A LIOUVILLE THEOREM FOR NONLOCAL EQUATIONS IN THE HEISENBERG GROUP

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Abstract. We establish a Liouville-type theorem for a subcritical nonlinear problem, involving a fractional power of the sub-Laplacian in the Heisenberg group. To prove our result we will use the local realization of fractional CR covariant operators, which can be constructed as the Dirichlet-to-Neumann operator of a degenerate elliptic equation in the spirit of Caffarelli and Silvestre [8], as established in [14]. The main tools in our proof are the CR inversion and the moving plane method, applied to the solution of the lifted problem in the half-space $\mathbb{H}^n \times \mathbb{R}^+$.

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1. Introduction

In these notes we establish a Liouville-type result for the following nonlinear and nonlocal problem in the Heisenberg group:

$$\mathcal{P}_{\frac{1}{2}} u = u^p \quad \text{in } \mathbb{H}^n. \tag{1.1}$$

Here $\mathcal{P}_{\frac{1}{2}}$ denotes a CR covariant operator of order $1/2$ in $\mathbb{H}^n$, whose principal symbols agree with the pure fractional power $1/2$ of the Heisenberg Laplacian $-\Delta_{\mathbb{H}}$ (see (2.1) for the precise definition of $-\Delta_{\mathbb{H}}$). In [14] the authors study CR covariant operators of fractional orders on orientable and strictly pseudoconvex CR manifolds. In this context, the Heisenberg group $\mathbb{H}^n$ plays the same role as $\mathbb{R}^n$ in conformal geometry, see also [24].

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Given a Kähler-Einstein manifold $\mathcal{X}$, CR covariant operators of fractional order γ are pseudodifferential operators whose principal symbol agrees with the pure fractional powers of the CR sub-Laplacian on the boundary $\mathcal{M} = \partial\mathcal{X}$. They can be defined using scattering theory, as done in [12, 20, 19, 18].

One of the main results in [14] establishes that it is possible to characterize fractional CR covariant operators on some CR manifold $\mathcal{M} = \partial\mathcal{X}$, as the Dirichlet-to-Neumann map of a degenerate elliptic equation in the interior of $\mathcal{X}$ as in [8].

To construct fractional CR covariant operators in the specific case of the Heisenberg group, $\mathbb{H}^n$ is identified with the boundary of the Siegel domain in $\mathbb{R}^{2n+2}$ (see Section 2 for the precise definition) and it is crucial to use its underlying complex hyperbolic structure (see [14]).

Since it will be of utmost importance in the sequel, we recall here the extension result proven in [14].

Theorem 1.1 (see Theorem 1.1 in [14]). *Let $\gamma \in (0,1)$, $a = 1-2\gamma$. For each $u \in C^\infty(\mathbb{H}^n)$, there exists a unique solution U for the extension problem*

$$\begin{cases} \dfrac{\partial^2 U}{\partial \lambda^2} + \dfrac{a}{\lambda}\dfrac{\partial U}{\partial \lambda} + \lambda^2 \dfrac{\partial^2 U}{\partial t^2} + \dfrac{1}{2}\Delta_{\mathbb{H}} U = 0 & \text{in } \widehat{\mathbb{H}}^n_+ := \mathbb{H}^n \times \mathbb{R}^+, \\ U = u & \text{on } \mathbb{H}^n \times \{\lambda = 0\}. \end{cases} \tag{1.2}$$

Moreover,

$$\mathcal{P}_\gamma u = -c_\gamma \lim_{\lambda \to 0} \lambda^a \frac{\partial U}{\partial \lambda},$$

where c_γ is a constant depending only on γ which precise value is given by

$$c_\gamma = -\frac{\Gamma(\gamma)}{\gamma\Gamma(-\gamma)} \cdot 2^{2\gamma-1}.$$

Observe that, differently from the extension result established in [13], here we have the additional term $\lambda^2 \frac{\partial^2 U}{\partial t^2}$ which appears when one considers CR fractional sub-Laplacian. When $a = 1/2$ the equation in (1.2) satisfied by U becomes:

$$\frac{\partial^2 U}{\partial \lambda^2} + \lambda^2 \frac{\partial^2 U}{\partial t^2} + \frac{1}{2}\Delta_{\mathbb{H}} U = 0, \tag{1.3}$$

and we have

$$\mathcal{P}_{\frac{1}{2}} u = -c_{\frac{1}{2}} \lim_{\lambda \to 0} \frac{\partial U}{\partial \lambda}.$$

Replacing λ by $\sqrt{2}\lambda$ in (1.3), we will consider the operator

$$\mathcal{L} = \Delta_{\mathbb{H}} + \frac{\partial^2}{\partial\lambda^2} + 4\lambda^2\frac{\partial^2}{\partial t^2}. \tag{1.4}$$

Our Liouville-type theorem is the analogue, for the fractional operator $\mathcal{P}_{\frac{1}{2}}$, of a result contained in [4], for the sublaplacian $\Delta_{\mathbb{H}}$. In [4], the authors establish a nonexistence result for a class of positive solution of the equation

$$-\Delta_{\mathbb{H}}u = u^p, \tag{1.5}$$

for p subcritical (i.e. $0 < p < \frac{Q+2}{Q-2}$, where $Q = 2n+2$ denotes the homogeneous dimension of $\mathbb{H}^n$). The technique they used is based on the moving plane method (which goes back to Alexandrov [1] and Serrin [25]), adapted to the Heisenberg group setting. This method requires two basic tools: the maximum principle and invariance under reflection with respect to a hyperplane. Since the operator $-\Delta_{\mathbb{H}}$ is not invariant under the usual reflection, in [4] a new reflection, called H-reflection, was introduced. Since it will be important in the sequel, we recall here the definition of H-reflection.

Definition 1.2. *For any $\xi = (x, y, t) \in \mathbb{H}^n$, we consider the plane $T_\mu := \{\xi \in \mathbb{H}^n \ : \ t = \mu\}$. We define*

$$\xi_\mu := (y, x, 2\mu - t),$$

to be the H-reflection of ξ with respect to the plane T_μ.

Due to the use of this reflection, the proof of the nonexistence result in [4] requires the solution u of (1.5) to be cylindrical, that is, $u(x, y, t) = u(r_0, t)$ must depend only on r_0 and t where $r_0 = (|x|^2 + |y|^2)^{\frac{1}{2}}$.

We can now state our main result, which is the analogue for the operator $\mathcal{P}_{\frac{1}{2}}$ of the Liouville result contained in [4].

Theorem 1.3. *Let $0 < p < \frac{Q+1}{Q-1}$, where $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}^n$. Then there exists no cylindrical solution $u \in C^2(\mathbb{H}^n)$ of*

$$\begin{cases} \mathcal{P}_{\frac{1}{2}}u = u^p & \text{in } \mathbb{H}^n, \\ u > 0 & \text{in } \mathbb{H}^n. \end{cases} \tag{1.6}$$

Using the local formulation (1.2) established in [14], the above theorem will follow as a corollary of the following Liouville-type result for a nonlinear Neumann problem in the half-space $\mathbb{H}^n \times \mathbb{R}^+$.

