HARMONIC ANALYSIS TECHNIQUES IN SEVERAL COMPLEX VARIABLES
SU UN' APPLICAZIONE DELL' ANALISI ARMONICA REALE ALL'ANALISI COMPLESSA IN PIU' VARIABILI

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Abstract. We give a survey of recent joint work with E. M. Stein (Princeton University) concerning the application of suitable versions of the T(1)-theorem technique to the study of orthogonal projections onto the Hardy and Bergman spaces of holomorphic functions for domains with minimal boundary regularity.

Sunto. Questo resoconto offre una sintesi di una serie di recenti collaborazioni con E. M Stein (Princeton University) sull'applicazione del celeberrimo teorema T(1) allo studio delle proiezioni ortogonali sugli spazi di Hardy e di Bergman per funzioni olomorfe su domini dotati di minima regolarita' al bordo.

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1. Introduction

This is a summary of recent work [59]-[64] concerning the L^p-regularity properties of *orthogonal* projections (Bergman projection, Szegő projection) onto L^2-closed subspaces of *holomorphic functions* (Bergman space, holomorphic Hardy space) for bounded domains $D \subset \mathbb{C}^n$ with *minimal* boundary regularity. Regularity properties of the Szegő and Bergman projections, in particular L^p-regularity, have been the object of considerable

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interest for more that 40 years. When the boundary of the domain D is sufficiently smooth, decisive results were obtained in the following settings: *(a)*, when D is strongly pseudoconvex [49], [65]; *(b)*, when $D \subset \mathbb{C}^2$ and its boundary is of finite type [69], [78]; *(c)*, when $D \subset \mathbb{C}^n$ is convex and its boundary is of finite type [70], [72]; and *(d)*, when $D \subset \mathbb{C}^n$ is of finite type and its Levi form is diagonalizable [19]. Related results include [1], [4], [7], [8], [16], [29], [33]-[37], [39], [40], [38], [56], [78], [84], [85], [89].

It should be noted that several among these works depend on good estimates or explicit formulas for the Szegő or Bergman kernels. In our non-smooth setting these are unavailable and we have to proceed via a different framework, by pursuing a theory of singular integral operators with *holomorphic* kernel that blends the complex structure of the ambient domain with the Calderòn-Zygmund theory for singular integrals on non-smooth domains in $\mathbb{R}^{2n}$. Our present task is to highlight the main threads linking the various themes in [59]-[64] and convey a general idea of the methods of proof (and at times we will sacrifice technical detail in favor of a more streamlined exposition). While most of the proofs are deferred to [59]-[64], here we indicate references to the specific statements therein.

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2. The Szegő projection

2.1. **Motivation and context.** Our starting point is the seminal work by Calderòn [17], Coifman-McIntosh and Meyer [23] and David [25] on the $L^p(\Gamma)$-regularity of the classical Cauchy integral for a planar curve $\Gamma \subset \mathbb{C}$, in the situation when Γ is the boundary of a domain $D \subset \mathbb{C}$ (and we will write $\Gamma = bD$):

$$\mathcal{C}f(z) = \frac{1}{2\pi i}\int\limits_{w\in bD} f(w)\,\frac{dw}{w-z}, \quad z \in \overline{D}. \tag{2.1}$$

For $z \in bD$ we interpret (2.1) as a singular integral in the "principal value" sense, see [22, (1.1)]. The situation when bD is of class $C^{1,\alpha}$ (with $\alpha > 0$) can be easily reduced to the classical setting of the Hilbert transform operator [22, Section 1.1, Example 8]. However dealing with the case when bD is of class C^1 and more generally Lipschitz, required new ideas that ultimately led to the so-called "T(1)-theorem" technique [26] for a more general class of singular integral operators[1], and to applications to the study of analytic capacity [90] as well as the solution of the Vitushkin conjecture [91]. In the setting of higher dimension (that is, for a Lipschitz domain $D \subset \mathbb{R}^N$ with $N \geq 2$), the Cauchy integral and the related singular integral operators collectively known as *boundary layer potentials*[2] provide the solution to various boundary value problems for harmonic functions. Here we are especially interested in the *L^p-Dirichlet problem for harmonic functions*: given $u : bD \to \mathbb{R}$ with $u \in L^p(bD, d\sigma)$, where $d\sigma$ is the induced Lebesgue measure on bD, find $U : D \to \mathbb{R}$ such that

