C^{1+\theta}([0,\tau];Y) \times C^{1+\theta}([0,\tau];\mathbb{C})$.

In order to apply Lemma 2.2 we need to characterize the space $Y^{\theta}_{A^{-1},1-C}$. Since

$$(\lambda A^{-1}-1+C)^{-1} = \left(I-(\lambda A^{-1}-1)^{-1}C\right)^{-1}A(\lambda-A)^{-1},$$

it follows that

$$Y^{\theta}_{A^{-1},1-C} = Y^{\theta}_{A} = (X, D(A))_{\theta_0,\infty}.$$

This allows to write the result as follows

Theorem 3.2. *Suppose* $(-A)^{\theta-1}B \in \mathcal{L}(U,Y)$, $0<\theta<1$. *Suppose* $g \in C^2([0,\tau];\mathbb{C})$, $(I-C)y_0 + \dfrac{g'(0)}{\Phi[A^{-1}Bu]}A^{-1}Bu = A^{-1}g_0$, *with* $g_0 \in (Y, D(A))_{\theta,\infty}$. *Then problem (1)-(3) admits a unique solution* $(y,f) \in C^{\theta}([0,\tau];Y) \times C([0,\tau];\mathbb{C})$, f *is differentiable and* $f' \in B([0,\tau];\mathbb{C})$.

Next step consists of applying directly Lemma 2.3 to problem (1)-(3).

Theorem 3.3. *Suppose* $(-A)^{\theta-1}B \in \mathcal{L}(U,Y)$, *so that* $A^{-1}Bu = (-A)^{-\theta}(-A)^{\theta-1}Bu \in D\big((-A)^{\theta}\big) = [X, D(A)]_{\theta} \subseteq (X, D(A))_{\theta,\infty}$, $\Phi[A^{-1}y_0] = g(0)$, $y_0 + A^{-1}Bu[g'(0) - \Phi[y_0]]/\Phi[A^{-1}Bu] \in D(A)$, $g \in C^2([0,\tau];\mathbb{C})$. *(Notice that if* $A^{-1}y_1 = y_0 + A^{-1}Bu[g'(0) - \Phi[y_0]]/\Phi[A^{-1}Bu]$ *then* $\Phi[A^{-1}y_1] = g'(0)$ *). Suppose* $\Phi[A^{-1}Bu] \neq 0$ *too. Then there is a unique solution* $(y,f) \in C^1([0,\tau];Y) \times C^1([0,\tau];\mathbb{C})$ *such that* $\big(A^{-1}y'\big)' - f'(.)A^{-1}Bz \in C^{\theta}([0,\tau];Y) \cap B\big([0,\tau];(X, D(A))_{\theta,\infty}\big)$.

Now we devote ourselves to generalized solutions of

$$\frac{d}{dt}\big(A^{-1}y(t)\big) = y(t) + f(t)A^{-1}Bz, \quad 0 \le t \le \tau, \tag{18}$$

$$\big(A^{-1}y\big)(0) = A^{-1}y_0, \tag{19}$$

$$\Phi[A^{-1}y(t)] = g(t), \quad 0 \le t \le \tau. \tag{20}$$

Then less regularity is required. As a consequence of Lemma 2.4, we conclude the following theorem

Theorem 3.4. *Let A be the generator of an analytic semigroup in Y, $(-A)^{\theta-1}B \in \mathcal{L}(U,Y)$, $0<\theta<1$, $\Phi \in X^*$, $\Phi[A^{-1}y_0] = g(0)$, $g \in C^1([0,\tau];\mathbb{C})$, $\Phi[A^{-1}Bu] \neq 0$. Then problem (18)-(20) admits a unique solution (y,f) such that $A^{-1}y \in C^1([0,\tau];Y)$, $f \in C([0,\tau];\mathbb{C})$, $y \in C^\theta([0,\tau];Y) \cap B\big([0,\tau];(X,D(A))_{\theta,\infty}\big)$*