Theorem 1.4. *Let $0 < p < \frac{Q+1}{Q-1}$ and let $U \in C^2(\mathbb{H}^n \times \mathbb{R}^+) \cap C^1(\overline{\mathbb{H}^n \times \mathbb{R}^+})$ be a nonnegative solution of*

$$\begin{cases} \dfrac{\partial^2 U}{\partial \lambda^2} + 4\lambda^2 \dfrac{\partial^2 U}{\partial t^2} + \Delta_{\mathbb{H}} U = 0 & \text{in } \mathbb{H}^n \times \mathbb{R}^+, \\ -\dfrac{\partial U}{\partial \lambda} = U^p & \text{on } \mathbb{H}^n \times \{\lambda = 0\}. \end{cases} \tag{1.7}$$

Suppose that $U(x,y,t,\lambda) = U(r_0,t,\lambda)$, that is U depends only on r_0, t, λ, where $r_0 = (|x|^2+|y|^2)^{\frac{1}{2}}$. Then $U \equiv 0$.

In the Euclidean case, classical nonexistence results for subcritical nonlinear problems in the all space $\mathbb{R}^n$ are contained in [17] and [9]. Analogue results for nonlinear Neumann problems in the half-space $\mathbb{R}^n_+$ where established in [22, 23], using the methods of moving planes and moving spheres.

In the Heisenberg group setting there are several papers concerning nonexistence results for problem (1.5). In [15] some nonexistence results for positive solutions of (1.5) when p is subcritical were established, under some integrability conditions on u and ∇u. In [21, 27] similar nonexistence results for positive solutions of (1.5) in the half-space are established for the critical exponent $p = \frac{Q+2}{Q-2}$. In [5], a Liouville-type result for solution of (1.5) is proved without requiring any decay condition on u, but only for $0 < p < \frac{Q}{Q-2}$. In [4] the authors extend this last result to any $0 < p < \frac{Q+2}{Q-2}$ but only in the class of cylindrical solution. A last more recent result in this context was proven in [28], who established that there are no positive solution of (1.5) for $0 < p < \frac{Q(Q+2)}{(Q-1)^2}$. This result uses a different technique, based on the vector field method, and improves the results contained in [15] and [5], since it does not require any decay on the solution u and it improves the exponent p. Nevertheless it seems that it does not allow to reach the optimal exponent $\frac{Q+2}{Q-2}$ (observe that $\frac{Q}{Q-2} < \frac{Q(Q+2)}{(Q-1)^2} < \frac{Q+2}{Q-2}$).

In this note we aim to establish a first Liouville-type result for a CR fractional power of $-\Delta_{\mathbb{H}}$; this is, to our knowledge, the first nonexistence result in this fractional setting.

Let us comment now the basic tools in the proof of our main result. Following [4], in order to get a nonexistence result, we combine the method of moving planes with the CR inversion of the solution u.

The CR inversion was introduced in [24], and it is the analogue of the Kelvin transform, in the Heisenberg group context. In Section 3 we will give the precise definition of CR inversion and we will show which problem is satisfied by the CR inversion of a solution of (1.7).

As said before, the moving plane method is based on several version of the maximum principles. More precisely we will need to prove that our operator satisfies a weak maximum principle and two versions of the Hopf Lemma (see Propositions 4.3 and 4.6).

The note is organized as follows:

- in Section 2 we recall some basic facts on the Heisenberg group and we will introduce the fractional CR operator $\mathcal{P}_{\frac{1}{2}}$;
- in Section 3 we will introduce the CR inversion of a function u and state a lemma concerning the CR inversion of a solution of our problem (1.7);
- in Section 4 we establish a maximum principles and Hopf Lemma for our operator, which will be basic tools in the method of moving plane;
- in Section 5 we will prove our main result (Theorem 1.4).

2. Preliminary facts on the Heisenberg group

In this section we recall some basic notions and properties concerning the Heisenberg group (see Chapter 3 in [6] and Chapter XII in [26]).

We will denote the points in $\mathbb{H}^n$ using the notation $\xi = (x, y, t) = (x_1, ..., x_n, y_1, ..., y_n, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. The Heisenberg group $\mathbb{H}^n$ is the space $\mathbb{R}^{2n+1}$ endowed with the group law $\circ$ defined in the following way:

$$\hat{\xi} \circ \xi := (\hat{x} + x, \hat{y} + y, \hat{t} + t + 2\sum_{j=1}^{n}(x_j\hat{y}_j - y_j\hat{x}_j)).$$

The natural dilation of the group is given by $\delta_\ell(\xi) := (\ell x, \ell y, \ell^2 t)$, and it satisfies $\delta_\ell(\hat{\xi} \circ \xi) = \delta_\ell(\hat{\xi}) \circ \delta_\ell(\xi)$.

In $\mathbb{H}^n$ we will consider the gauge norm defined as

$$|\xi|_{\mathbb{H}} := \big[\big(\sum_{i=1}^{n}(x_i^2+y_i^2)\big)^2+t^2\big]^{\frac{1}{4}},$$

which is homogeneous of degree one with respect to δ_ℓ. Using this norm, one can define the distance between two points in the natural way:

$$d_{\mathbb{H}}(\hat{\xi},\xi)=|\hat{\xi}^{-1}\circ\xi|_{\mathbb{H}},$$

where $\hat{\xi}^{-1}$ denotes the inverse of $\hat{\xi}$ with respect to the group action. We denote the ball associated to the gauge distance by

$$B_{\mathbb{H}}(\xi_0,R):=\{\xi\in\mathbb{H}^n\ :\ d_{\mathbb{H}}(\xi,\xi_0)<R\}.$$

Denoting by $|A|$ the Lebesgue measure of the set A, we have that

$$|B_{\mathbb{H}}(\xi_0,R)|=|B_{\mathbb{H}}(0,R)|=R^Q|B_{\mathbb{H}}(0,1)|.$$

Here $Q=2n+2$ denotes the homogeneous dimension of $\mathbb{H}^n$.