$$
(2.2) \qquad \begin{cases} U_{xx} + U_{yy} \; =: \; \Delta U(z) = 0, & \text{if } z \in D \\ \lim\limits_{z \to w} U(z) =: U^+(w) = u(w), & \text{if } w \in bD \\ \|\mathcal{N}(U)\|_{L^p(bD,d\sigma)} \leq \; C\|u\|_{L^p(bD,d\sigma)}\,, & \end{cases}
$$

where the limit that defines the boundary value U^+ is to be suitably interpreted (for instance, as a "non-tangential limit" [47, page 24]) and $\mathcal{N}(U)$ denotes the so-called "non-tangential maximal function" for U, see [47, page 13] and references therein. The solution of (2.2) can be expressed in terms of the aforementioned boundary layer potentials acting on the data u. As it turned out, the size of the p-range for which existence, uniqueness and L^p-regularity of the solution occur, is related to the size of the Lipschitz constant of the domain [47, Theorem 2.2.2]. (We may think of C^1- or smoother domains as having Lipschitz constant equal to zero.)

2.2. **Regularity of the Szegő projection: statement of the problem.** We are interested in the *holomorphic* analog of the problem (2.2) for domains with minimal regularity, which we presently recall in the situation when $D \subset \mathbb{C} \equiv \mathbb{R}^2$ (the planar setting). The L^p-Dirichlet problem for *holomorphic* functions on a planar domain $D \subset \mathbb{C}$ is stated as

[1]that include the the Cauchy integral (2.1) as a prototype.

[2]namely, the single layer potential and the double layer potential operators, see [47, Section 2.2] and references therein.

follows. Given $g : bD \to \mathbb{C}$ in $L^p(bD, d\sigma)$, find $G : D \to \mathbb{C}$ such that

$$
\begin{cases}
1/2(G_x - iG_y) \ =: \ \overline{\partial} G(z) = 0, & \text{if } z \in D \\
\lim\limits_{z \to w} G(z) =: G^+(w) = g(w), & \text{if } w \in bD \\
\|\mathcal{N}(G)\|_{L^p(bD,d\sigma)} \leq \ C\|g\|_{L^p(bD,d\sigma)}, &
\end{cases}
\tag{2.3}
$$

where we adopt the convention that $z \in D$ is expressed as $z = x + iy$. It is clear that if G solves (2.3) with data $g = u + iv$ then e.g., $U := \operatorname{Re} G$ solves (2.2) with data u. However, in contrast with the situation for (2.2), the natural data space for (2.3) is not $L^p(bD, d\sigma) + iL^p(bD, \sigma)$: it is instead the Hardy Space of holomorphic functions $H^p(D)$ (aka *Smirnov Class*)

$$
H^p(D) := \left\{ F \,\middle|\, \overline{\partial} F(z) = 0, z \in D, \ \sup_{\epsilon > 0} \int\limits_{z \in bD_\epsilon} |F(z)|^p d\sigma_\epsilon(z) < +\infty \right\}
$$

where $\{D_\epsilon\}_\epsilon$ is any family of (say, rectifiable) subdomains of D with $D_\epsilon \uparrow D$.

In fact $H^p(D)$ can be identified with a *proper* subspace of $L^p(bD, d\sigma) + iL^p(bD, d\sigma)$ which we denote $\mathcal{H}^p(bD, d\sigma)$. More precisely, we invoke the well-known fact that functions in $H^p(D)$ have non-tangential limits that belong to $L^p(bD, d\sigma)$, see [28] and [87], and then we identify $\mathcal{H}^p(bD, d\sigma)$ with the space $\{F^+ \mid F \in H^p(D)\}$. Returning to the Dirichlet problem for holomorphic functions, one thus needs $g = F^+$ for some $F \in H^p(D)$ and if this is the case then $G := F$ solves (2.3).