We also observe that under the assumption $(-A)^{\theta-1}B \in \mathcal{L}(U,Y)$, and for a given continuous $f(t)$, the candidate solution of

$$\text{(P)} \qquad \begin{cases} y'(t) = Ay(t) + f(t)Bu, \quad t \in [0,\tau], \\ \quad y(0) = y_0 \end{cases}$$

is

$$\begin{aligned} y(t) &= e^{tA}y_0 + \int_0^t e^{(t-s)A} f(s)Bu\,ds, \\ &= e^{tA}y_0 + \int_0^t (-A)^{1-\theta} e^{(t-s)A}(-A)^{\theta-1} f(s)Bu\,ds. \end{aligned}$$

It is easy to see, using the interpolation result, see [13], that

$$\big[D\big((-A)^\alpha\big), D\big((-A)^\beta\big)\big]_\theta = D\big((-A)^{(1-\theta)\alpha+\theta\beta}\big), \quad 0<\theta<1, \quad \alpha,\ \beta>0$$

$$\|(-A)^{1-\theta}e^{tA}\|_{\mathcal{L}(X)} \le \frac{c}{t^{1-\theta}}\,,$$

so that $y(t)$ is well defined. We could also observe that

$$\int_0^t (-A)^{1-\theta} e^{(t-s)A}(-A)^{\theta-1} f(s)Bu\,ds \in D(A)$$

provided that $(-A)^{\theta-1}Bu \in (Y,D(A))_{1-\theta,\infty}$, but y, in general, is not differentiable. However,

$$(-A)^{\theta-1}y(t) = e^{tA}(-A)^{\theta-1}y_0 + \int_0^t e^{(t-s)A}(-A)^{\theta-1}f(s)Bu\,ds,$$

so that

$$\frac{d}{dt}(-A)^{\theta-1}y(t) = A(-A)^{\theta-1}y(t) + f(t)(-A)^{\theta-1}Bu.$$

Therefore, we could solve the identification problem

$$\begin{aligned} &\frac{d}{dt}\big((-A)^{\theta-1}y(t)\big) = A(-A)^{\theta-1}y(t) + (-A)^{\theta-1}f(t)Bu, \\ &(-A)^{\theta-1}y(0) = (-A)^{\theta-1}y_0, \\ &\Phi\left[(-A)^{\theta-1}y(t)\right] = g(t). \end{aligned}$$

Notice that the above differential equation is obtained by applying $(-A)^{\theta-1}$ to problem (P). On the other hand, the change of variable $(-A)^{\theta-1}y(t) = x(t)$ reduces this inverse problem to the usual problem

$$x'(t) = Ax(t) + (-A)^{\theta-1}Buf(t), \tag{21}$$

$$x(0) = (-A)^{\theta-1}y_0, \tag{22}$$

$$\Phi\left[x(t)\right] = g(t). \tag{23}$$

By using the results in [1], if $(-A)^{\theta-1}Bu \in (Y, D(A))_{\delta,\infty}$, $\delta \in (0,1)$ $\big($in particular, $(-A)^{\delta+\theta-1}B \in \mathcal{L}(U;Y)$ $\big)$, $y_0 \in D\big((-A)^{\delta+\theta-1}\big)$, $\Phi[(-A)^{\theta-1}Bu] \neq 0$, $g \in C^1([0,\tau];\mathbb{C})$, $\Phi[(-A)^{\theta-1}y_0] = g(0)$, then problem (21)-(23) admits a unique solution (x, f) such that x', $Ax \in C^\delta([0,\tau];Y)$, $f \in C^\delta([0,\tau];\mathbb{C})$, $x' \in B\big([0,\tau];(Y,D(A))_{\delta,\infty}\big)$, and hence the solution y to the given problem has the regularity $\dfrac{d}{dt}\big((-A)^{\theta-1}y\big)$, $(-A)^\theta y \in C^\delta([0,\tau];Y)$, $\dfrac{d}{dt}\big((-A)^{\theta-1}y\big) \in B\big([0,\tau];(Y,D(A))_{\delta,\infty}\big)$.