For every $j=1,\cdots,n$, we denote by X_j, Y_j, and T the following vector fields:

$$X_j=\frac{\partial}{\partial x_j}+2y_j\frac{\partial}{\partial t},\quad Y_j=\frac{\partial}{\partial y_j}-2x_j\frac{\partial}{\partial t},\quad T=\frac{\partial}{\partial t}.$$

They form a basis of the Lie Algebra of left invariant vector fields. Moreover, an easy computions shows that $[X_k,Y_j]=-4\delta_{kj}T$. The Heisenberg gradient of a function f is given by

$$\nabla_{\mathbb{H}}f=(X_1f,\cdots,X_nf,Y_1f,\cdots,Y_nf).$$

Finally, we define the sublaplacian as

$$\Delta_{\mathbb{H}}:=\sum_{j=1}^{n}(X_j^2+Y_j^2). \tag{2.1}$$

It can be written also in the form $\Delta_{\mathbb{H}}=\mathrm{div}(\overline{A}\nabla^T)$, where $\overline{A}=\overline{a}_{kj}$ is the $(2n+1)\times(2n+1)$ symmetric matrix given by $\overline{a}_{kj}=\delta_{kj}$ for $k,j=1,...,2n$, $\overline{a}_{j(2n+1)}=\overline{a}_{(2n+1)j}=2y_j$ for $j=1,...,n$, $\overline{a}_{j(2n+1)}=\overline{a}_{(2n+1)j}=-2x_j$ for $j=n+1,...,2n$ and $\overline{a}_{(2n+1)(2n+1)}=4(|x|^2+|y|^2)$. It is easy to observe that $\overline{A}$ is positive semidefinite for any $(x,y,t)\in\mathbb{H}^n$. This operator is degenerate elliptic, and it is hypoelliptic since it satisfies the Hormander condition.

We pass now to describe CR covariant operators of fractional orders in $\mathbb{H}^n$. For more precise notions of CR geometry and for the construction of CR covariant fractional powers of the sub-Laplacian on more general CR manifolds, we refer to [14] and references therein. Here we just consider the case of the Heisenberg group, since it is the one of interest.

Introducing complex coordinates $\zeta = x+iy \in \mathbb{C}^n$, we can identify the Heisenberg group $\mathbb{H}^n$ with the boundary of the Siegel domain $\Omega_{n+1} \subset \mathbb{C}^{n+1}$, which is given by

$$\Omega_{n+1} := \{(\zeta_1, \dots, \zeta_{n+1}) = (\zeta, \zeta_{n+1}) \in \mathbb{C}^n \times \mathbb{C} \,|\, q(\zeta, \zeta_{n+1}) > 0\},$$

with

$$q(\zeta, \zeta_{n+1}) = \operatorname{Im} \zeta_{n+1} - \sum_{j=1}^{n} |\zeta_j|^2,$$

through the map $(\zeta, t) \in \mathbb{H}^n \to (\zeta, t + i|\zeta|^2) \in \partial\Omega_{n+1}$. It is possible to see that $\mathcal{X} = \Omega_{n+1}$ is a Kähler-Einstein manifold, endowed with a Kähler metric g_+, which can be identified with the complex hyperbolic space. The boundary manifold $\mathcal{M} = \partial\Omega_{n+1}$ inherits a natural CR structure from the complex structure of the ambient manifold.

Scattering theory tells us that for $s \in \mathbb{C}$, $\operatorname{Re}(s) \geq \frac{m}{2}$, and except for a set of exceptional values, given f smooth on $\mathcal{M}$, the eigenvalue equation

$$-\Delta_{g^+} u - s(m-s)u = 0, \quad \text{in } \mathcal{X}$$

has a solution u with the expansion

$$\begin{cases} u = q^{(m-s)} F + q^s G \quad \text{for some} \quad F, G \in C^\infty(\overline{\mathcal{X}}), \\ F|_{\mathcal{M}} = f. \end{cases}$$

The scattering operator is defined as

$$S(s) : \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})$$

by

$$S(s)f := G|_{\mathcal{M}}.$$

We set $s = \frac{m+\gamma}{2}$, for $\gamma \in (0, m)\backslash\mathbb{N}$. The conformal fractional sub-Laplacian on $\mathbb{H}^n$ is defined in the following way:

$$\mathcal{P}_\gamma f = C_\gamma S(s) f, \tag{2.2}$$

for a constant

$$C_\gamma = 2^{2\gamma-1}\frac{\Gamma(\gamma)}{\gamma\Gamma(-\gamma)}.$$

For $\gamma = 1$ and $\gamma = 2$ we have:

$$P_1 = -\Delta_{\mathbb{H}} \quad \text{and} \quad P_2 = \Delta_{\mathbb{H}}^2 + T^2.$$

A crucial property of $\mathcal{P}_\gamma$ is its conformal covariance.

As explained in the introduction, one of the main result in [14], is the characterization of these fractional operators via the extension problem (1.2). Throughout this paper, we will work on this lifted problem in the extended space, let us introduce some notations also in this setting.

Analougsly to $\mathbb{H}^n$, in $\widehat{\mathbb{H}}^n$ we define the following group low (that for simplicity of notation we still denote by $\circ$):
for $z = (x_1, \cdots, x_n, y_1, \cdots, y_n, t, \lambda) \in \widehat{\mathbb{H}}^n$ and $\hat{z} = (\hat{x}_1, \cdots, \hat{x}_n, \hat{y}_1, \cdots, \hat{y}_n, \hat{t}, \hat{\lambda}) \in \widehat{\mathbb{H}}^n$, we set

$$\hat{z} \circ z := (\hat{x} + x, \hat{y} + y, \hat{t} + t + 2\sum_{j=1}^n (x_j\hat{y}_j - y_j\hat{x}_j), \hat{\lambda} + \lambda).$$

Moreover we consider the norm given by

$$|z|_{\widehat{\mathbb{H}}^n} := [(|x|^2 + |y|^2 + \lambda^2)^2 + t^2]^{\frac{1}{4}}.$$

Finally we denote the distance $d_{\widehat{\mathbb{H}}}$ between z and $\hat{z}$, by

$$d_{\widehat{\mathbb{H}}}(z, \hat{z}) := |\hat{z}^{-1} \circ z|_{\widehat{\mathbb{H}}^n}.$$

Observe that when $\lambda = \hat{\lambda} = 0$, that is z and $\hat{z}$ belong to $\mathbb{H}^n$, $d_{\widehat{\mathbb{H}}}(z, \hat{z}) = d_{\mathbb{H}}(z, \hat{z})$. Moreover, given $\bar{z} \in \widehat{\mathbb{H}}^n$ we set

$$\mathcal{B}(\bar{z}, R) = \{z \in \mathbb{C}^{n+1} \mid d_{\widehat{\mathbb{H}}}(z, \bar{z}) < R\}$$

and for any $z_0 \in \mathbb{H}^n \times \{0\}$ we denote

$$\mathcal{B}^+(z_0, R) = \{z \in \mathbb{H}^n \times \mathbb{R}^+ \mid d_{\widehat{\mathbb{H}}}(z, z_0) < R\,, \lambda > 0\}.$$

The operator $\mathcal{L}$, writing explicitly all the terms, becomes

$$\mathcal{L} = \frac{\partial^2}{\partial\lambda^2} + \sum_{j=1}^n \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4y_j\frac{\partial^2}{\partial x_j \partial t} - 4x_j\frac{\partial^2}{\partial y_j \partial t}\right)$$

$$+ 4(\lambda^2 + \sum_{j=1}^{n}(x_j^2 + y_j^2))\frac{\partial^2}{\partial t^2}.$$

Also in this case, we can write $\mathcal{L} = \operatorname{div}(A\nabla^T)$, where now A is the $(2n+2) \times (2n+2)$ symmetric matrix given by $a_{kj} = \delta_{kj}$ if $k, j = 1, \cdots, 2n$, $a_{j(2n+1)} = a_{(2n+1)j} = 2y_j$ if $j = 1, \cdots, n$, $a_{j(2n+1)} = a_{(2n+1)j} = -2x_j$ if $j = n+1, \cdots, 2n$, $a_{(2n+1)(2n+1)} = 4(|x|^2 + |y|^2 + \lambda^2)$, $a_{(2n+2)(2n+2)} = 1$, $a_{j(2n+2)} = a_{(2n+2)j} = 0$ if $j = 1, \cdots, 2n$.