The holomorphic Hardy space $\mathcal{H}^p(bD, d\sigma)$ is a closed subspace of $L^p(bD, d\sigma)$ (a fact that can be seen e.g., by applying the Cauchy formula on small discs + the co-area formula [31]); thus for the exponent $p = 2$ the theory of Hilbert spaces grants the existence of a unique, *orthogonal* projection $\mathcal{S}$: $L^2(bD, d\sigma) \to H^2(bD, d\sigma)$, known as the *Szegő projection*, which is a singular integral operator characterized by the following properties:

$$
\mathcal{S}^2 = \mathcal{S}, \quad \mathcal{S}^* = \mathcal{S}, \quad \|\mathcal{S}\|_{L^2(bD,\sigma) \to L^2(bD,d\sigma)} = 1 \, .
$$

(Here $\mathcal{S}^*$ denotes the adjoint of $\mathcal{S}$ taken with respect to the inner product in $L^2(bD, d\sigma)$.)

These properties in particular indicate that the Szegő projection is the natural solution operator for (2.3) in the case when $p = 2$. On the other hand, solving (2.3) in $L^p(bD, d\sigma)$ for $p \neq 2$ is a much harder problem, and one that is ultimately related to the

L^p-Regularity problem for $\mathcal{S}$: *under* minimal *assumptions on* D*, find the* maximal *exponent* $P = P(D) \in [2, +\infty]$ *such that* $\mathcal{S} : L^p(bD, d\sigma) \to L^p(bD, d\sigma)$ *is bounded* for all $P' < p < P$.

By symmetry considerations (the fact that $\mathcal{S}^* = \mathcal{S}$) we have that P and P' must be conjugate exponents (namely, $1/P + 1/P' = 1$).

We point out that the problem (2.3) and the L^p-regularity problem for $\mathcal{S}$ can also be stated in higher dimension, that is for $D \subset \mathbb{C}^n$: in this setting the quantity $\overline{\partial}G$ is interpreted as a differential form of type $(0,1)$ and the condition that $\overline{\partial}G = 0$ is then equivalent to the requirement that

$$G_{x_j} - iG_{y_j} = 0 \quad \text{for all} \quad j = 1, \ldots, n$$

where $x_j + iy_j = z_j$ with $j = 1, \ldots n$ and $z = (z_1, \ldots, z_n) \in D$. (Everything else in (2.3), also the definitions pertaining the Hardy spaces $\mathcal{H}^p(bD, d\sigma)$ and the L^p-regularity problem for $\mathcal{S}$, are meaningful regardless of the size of the dimension[3]).

2.3. **Regularity of the Szegő projection: case of planar domains.** It turns out that in complex dimension $n = 1$, that is for a bounded and simply connected domain $D \subset \mathbb{C}$, the size of the maximal interval (P', P) is related to the boundary regularity of D. Specifically, we have the following results:

1. If $D \subset \mathbb{C}$ is Vanishing Chord-Arc (e.g., D of class C^1), then $P = +\infty$, [59, Theorem 2.1 (1)] (see also [57]).
2. If $D \subset \mathbb{C}$ is Lipschitz with constant M, then
$$P = 2\left(1 + \frac{\pi}{2 \arctan M}\right) > \ 4\,,$$
and the interval determined by such P is optimal within the Lipschitz category, [59, Theorem 2.1 (2), and page 69].
3. If $D \subset \mathbb{C}$ is a rectifiable local graph, then $P = 4$, [59, Theorem 2.1 (3)].
4. If $D \subset \mathbb{C}$ is Ahlfors-Regular, then $P = 2 + \epsilon$ for some $\epsilon = \epsilon(D) > 0$, [59, Theorem 2.1 (4)].

[3]A more general version of these problem can be stated in which the data g is a differential form of degree $0 \le r \le 2n - 1$ and includes (2.3) as the special case: $r = 0$, but we will not pursue it here.

5. There is a rectifiable domain $D_0 \subset \mathbb{C}$, see [9], such that

$$\mathcal{S} : L^p(bD_0, \sigma) \to L^p(bD_0, \sigma) \iff p = 2.$$

The methods of proof for all these results rest on the existence of a conformal map $\psi : \mathbb{D}_1(0) = \{|z| < 1\} \to D$ (namely, the original problem for $\mathcal{S}$ is reduced to a *weighted* problem for $\mathcal{S}_0$= the Szegő projection of $\mathbb{D}_1(0)$ with weight $\omega = |\psi'|^{1-p/2}$ to which one may apply the theory of Muckenhoupt [88]) and thus are not applicable to higher dimension that is, to the situation when $D \subset \mathbb{C}^n$ and $n \geq 2$.