4. Second-Order Problem

We establish correspondingly an approach for the second-order problem of parabolic type

$$y'' = B_1 y' + Ay + f(t)Bz, \quad 0 \le t \le \tau \tag{24}$$

$$y(0) = y_0, \quad y'(0) = y_1 \tag{25}$$

in a Hilbert space H, A, B_1 and B being suitable closed operators. Under certain hypotheses, problem (24)-(25) is transformed to a first-order problem

$$\frac{d}{dt}\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 0 & I \\ A & B_1 \end{bmatrix}\begin{bmatrix} y \\ x \end{bmatrix} + f(t)\begin{bmatrix} 0 \\ Bz \end{bmatrix},$$

$$\begin{bmatrix} y \\ x \end{bmatrix}(0) = \begin{bmatrix} y_0 \\ x_0 \end{bmatrix}$$

of parabolic type in a suitable space $W \times Y$, W a Hilbert space, but B operates from U into $D(A^*)'$. Notice that

$$\begin{bmatrix} 0 & I \\ A & B_1 \end{bmatrix}^{-1} = \begin{bmatrix} -A^{-1}B_1 & A^{-1} \\ I & 0 \end{bmatrix},$$

so that,

$$\begin{bmatrix} -A^{-1}B_1 & A^{-1} \\ I & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} + f(t) \begin{bmatrix} A^{-1}Bz \\ 0 \end{bmatrix}.$$

Let $\Psi \in H^*$ and introduce $\Phi \in H^* \times H^*$,

$$\Phi \begin{bmatrix} y \\ x \end{bmatrix} = \Psi[y] + \Psi\left[A^{-1}B_1 x\right],$$

where it is assumed that $A^{-1}B_1$ extends to a bounded operator on Y. Then

$$\Phi \begin{bmatrix} (-A)^{-1}B_1 y + A^{-1}x \\ y \end{bmatrix} = \Psi\left[(-A)^{-1}B_1 y + A^{-1}x\right] + \Psi\left[A^{-1}B_1 y\right].$$

If the information is $\Psi\left[A^{-1}y\right] = g(t)$, we conclude that $\Psi\left[A^{-1}x\right] = g'(t)$. Moreover,

$$\Phi \begin{bmatrix} (-A)^{-1}Bz \\ 0 \end{bmatrix} = \Psi[A^{-1}Bz].$$

Moreover, Theorems 1-4 apply immediately.

Finally, the treatment of the inverse problem like

$$\begin{aligned}
&\frac{d}{dt}\big(A^{-1}y(t)\big) = y(t) + f(t)A^{-1}Bz, \quad 0 \le t \le \tau, \\
&\big(A^{-1}y\big)(0) = A^{-1}y_0, \\
&\Phi[A^{-1}y(t)] = g(t), \quad 0 \le t \le \tau
\end{aligned}$$

is trivial because the change of variable $A^{-1}y = \xi$ transforms it into the desired problem

$$\begin{aligned}
&\frac{d\xi}{dt} = A\xi(t) + f(t)A^{-1}Bz, \quad 0 \le t \le \tau, \\
&\xi(0) = A^{-1}y_0, \\
&\Phi[\xi(t)] = g(t), \quad 0 \le t \le \tau.
\end{aligned}$$

5. Examples and Applications

Here we give some concrete examples from control theory explaining the abstract results. Our standard reference is the monograph [9].

Application 1. *Heat Equation with Dirichlet Boundary Control*

Consider the initial boundary value problem in an open boundary. Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, with sufficiently smooth boundary $\partial\Omega = \Gamma$

$$\begin{aligned} &\frac{\partial y}{\partial t} = \Delta y + c^2 y, \quad \text{in } (0,\tau) \times \Omega \equiv Q \\ &y(0,\cdot) = y_0 \quad \text{in } \Omega \\ &y|_\Sigma = f(t)u(\cdot) \quad \text{in } (0,\tau) \times \Gamma \equiv \Sigma. \end{aligned}$$