In the sequel it will be useful to express $\mathcal{L}$ for cylindrical and radial functions. Let

$$r = (|x|^2 + |y|^2 + \lambda^2)^{1/2},$$
$$\rho = (r^4 + t^2)^{1/4}.$$

Suppose that Ψ is a radial function, that is, Ψ depends only on ρ; then a direct computations gives:

$$\mathcal{L}\Psi(\rho) = \frac{r^2}{\rho^2}\left(\frac{\partial^2\Psi(\rho)}{\partial\rho^2} + \frac{Q}{\rho}\frac{\partial\Psi(\rho)}{\partial\rho}\right). \tag{2.3}$$

In a similar way, we deduce that for cylindrical symmetric functions $\phi = \phi(r, t)$, we have

$$\mathcal{L}\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{Q-2}{r}\frac{\partial\phi}{\partial r} + 4r^2\frac{\partial^2\phi}{\partial t^2}.$$

Using the radial form (2.3) for $\mathcal{L}$, an easy computation yelds the following lemma.

Lemma 2.1 (See [11])**.** *For $\rho \neq 0$, let $\psi(\rho) = \frac{1}{\rho^{Q-1}} = \frac{1}{\rho^{2n+1}}$. Then we have that*

$$\begin{cases} \mathcal{L}\psi(\rho) = 0 & \in \ \mathbb{H}^n \times \mathbb{R}^+ \setminus \{0\} \\ -\frac{\partial}{\partial\lambda}\psi(\rho) = 0 & \text{on } \mathbb{H}^n \setminus \{0\} \times \{\lambda = 0\}. \end{cases} \tag{2.4}$$

3. CR inversion

Following [4] and [24], we define the CR inversion in the half-space $\mathbb{H}^n \times \mathbb{R}^+$.

For any $(x, y, t, \lambda) \in \mathbb{H}^n \times \mathbb{R}^+$, let as before $r = (|x|^2 + |y|^2 + \lambda^2)^{\frac{1}{2}}$ and $\rho = (r^4 + t^2)^{\frac{1}{4}}$. We set

$$\widetilde{x}_i = \frac{x_i t + y_i r^2}{\rho^4}, \quad \widetilde{y}_i = \frac{y_i t - x_i r^2}{\rho^4}, \quad \widetilde{t} = -\frac{t}{\rho^4}, \quad \widetilde{\lambda} = \frac{\lambda}{\rho^2}.$$

The CR inversion of a function U defined on $\mathbb{H}^n \times \mathbb{R}^+$, is given by

$$v(x, y, t, \lambda) = \frac{1}{\rho^{Q-1}} U(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{\lambda}).$$

The following lemma shows which equation is satisfied by the CR inversion of a solution of problem (1.7). We refer to [11] for the proof.

Lemma 3.1. *(see* [11]*) Suppose that* $U \in C^2(\mathbb{H}^n \times \mathbb{R}^+ \setminus \{0\}) \cap C(\overline{\mathbb{H}^n \times \mathbb{R}^+} \setminus \{0\})$ *is a solution of* (1.7)*. Then the CR inversion* v *of* U *satisfies*

$$\begin{cases} \mathcal{L}v = 0 & \text{in } \mathbb{H}^n \times \mathbb{R}^+ \setminus \{0\} \\ -\dfrac{\partial v}{\partial \lambda} = \rho^{p(Q-1)-(Q+1)} v & \text{on } \mathbb{H}^n \setminus \{0\} \times \{\lambda = 0\}. \end{cases} \tag{3.1}$$

4. Maximum principles and Hopf Lemma

Basic tools in the method of moving planes are the maximum principle and Hopf Lemma. For the validity of the maximum principle and Hopf Lemma for the sublaplacian in homogeneous Carnot groups, we refer to [6] (Chapter 5, Appendix A). We start by recalling the classical maximum principle for Hörmander-type operators due to Bony [7].

Proposition 4.1. [7] *Let $\mathcal{D}$ be a bounded domain in $\widehat{\mathbb{H}}^n$, let Z be a smooth vector field on $\mathcal{D}$ and let a be a smooth nonnegative function. Assume that $U \in C^2(\mathcal{D}) \cap C^1(\overline{\mathcal{D}})$ is a solution of*

$$\begin{cases} -\mathcal{L}U + Z(z)U + a(z)U \geq 0 & \text{in } \mathcal{D}, \\ U \geq 0 & \text{on } \partial\mathcal{D}. \end{cases} \tag{4.1}$$

Then $U \geq 0$ in $\mathcal{D}$.

We prove now two Hopf Lemmas. The first one is Hopf Lemma for the operator $\mathcal{L}$ in a subset $\mathcal{V}$ of $\widehat{\mathbb{H}}^n$. We remind that the analogue result for the Heisenberg Laplacian $\Delta_{\mathbb{H}}$ in $\mathbb{H}^n$ was established in [3].

We start with the following definition.

Definition 4.2. *Let $\mathcal{D} \subset \widehat{\mathbb{H}}^n$. We say that $\mathcal{D}$ satisfies the interior $d_{\widehat{\mathbb{H}}}$-ball condition at $P \in \partial\mathcal{D}$ if there exists a constant $R > 0$ and a point $z_0 \in \mathcal{D}$, such that the ball $\mathcal{B}(z_0, R) \subset \mathcal{D}$ and $P \in \partial\mathcal{B}(z_0, R)$, where $\mathcal{B}(z_0, R) = \{z \in \widehat{\mathbb{H}}^n \mid d_{\widehat{\mathbb{H}}}(z, z_0) < R\}$.*

Lemma 4.3. *Let $\mathcal{D} \subset \widehat{\mathbb{H}}^n$ satisfy the interior $d_{\widehat{\mathbb{H}}}$-ball condition at the point $P_0 \in \partial\mathcal{D}$ and let $U \in C^2(\mathcal{D}) \cap C^1(\overline{\mathcal{D}})$, be a solution of*

$$-\mathcal{L}U \geq c_1(z)U \text{ in } \mathcal{D} \tag{4.2}$$

with $c_1 \in L^\infty(\mathcal{D})$. Suppose that $U(z) > U(P_0) = 0$ for every $z \in \mathcal{D}$.