On the other hand, item 1. can also be studied via a conformal map-free argument that relies on the comparison of $\mathcal{S}$ with the Cauchy integral $\mathcal{C}$. We point out that for the Cauchy integral boundedness in L^2 *implies* boundedness in L^p for $1 < p < \infty$, see [88], and so in general we have $\mathcal{S} \neq \mathcal{C}$.

The approach to the analysis of the Szegő projection that we are about to describe was first formulated for the case when $D \subset \mathbb{C}$ is smooth, see [49] and [50], and the comparison of $\mathcal{C}$ and $\mathcal{S}$ hinged on the following facts:

(a.) Each of $\mathcal{C}$ and $\mathcal{S}$ is a *projection*[4]: $L^2(bD, d\sigma) \to \mathcal{H}^2(bD, d\sigma)$.

(b.) $\mathcal{S}$ is self-adjoint while $\mathcal{C}$ (in general) is not[5].

(c.) Each of $\mathcal{C}$ and $\mathcal{S}$ is *bounded*: $L^2(bD, d\sigma) \to L^2(bD, d\sigma)$.

(d.) The kernel of the operator $\mathcal{A}^{(\sigma)} := \mathcal{C}^* - \mathcal{C}$, where $\mathcal{C}^*$ denotes the formal adjoint of $\mathcal{C}$ in $L^2(bD, d\sigma)$, is *"small"* if D is sufficiently smooth (a cancellation of singularities occurs by performing a second-order Taylor expansion at $w = z$).

Then one has the following identities on $L^2(bD, d\sigma)$:

$$\mathcal{S}\mathcal{C} = \mathcal{C} \quad \text{and} \quad \mathcal{C}\mathcal{S} = \mathcal{S}\,, \quad \text{by item (a.)}\,.$$

Taking $L^2(bD, d\sigma)$-adjoints of the second identity above, we get

$$\mathcal{S}\mathcal{C}^* = \mathcal{S}\,, \quad \text{by item (b.)}.$$

[4]that is, $\mathcal{S} \circ \mathcal{S} = \mathcal{S}$ and $\mathcal{C} \circ \mathcal{C} = \mathcal{C}$.

[5]unless D is a disc, see [50].

Subtracting the first of the two identities above from the latter we obtain

$$\mathcal{S}\,[I - \mathcal{A}^{(\sigma)}] = \mathcal{C} \quad \text{in} \quad L^2(bD, d\sigma) \tag{2.4}$$

where I denotes the identity operator: $L^2(bD, d\sigma) \to L^2(bD, d\sigma)$. Now using `(c.)` and the fact that $(\mathcal{A}^{(\sigma)})^* = -\mathcal{A}^{(\sigma)}$ (recall that $\mathcal{A}^{(\sigma)} = \mathcal{C}^* - \mathcal{C}$) it is not hard to see that the operator $I - \mathcal{A}^{(\sigma)}$ is invertible on $L^2(bD, d\sigma)$ with bounded inverse, and we conclude that the identity

$$\mathcal{S} = \mathcal{C}\,[I - \mathcal{A}^{(\sigma)}]^{-1} \quad \text{holds in} \quad L^2(bD, d\sigma). \tag{2.5}$$

However, by item `(d.)` (which holds if D is smooth) the operator $\mathcal{A}^{(\sigma)}$ is in fact compact in $L^p(bD, d\sigma)$ for $1 < p < \infty$, and by the closed graph theorem it follows that $I - \mathcal{A}^{(\sigma)}$ is invertible in $L^p(bD, d\sigma)$ with bounded inverse, see [59, page 65]. It follows that the right-hand side of (2.5) is a well-defined and bounded operator: $L^p(bD, d\sigma) \to L^p(bD, d\sigma)$ for $1 < p < \infty$, and we conclude from the above that $\mathcal{S}$ extends to a bounded operator on $L^p(bD, d\sigma)$ for $1 < p < \infty$, thus solving the L^p-regularity problem for $\mathcal{S}$ with $P = \infty$, whenever $D \subset \mathbb{C}$ is smooth.