We want to determinate (y, f) from the additional information

$$\Phi[y(t,.)] = \int_\Omega \eta(x) y(t,x)\, dx = g(t), \quad 0 \leq t \leq \tau,$$

where $\eta \in L^2(\Omega)$ is known together with $g \in C([0,\tau];\mathbb{C})$. Introducing operator A by

$$Ah = \Delta h + c^2 h, \quad D(A) = H^2(\Omega) \cap H_0^1(\Omega).$$

Then it is seen in [9], p. 181, that our problem is reduced to the abstract form

$$\frac{dy}{dt} = Ay(t) + f(t)Bu, \quad 0 \leq t \leq \tau, \tag{26}$$

$$y(0) = y_0, \tag{27}$$

where

$$Bu = -ADu, \quad B : L^2(\Gamma) \to [D(A)]', \tag{28}$$

D being the Dirichlet map defined by

$$h = Dg \text{ if and only if } (\Delta + c^2)h = 0 \text{ in } \Omega, \quad h|_\Gamma = g.$$

A is the isomorphic extension $L^2(\Omega) \to [D(A)]'$ of the operator A introduced previously. An important role is played by the operator A_D defined by

$$A_D h := -\Delta h, \quad D(A_D) = H^2(\Omega) \cap H_0^1(\Omega).$$

Now it is known that A generates an analytic semigroup in $L^2(\Omega) = H$. Moreover,

$$(-A+\omega)^{-\gamma}B \in \mathcal{L}(L^2(\Gamma), L^2(\Omega))$$

provided that $\gamma = 3/4 + \epsilon, \quad \forall \epsilon > 0$. Therefore, all our results apply to (26), (27) under the additional information

$$\Phi[(-A)^{-1}y(t)] = g(t),$$

and thus the basic assumption relies to $\Phi[(-A)^{-1}Bu] \neq 0$.

Application 2. *Heat Equation with Neumann Boundary Control.*

Here we use [9], Section 3.3, pp. 194-204. The system is analogous to the previous one, but now the control function u acts on Neumann boundary conditions:

$$\frac{\partial y}{\partial t} = (\Delta + c^2)y, \quad \text{in } (0,\tau)\times\Omega \equiv Q, \tag{29}$$

$$y(0,\cdot) = y_0 \quad \text{in } \Omega, \tag{30}$$

$$\frac{\partial y}{\partial \nu}\Big|_{\Sigma} = f(t)u \quad \text{in } (0,\tau)\times\Gamma \equiv \Sigma. \tag{31}$$

The additional information

$$\Phi\left[(-A+\omega)^{-1}\right] = g(t), \quad 0 \le t \le \tau, \tag{32}$$

where $g \in C([0,\tau];\mathbb{C})$ and Φ as above. Other examples could be discussed for second-order equations in time, taking into account [9], Section 4. We refer to the paper [2]. The results could be applied to concrete problems described in [10], [11], [12].

Application 3. *The Strongly Damped Wave Equation with Point Control and Dirichlet BC.*

Consider, for $\rho > 0$, the wave equation with a strong degree damping

$$w_{tt} - \Delta w - \rho\Delta w_t = f(t)\delta(x-x^0)u, \quad \text{in } (0,\tau]\times\Omega = Q, \tag{33}$$

$$w(0,\cdot) = w_0, \quad w_t(0,\cdot) = w_1 \quad \text{in } \Omega, \tag{34}$$

$$w|_{\Sigma} \equiv 0 \quad \text{in } (0,\tau]\times\Gamma, \tag{35}$$

$$\int_{\Omega} h(x)\Delta^{-1}w(t,x)\,dx = g(t), \quad 0 \le t \le \tau, \tag{36}$$

where $x^0 \in \Omega, \quad h \in L^2(\Omega)$.

One can easily translate (33)-(35) into

$$w_{tt} + \mathcal{A}w + \rho\mathcal{A}w_t = f(t)\delta(x - x^0)u,$$

with

$$\mathcal{A} = -\Delta, \quad D(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega).$$

By choosing $H = D(\mathcal{A}^{1/2}) \times L^2(\Omega), \quad U = \mathbb{R}$, our problem can be reduced to

$$w' = Aw + f(t)Bu,$$

where

$$A = \begin{bmatrix} 0 & I \\ -\mathcal{A} & -\rho\mathcal{A} \end{bmatrix}, \quad Bu = \begin{bmatrix} 0 \\ \delta(x - x^0)u \end{bmatrix}.$$

Then A generates an analytic semigroup in H, but only for $n = 1$ we have $(-A)^{-1/2}Bu \in H$.

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