Then

$$\lim_{s\to 0} \frac{U(P_0) - U(P_0 - s\nu)}{s} < 0.$$

where ν is the outer normal to $\partial\mathcal{D}$ in P_0.

Remark 4.4. *We observe that if the function c_1 in Lemma 4.3 is identically zero, then we can drop the assumption $U(P_0) = 0$.*

Proof. By assumption, there exist a point $z_0 = (\hat{x}_1, ..., \hat{x}_n, \hat{y}_1, ..., \hat{y}_n, \hat{t}, \hat{\lambda})$ and a radius $R > 0$ such that the ball $\mathcal{B}(z_0, R) \subset \mathcal{D}$ and $P_0 \in \partial\mathcal{B}(z_0, R)$.

We consider the function

$$\psi = Ue^{-K(x_1-\hat{x}_1)^2}, \text{ for } K > 0. \tag{4.3}$$

An easy computation yields

$$\frac{\partial^2\psi}{\partial x_1^2} = e^{-K(x_1-\hat{x}_1)^2}[4K^2(x_1-\hat{x}_1)^2U - 2KU - 4K(x_1-\hat{x}_1)\frac{\partial U}{\partial x_1} + \frac{\partial^2 U}{\partial x_1^2}],$$

and

$$\begin{aligned}
&\frac{\partial^2\psi}{\partial x_j^2} = e^{-K(x_1-\hat{x}_1)^2}\frac{\partial^2 U}{\partial x_j^2}, \quad \frac{\partial^2\psi}{\partial y_k^2} = e^{-K(x_1-\hat{x}_1)^2}\frac{\partial^2 U}{\partial y_k^2}, \\
&\frac{\partial^2\psi}{\partial \lambda^2} = e^{-K(x_1-\hat{x}_1)^2}\frac{\partial^2 U}{\partial \lambda^2},
\end{aligned}$$

where $j = 2, \cdots, n, k = 1, \cdots, n$.

Moreover,

$$\begin{aligned}
&\frac{\partial^2\psi}{\partial x_1 \partial t} = e^{-K(x_1-\hat{x}_1)^2}[-2K(x_1-\hat{x}_1)\frac{\partial U}{\partial t} + \frac{\partial^2 U}{\partial x_1 \partial t}], \\
&\frac{\partial^2\psi}{\partial x_j \partial t} = e^{-K(x_1-\hat{x}_1)^2}\frac{\partial^2 U}{\partial x_j \partial t}.
\end{aligned}$$

Therefore, we have

$$\mathcal{L}\psi + 4K(x_1 - \hat{x}_1)X_1\psi = e^{-K(x_1-\hat{x}_1)^2}[-4K^2(x_1 - \hat{x}_1)^2U + \mathcal{L}U - 2KU]$$

and hence for K sufficiently large, we deduce

$$-\mathcal{L}\psi - 4K(x_1 - \hat{x}_1)X_1\psi \geq 0.$$

We introduce now the function $\phi = e^{-\alpha R^2} - e^{-\alpha\rho^2}$, where $\rho = d_{\widehat{\mathbb{H}}}(z, z_0)$, and $0 < \rho < R$. Since ϕ depends only on the distance from z_0, and $\mathcal{L}$ and X_1 are invariant with respect to the group action in $\widehat{\mathbb{H}}^n$, we can use formula (2.3) where now $\rho = d_{\widehat{\mathbb{H}}}(z_0^{-1} \circ z, 0)$, and the factor $\frac{r^2}{\rho^2}$ is replaced by the function $G(z_0^{-1} \circ z)$, where

$$G(x_1, ..., x_n, y_1, ..., y_n, t, \lambda) := \frac{\sum_{j=1}^n (x_j^2 + y_j^2) + \lambda^2}{[(\sum_{j=1}^n (x_j^2 + y_j^2) + \lambda^2)^2 + t^2]^{\frac{1}{2}}}. \tag{4.4}$$

Choosing α sufficiently large, we have

$$\begin{aligned} &-\mathcal{L}\phi - 4K(x_1 - \hat{x}_1)X_1\phi \\ &= \left[G(z_0^{-1} \circ z)(4\alpha^2\rho^2 - 2(Q+1)\alpha)\right. \\ &\left.-8K\alpha(x_1 - \hat{x}_1)\rho X_1\rho\right] e^{-\alpha\rho^2} \geq 0, \end{aligned}$$

Let $\mathcal{A} := \mathcal{B}(z_0, R) \setminus \mathcal{B}(z_0, R_1)$ for $0 < R_1 < R$. For ε small enough

$$\psi(z) + \varepsilon\phi(z) \geq 0 \quad \text{in } \partial\mathcal{A} := \partial\mathcal{B}(z_0, R) \cup \partial\mathcal{B}(z_0, R_1).$$

Then, by Proposition 4.1 we obtain that $\psi(z) + \varepsilon\phi(z) \geq 0$ in $\mathcal{A}$.
Therefore, using that $\psi(P_0) = \phi(P_0) = 0$, we deduce that for s small

$$\psi(P_0) - \psi(P_0 - s\nu) + \varepsilon(\phi(P_0) - \phi(P_0 - s\nu)) \leq 0.$$

Using that ϕ is strictly increasing in ρ, we deduce

$$\lim_{s\to 0} \frac{\psi(P_0) - \psi(P_0 - s\nu)}{s} < 0,$$

which, using the definition of ψ (4.3), implies that

$$\lim_{s\to 0} \frac{U(P_0 - s\nu) - U(P_0)}{s} < 0.$$

□

For any $\Omega \subset \mathbb{H}^n$, we denote by $\mathcal{C}$ the infinite cylinder

$$\mathcal{C} = \Omega \times (0, +\infty).$$

Before proving our second Hopf Lemma, let us recall the notion of interior ball condition in the Heisenberg group.

Definition 4.5. *Let $\Omega \subset \mathbb{H}^n$. We say that Ω satisfies the interior Heisenberg ball condition at $\xi \in \partial\Omega$ if there exists a constant $R > 0$ and a point $\xi_0 \in \Omega$, such that the Heisenberg ball $B_{\mathbb{H}}(\xi_0, R) \subset \Omega$ and $\xi \in \partial B_{\mathbb{H}}(\xi_0, R)$.*

Lemma 4.6. *Let $\Omega \subset \mathbb{H}^n$ satisfy the interior Heisenberg ball condition at the point $P \in \partial\Omega$ and let $U \in C^2(\mathcal{C}) \cap C^1(\overline{\mathcal{C}})$, be a nonnegative solution of*

$$\begin{cases} -\mathcal{L}U \geq c_1(z)U & \text{in } \mathcal{C}, \\ -\partial_\lambda U \geq c_2(\xi)U & \text{on } \Omega, \end{cases} \tag{4.5}$$

with $c_1, c_2 \in L^\infty(\mathcal{C})$ and c_1 is nonnegative. Suppose that $U((P,0)) = 0$ and U is not identically null.