We remark that the steps `(a.)` – `(d.)` can be stated for *any* positive boundary measure μ (not just the induced Lebesgue measure σ) provided the orthogonal projection $\mathcal{S} \equiv \mathcal{S}_\mu$ is defined with respect to the duality induced by the measure μ, namely

$$(f, g) = \int\limits_{w \in bD} f(w)\overline{g}(w)\, d\mu(w).$$

2.4. **Regularity of the Szegő projection: dimension-induced obstructions.** The procedure described in the previous section is, in principle, dimension-free in the sense that it relies on the existence of "some" operator $\mathcal{C}$ that satisfies the four conditions `(a.)` through `(d.)`. In the setting of Section 2.3 (that is when $D \subset \mathbb{C}$ and D is sufficiently smooth) one takes $\mathcal{C}$ to be the Cauchy integral (2.1), and the proof of the crucial item `(a.)` then rests on the following two features of $\mathcal{C}$:

- `(i.)` The fact that *Cauchy kernel* $C(w, z)$ (that is the kernel of $\mathcal{C}$) is *universal* in the sense that its dependence on the domain D is effected only through the inclusion
$$j : bD \hookrightarrow \mathbb{C}\,.$$

Specifically, we have

$$C(w,z)=\frac{1}{2\pi i}\,j^*\!\left(\frac{dw}{w-z}\right)\qquad w,z\in\mathbb{C}\times\mathbb{C},\quad w\neq z\,,\tag{2.6}$$

where j^* is the so-called *pull-back by the inclusion map*, see e.g., [82, Section III.1.5].

(ii.) The fact that the *Cauchy kernel function* $1/(w-z)$ is (obviously) *holomorphic* in the parameter $z\in D$ whenever $w\in\mathbb{C}\setminus\overline{D}$, in particular for each fixed $w\in bD$.

In higher dimension both of these properties become highly problematic as the only known *universal* reproducing kernel is the *Bochner-Martinelli* kernel:

$$H(w,z)=\frac{(n-1)!}{(2\pi i)^n}\sum_{j=1}^{n} j^*\!\left(\frac{\overline{w}_j-\overline{z}_j}{|w-z|^{2n}}\,dw_j\bigwedge_{\nu\neq j} d\overline{w}_\nu\wedge dw_\nu\right),\quad w,z\in\mathbb{C}^n\times\mathbb{C}^n\,,w\neq z,\tag{2.7}$$

see e.g., [82, Lemma IV.1.5 (a)]. It is clear that $H(w,z)=C(w,z)$ when $n=1$, because in such case the coefficient in $H(w,z)$ is just $(\overline{w}-\overline{z})/|w-z|^2=1/(w-z)$.

On the other hand, when $n\geq 2$ the coefficients of the kernel (2.7) are obviously nowhere holomorphic, thus $H(w,z)$ is of no use in the analysis of the Szegő projection described in the previous section[6]: there is no canonical, higher dimensional *holomorphic* analog of the Cauchy kernel (2.6). Instead, one has to look into ad-hoc constructions that are tailored to certain specific features of the domain. More precisely, the existence of a higher-dimensional holomorphic analog of $C(w,z)$ is intimately connected with a geometric constraint on the domain, namely the requirement that D be *pseudoconvex* [82, Section II.2.10] or, equivalently, that D be a so-called *weak (or local) domain of holomorphy* [82, Section II.2.1]: for any $w\in bD$ there must be a function $f_w(z)$ that is holomorphic in $z\in D$ but cannot be extended holomorphically past w. While any planar domain $D\subset\mathbb{C}$ is obviously a weak domain of holomorphy[7], there are domains $D\subset\mathbb{C}^n$ ($n\geq 2$) with the property that *any* function holomorphic in D can be holomorphically extended to a larger domain $\Omega\supset D$ [82, Lemma II.2.2].

[6] The Bochner Martinelli integral for a general domain D cannot satisfy item (a.).

[7] for any $w\in bD$, simply take $f_w(z)$ to be the Cauchy kernel function, i.e. $f_w(z)=1/(w-z)$, $z\in D$.