Then

$$\partial_\nu U(P,0) < 0, \tag{4.6}$$

where ν is the outer normal to $\partial\Omega$ in P.

Proof. We follow the proof of Lemma 2.4 in [10].

By the strong maximum principle and by Lemma 4.3, we have that

$$U > 0 \quad \text{on } \mathcal{C} \cup \Omega. \tag{4.7}$$

Indeed, the strong maximum principle ensures that $U > 0$ in $\mathcal{C}$, moreover U cannot vanish at a point in Ω, otherwise at this point the Neumann condition would be violated by Lemma 4.3.

We start by proving the lemma in the case $c_1(z) = c_2(\xi) \equiv 0$.

Since Ω satisfies the interior Heisenberg ball condition at P, there exist $z_0 \in \Omega \times \{0\}$ and $R > 0$, such that the ball $\mathcal{B}^+(z_0, R)$ is contained in the cylinder $\mathcal{C}$ and $(\overline{\partial\mathcal{C} \cap \{\lambda > 0\}}) \cap \partial\mathcal{B}^+(z_0, R) = \{(P, 0)\}$. We consider the set

$$\mathcal{A} = \left(\mathcal{B}^+(z_0, R) \setminus \overline{\mathcal{B}^+(z_0, R/2)}\right) \cap \{\lambda > 0\}.$$

We observe that $\{(P, 0)\} = \partial\mathcal{A} \cap \overline{\partial\mathcal{C} \cap \{\lambda > 0\}}$.

For $z \in \mathcal{A}$ we consider the function $\eta(z) = e^{-\alpha\rho^2} - e^{-\alpha R^2}$, where $\rho = d_{\widehat{\mathbb{H}}}(z, z_0)$. Writing $\mathcal{L}$ in radial coordinate as in (2.3), we have that

$$\mathcal{L}\eta(\rho) = G(z_0^{-1} \circ z)\left(4\alpha^2\rho^2 - 2(Q+1)\alpha\right)e^{-\alpha\rho^2},$$

where G is defined as in (4.4).

Therefore, for α sufficiently large, we have that

$$-\mathcal{L}\eta \leq 0. \tag{4.8}$$

By (4.7) we deduce that $U > 0$ on $\partial\mathcal{B}^+(z_0, R/2) \cap \{\lambda \geq 0\}$. Hence, we may choose $\varepsilon > 0$ such that

$$U - \varepsilon\eta \geq 0 \quad \text{on} \;\; \partial\mathcal{B}^+(z_0, R/2) \cap \{\lambda \geq 0\}.$$

Claim: $U - \varepsilon\eta \geq 0$ in $\mathcal{A}$.

Indeed, using (4.8), we deduce that $-\mathcal{L}(U - \varepsilon\eta) \geq 0$ in $\mathcal{A}$. Hence, by the maximum principle, we have that the minimum of $U - \varepsilon\eta$ is attained only on $\partial\mathcal{A}$ (unless $U - \varepsilon\eta$ is constant). Now, on one side we have that

$$U - \varepsilon\eta \geq 0 \quad \text{on} \;\; \partial\mathcal{A} \cap \{\lambda > 0\}.$$

On the other side, since $\partial_\lambda\eta = 0$ on $\{\lambda = 0\}$, we deduce that $-\partial_\lambda(U - \varepsilon\eta) \geq 0$ on $\partial\mathcal{A} \cap \{\lambda = 0\}$.

Thus, using Lemma 4.3, we conclude that the minimum of $U - \varepsilon\eta$ cannot be achieved on $\left(\mathcal{B}^+(z_0, R) \setminus \overline{\mathcal{B}^+(z_0, R)/2)}\right) \cap \{\lambda = 0\}$. This reaches the claim.

Finally, since $(U - \varepsilon\eta)((P, 0)) = 0$, we deduce that $\partial_\nu(U - \varepsilon\eta)((P, 0)) \leq 0$, which in turn implies that $\partial_\nu U((P, 0)) < 0$ using that $\partial_\nu\eta((P, 0)) < 0$. This concludes the case $c_1(z) = c_2(\xi) \equiv 0$.

In the general case, we introduce the function $v = e^{-\beta\lambda}U$ and we compute

$$\begin{aligned} -\mathcal{L}v &= -e^{-\beta\lambda}\mathcal{L}U + 2\beta e^{-\beta\lambda}U_\lambda - \beta^2 v \\ &\geq c_1(z) - \beta^2 v + 2\beta e^{-\beta\lambda}(\beta e^{\beta\lambda}v + e^{\beta\lambda}v_\lambda) \\ &= c_1(z) + \beta^2 v + 2\beta v_\lambda. \end{aligned}$$

Therefore, we have

$$-\mathcal{L}v - 2\beta v_\lambda \geq c_1(z) + \beta^2 v \geq 0.$$

Moreover, for β large enough

$$-\partial_\lambda v \geq (\beta + c_2(z))v \geq 0.$$

We can apply the first part of the proof to the function v, noting that the same argument works when the operator $\mathcal{L}$ is replaced by $\mathcal{L} + 2\beta\partial_\lambda$.

$\square$

5. Proof of Theorems 1.3 and 1.4

In this last section we give the proof of our Liouville-type result.

We recall that $U = U(|(x,y)|, t, \lambda)$ is a solution of

$$\begin{cases} \mathcal{L}U = 0 & \text{in } \widehat{\mathbb{H}}^n_+ := \mathbb{H}^n \times \mathbb{R}^+, \\ -\partial_\lambda U = U^p & \text{on } \mathbb{H}^n = \partial\widehat{\mathbb{H}}^n_+, \\ U > 0 & \text{in } \widehat{\mathbb{H}}^n_+. \end{cases} \tag{5.1}$$

First of all, we consider the CR inversion of U. For $z = (x, y, t, \lambda) \in \widehat{\mathbb{H}}^n$, let

$$w(z) = \frac{1}{\rho^{Q-1}} U(\widetilde{z}),$$

where $\widetilde{z} = \frac{1}{\rho^4}\left(xt + yr^2, yt - xr^2, -t, \lambda\rho^2\right)$, $r = \left(\sum_{j=1}^n (x_j^2 + y_j^2) + \lambda^2\right)^{\frac{1}{2}}$ and $\rho(z) = (r^4 + t^2)^{\frac{1}{4}} = d_{\widehat{\mathbb{H}}}(z, 0)$. We have seen in Lemma 3.1 that w satisfies

$$\begin{cases} \mathcal{L}w = 0 & \text{in } \widehat{\mathbb{H}}^n_+ \setminus \{0\}, \\ -\partial_\lambda w = \rho^{p(Q-1)-(Q+1)} w^p & \text{on } \mathbb{H}^n. \end{cases} \tag{5.2}$$

Observe that the function w could be singular at the origin and it satisfies $\lim_{\rho\to\infty} \rho^{Q-1}w(z) = U(0)$.

We start now applying the moving plane method. We will move a hyperplane orthogonal to the t-direction and use the H-reflection. More precisely, for any $\mu \leq 0$, let

$T_\mu = \{z \in \widehat{\mathbb{H}}^n_+ \mid t = \mu\}$, and $\Sigma_\mu = \{z \in \widehat{\mathbb{H}}^n_+ \mid t < \mu\}$. For $z \in \Sigma_\mu$, we define $z_\mu = (y, x, 2\mu - t, \lambda)$. To avoid the singular point, we consider

$$\widetilde{\Sigma}_\mu = \Sigma_\mu \setminus \{e_\mu\},$$

where $e_\mu = (0, 0, 2\mu, 0)$ is the reflection of the origin. Let

$$w_\mu(z) = w_\mu(|(x,y)|, t, \lambda) := w(|(x,y)|, 2\mu - t, \lambda) = w(y, x, 2\mu - t, \lambda) = w(z_\mu),$$

and

$$W_\mu(z) := w_\mu(z) - w(z) = w(z_\mu) - w(z), \quad z \in \Sigma_\mu.$$

By using the invariance of the operator under the CR transform as in Lemma 3.1 and the fact that $\rho(z_\mu) \leq \rho(z)$, we have that

$$\begin{cases} \mathcal{L}W_\mu = 0 & \text{in } \widehat{\mathbb{H}}^n_+, \\ -\partial_\lambda W_\mu \geq c(z,\mu)W_\mu & \text{on } \mathbb{H}^n, \end{cases}$$

where $c(z,\mu) = \frac{p\Psi_\mu^{p-1}}{\rho^{(Q+1)-p(Q-1)}}$ and $\Psi_\mu(z)$ is between $w(z)$ and $w_\mu(z)$. By the definition of w_μ and w, we have that $c(z,\mu) \approx C/\rho^2$ at infinity.

The following proposition contains the first crucial step in the method of moving planes.

Proposition 5.1. *Assume that $w \in C^2(\widehat{\mathbb{H}}^n_+) \cap C^1(\overline{\widehat{\mathbb{H}}^n_+}) \setminus \{0\}$ satisfies* (5.2)*. Then* (i) *For $\mu < 0$ with $|\mu|$ large enough, if $\inf_{\Sigma_\mu} W_\mu < 0$, then the infimum is attained at some point $z_0 \in \overline{\Sigma_\mu} \setminus \{e_\mu\}$.* (ii) *There exists an $R_1 > 0$ such that whenever $\inf_{\Sigma_\mu} W_\mu$ is attained at $z_0 \in \overline{\Sigma_\mu} \setminus \{e_\mu\}$ with $W_\mu(z_0) < 0$, then $\rho(z_0) = d_{\widehat{\mathbb{H}}}(z_0, 0) \leq R_1$.*

The proof is based on the construction of a comparison function and on the use of maximum principles and Hopf Lemmas. For the details we refer to [11]. We can prove now our main result Theorem 1.4.

Proof of Theorem 1.4. By Proposition 5.1, we deduce that for μ negative and large in absolute value we have that $W_\mu \geq 0$ in Σ_μ. Let us define $\mu_0 = \sup\{\mu < 0 \mid W_\sigma \geq 0 \text{ on } \Sigma_\sigma \setminus e_\sigma \text{ for all } \sigma < \mu\}$. We only need to prove that $\mu_0 = 0$. Suppose that $\mu_0 \neq 0$ by contradiction. By continuity, $W_{\mu_0} \geq 0$ in Σ_{μ_0}. By the maximum principle, we deduce that $W_{\mu_0} \equiv 0$ in Σ_{μ_0} or

$$W_{\mu_0} > 0 \quad \text{on} \quad \Sigma_{\mu_0} \cup (\partial\Sigma_{\mu_0} \cap \{\lambda = 0\} \cap \{t < \mu_0\}) \setminus \{e_{\mu_0}\}. \tag{5.3}$$

Using the Neumann condition satisfied by W_{μ_0} and the assumption $\mu_0 > 0$, we see that $W_{\mu_0} \equiv 0$ is impossible. Therefore (5.3) holds.

By the definition of μ_0 there exists $\mu_k \to \mu_0$, $\mu_0 < \mu_k < 0$ such that $\inf_{\Sigma_{\mu_k}} W_{\mu_k} < 0$.

We observe that for some positive b_1:

$$\min\left\{W_{\mu_0}(z) \mid z \in \partial\mathcal{B}^+(e_{\mu_0}, |\mu_0|/2) \cap \widehat{\mathbb{H}}^n_+\right\} = b_1.$$

From this fact, using a similar argument to the one of point i) in Proposition 5.1, we deduce that

$$W_{\mu_0} \geq b_1 \quad \text{in } \overline{\mathcal{B}^+(e_{\mu_0}, |\mu_0|/2)} \setminus \{e_{\mu_0}\}.$$

Therefore, we have that

$$\lim_{k\to\infty} \inf\left\{W_{\mu_k}(z) \mid z \in \mathcal{B}^+(e_{\mu_k}, |\mu_0|/2) \setminus \{e_{\mu_k}\}\right\} \geq b_1.$$

Using this bound and the fact that $W_{\mu_k}(z) \to 0$ as $\rho(z) \to \infty$, we deduce that for k large enough, the negative infimum of W_{μ_k} is attained at some point $z_k \in \overline{\Sigma_{\mu_k}} \setminus \mathcal{B}^+(e_{\mu_k}, |\mu_0|/2)$.

By Proposition 5.1 we know that the sequence $\{z_k\}$ is bounded and therefore, after passing to a subsequence, we may assume that $z_k \to z_0$. By (5.3) we have that $W_{\mu_0}(z_0) = 0$ and $z_0 \in \partial\Sigma_{\mu_0} \cap \{t = \mu_0\}$.

If $z_k \in \Sigma_{\mu_k}$ for an infinite number of k, then $\nabla W_{\mu_k}(z_k) = 0$, and therefore, by continuity

$$\nabla W_{\mu_0}(z_0) = 0. \tag{5.4}$$

If $z_0 \in \partial\Sigma_{\mu_0} \cap \widehat{\mathbb{H}}^n_+$, then by Lemma 4.3, we have that $\frac{\partial w}{\partial t}(z_0) < 0$, which gives a contradiction. Analogously, using Lemma 4.6, we get a contradiction if we assume that $z_0 \in \partial\Sigma_{\mu_0} \cap \{\lambda = 0\} \cap \{t = \mu_0\}$.

In the case in which $z_k \in \partial\Sigma_{\mu_k} \cap \{\lambda = 0\} \cap \{t < \mu_k\}$, we still have that the derivatives of W_{μ_k} at z_k in all directions except the λ direction vanish. Passing to the limit and arguing as above, we get a contradiction. Hence we have established that $\mu_0 = 0$. This implies that v is even in t, but since the origin 0 on the t-axes is arbitrary, we can perform the CR transform with respect to any point and then we conclude that w is constant in the direction t.

This shows that U is actually a solution of the following problem

$$\begin{cases} \Delta U = 0 & \text{in } \mathbb{R}^{2n+1}_+, \\ -\partial_\lambda U = U^p & \text{on } \mathbb{R}^{2n}. \end{cases} \tag{5.5}$$

Since $\frac{Q+1}{Q-1} = \frac{2n+3}{2n+1} < \frac{2n+1}{2n-1}$, we conclude the proof by using the standard Liouville type theorem for problem (5.5) (see [10, 22]).

□

The following lemma allows us to deduce Theorem 1.3 by Theorem 1.4.

Lemma 5.2. *Let $u \in C^2(\mathbb{H}^n)$ be cylindrically symmetric and positive (respectively non-negative). Then the corresponding solution U of the extension problem* (1.7) *is cylindrically simmetric, i.e. $U = U(r_0, t, \lambda)$ with $r_0 = \sqrt{x^2 + y^2}$, U is positive (respectively nonnegative) and moreover $U \in C^2(\widehat{\mathbb{H}}^n_+) \cap C^1(\widehat{\mathbb{H}}^n_+)$.*

The proof is technical and it uses the Fourier transform in the Heisenberg group. We refer to [11] and [14] for details.

We conclude with the proof of Theorem 1.3.

Proof of Theorem 1.3. By Lemma 5.2, we see that if u is a cylindrical function, that is $u = u(|(x, y)|, t)$, then its extension U satisfying (1.2) is also cylindrical in the all halfspace $\mathbb{H}^n \times \mathbb{R}^+$, in the sense that $U = U(|(x, y)|, t, \lambda)$. Using this fact, the conclusion follows as a corollary of Theorem 1.4.

□

References

[1] Alexandrov, A.D.; *Uniqueness theorems for surfaces in the large I,* Vestnik Leningrad Univ. Math. 11 (1956), 5-17.

[2] Berestycki H.; Nirenberg L.; *On the method of moving planes and the sliding method*, Bol. Soc. Brasil. Mat. **22** (1991), 1-37.

[3] Birindelli, I.; Cutri, Al., *A semi-linear problem for the Heisenberg Laplacian*, Rend. Sem. Mat. Univ. Padova 94 (1995), 137–153.

[4] Birindelli, I.; Prajapat, J.; *Nonlinear Liouville theorems in the Heisenberg group via the moving plane method*, Comm. Part. Diff. Equa. **24** (1999), no. 9-1, 1875-1890.

[5] Birindelli, I.; Capuzzo Dolcetta, I.; Cutri, A.; *A Liouville theorems for semilinear equations on the Heisenberg group*, Ann. Inst. H. Poin. Anal. Non Lin. **14** (1997), no. 3, 295-308.

[6] Bonfiglioli, A. ; Lanconelli, E. ; Uguzzoni F. ; *Stratified Lie Groups and Potential Theory for Their Sub-Laplacians*, Springer Monogr. Math.Springer, Berlin (2007).

[7] Bony, J.M.; Principe du maximum, inegalite de Harnack et unicite du probleme de Cauchy pour les operateurs elliptiques degeneres. Ann. Inst. Fourier (Grenoble) 19 (1969) 277-304.

[8] Caffarelli, L.; Silvestre, L., *An extension problem related to the fractional Laplacian*, Comm. in Part. Diff. Equa. **2** (2007), 1245-1260.

[9] Chen W.; Li C., *Classification of solutions of some nonlinear elliptic equations*, Duke Math. J. **63** (1991), 615–622.

[10] Chipot M.; Chlebík M.; Fila M.; Shafrir I.; *Existence of positive solutions of a semilinear elliptic equation in $\mathbb{R}^n_+$ with a nonlinear boundary condition*, J. Math. Anal. Appl. **223** (1998), 429–471.

[11] Cinti, E.; Tan, J.; *A nonlinear Liouville theorem for fractional equations in the Heisenberg group.* Preprint.

[12] Epstein, C.; Melrose, R.; Mendoza, G.; *Resolvent of the Laplacian on pseudoconvex domains*, Acta Math. bf 167 (1991), 1-106.

[13] Ferrari, F; Franchi, B; *Harnack inequality for fractional laplacians in Carnot groups*, preprint.

[14] Frank, R.; Gonzalez M.d.M; Monticelli, D.; Tan, J.; *Conformal fractional Laplacians on the Heisenberg group*, Advances in Mathematics, 270 (2015) 97–137.

[15] Garofalo, N; Lanconelli, E: *Existence and nonexistence results for semilinear equations on the Heisenberg group*, Indiana Univ. Math. J. **41** (1992), 71-98.

[16] Gidas B.; Ni W.-M.; Nirenberg L.; *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209-243.

[17] Gidas B., Spruck J. ; *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure App. Math. **35** (1982), 528–598.

[18] Gover A., Graham C.R.; *CR invariant powers of the sub-Laplacian*, J. Reine Angew. Math. **583** (2005),1–2,.

[19] Guillarmou, C. Sa Barreto, A. *Scattering and inverse scattering on ACH manifolds*, J. Reine Angew. Math. **622** (2008) 1–55.

[20] Hislop, P.; Perry, P.; Tang, S., *CR-invariants and the scattering operator for complex manifolds with boundary*, Anal. PDE **1** (2008) 197–227.

[21] Lanconelli E., Uguzzoni F., *Asymptotic behavior and non-existence theorems for semilinear Dirichlet problems involving critical exponent on unbounded domains of the Heisenberg group*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. **(8) 1** (1998), no. 1, 139-168.

[22] Li, Y.Y.; Zhang, L.; *Liouville-type theorems and Harnack-type inequalities for semilinear elliptic equations*, J. Anal. Math. **90** (2003), 27-87.

[23] Li, Y.Y.; Zhu, M.; *Uniqueness theorems through the method of moving spheres*, Duke Math. J. **80** (1995), 383-417.

[24] Jerison, D.S.; Lee, J.M.; *The Yamabe problem on CR manifolds*, J. Diff. Geo. **25**, (1987), 167–197.

[25] Serrin, J.; A symmetry problem in potential theory, Arch. Ration. Mech. Anal. **43** (1971), 304-318.

[26] Stein, E. M., *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, 43. Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, (1993).

[27] Uguzzoni, F., *A non-existence theorem for a semilinear Dirichlet problem involving critical exponent on halfspaces of the Heisenberg group*, NoDEA Nonlinear Diff. Equa. Appl. **6** (1999), 191-206.

[28] Xu, L; *Semi-linear Liouville theorems in the Heisenberg group via vector field methods*, J. Differential Equations **247** (2009), 2799-2820.

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