2.5. **Regularity of the Szegő projection in higher dimension: the case of smooth domains.** The Cauchy-Fantappiè theory (see [61, Section 4] and references therein) provides an algebraic framework to construct explicit, higher dimensional holomorphic analogues of the Cauchy kernel (2.6) for any bounded, *strongly* (equivalently, *strictly*) pseudoconvex domain $D \subset \mathbb{C}^n$, see [82, Section II.2.8]. The kernel construction and the proof of the corresponding conditions `(a.)` through `(d.)` were first carried out in [42], [49] and [81] and dealt with the case when the strongly pseudoconvex domain D is smooth (of class C^3 or better). In this section we describe the construction in such setting (see also [48] and [61]).

For D strongly pseudoconvex we write $D = \{\rho(z) < 0\}$ where $\rho : \mathbb{C}^n \to \mathbb{R}$ is a strictly plurisubharmonic *defining function* for D (see [82, Sections II.2.3 and II.2.7]), which is taken to be of class C^3 or better. For fixed $z \in D$, we consider the following differential form of type $(1, 0)$ in the variable w

$$\eta(w, z) = \sum_{j=1}^{n} \eta_j(w, z)\, dw_j$$

where we have set

$$\eta_j(w, z) = \chi_0(w, z)\left(\frac{\partial \rho}{\partial \zeta_j}(w) - \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 \rho(w)}{\partial \zeta_i \partial \zeta_j}(w_i - z_i)\right) + (1 - \chi_0(w, z))(\overline{w}_j - \overline{z}_j) \tag{2.8}$$

with χ_0 a smooth cutoff function supported in $\{|w - z| < \delta\}$.
Now $\eta(w, z)$ is a *generating form at* w in the sense that the complex-valued function of z

$$\langle \eta(w, z), w - z\rangle := \sum_{j=1}^{n} \eta_j(w, z)\,(w_j - z_j)$$

is bounded below by $|w - z|^2$ for any $z \in \overline{D}$, see [61, Section 4]. More precisely we have

$$\mathrm{Re}\langle \eta(w, z), w - z\rangle \geq c|w - z|^2, \quad w \in bD,\ z \in \overline{D}\,. \tag{2.9}$$

(We point out that the validity of this inequality when z is close to w is a consequence of the strict plurisubharmonicity of ρ, see [61, Lemma 4].) The Cauchy-Fantappiè theory then grants that the kernel

$$\widetilde{C}(w, z) = \frac{1}{(2\pi i)^n}\frac{\eta \wedge (\overline{\partial}_w \eta)^{n-1}(w, z)}{\langle \eta(w, z), w - z\rangle^n} \tag{2.10}$$

reproduces holomorphic functions (more precisely, the induced singular integral fixes the space $\mathcal{H}^2(bD, d\sigma)$) and it is clear that $\widetilde{C}(w,z)$ satisfies property `(b.)`, see [61, Section 4] and references therein[8]. On the other hand, it is apparent from (2.8) that, as a function of $z \in D$, this kernel is holomorphic only for z near $w \in bD$. Thus, in order to achieve the crucial condition `(a.)` one needs to modify $\widetilde{C}(w,z)$ by adding a correction term that will make the kernel globally holomorphic:

$$C(w,z) \;=\; \widetilde{C}(w,z) \;+\; C_\rho(w,z)\,. \tag{2.11}$$

The correction $C_\rho(w,z)$ is obtained either by solving a $\overline{\partial}$-problem (in the z-variable) on a strongly pseudoconvex, smooth domain Ω that contains $\overline{D}$, see [49] and [61, Section 8], or by solving a Cousin problem as in [42] and [81]. The resulting kernel (2.11) will be globally holomorphic and the corresponding operator (still denoted $\mathcal{C}$) will satisfy properties `(a.)` and `(b.)`.

We point out that the procedure described up to this point can be carried out under the weaker assumption that the domain (that is the defining function ρ) be of class C^2 [82, Section V.1.1]: it was in order to prove the remaining properties `(c.)` and `(d.)` that one needed more regularity. Specifically, in the setting of [49] one needed to assume that ρ be of class C^3, and the proof of item `(c.)` (the $L^2(bD, d\sigma)$-regularity of the holomorphic Cauchy integral $\mathcal{C}$) could then be achieved via an "osculation by model domain" technique. The basic idea is that there is a strongly pseudoconvex and smooth "model" domain D_0 for which the operator $\mathcal{C}_0$ constructed as in (2.10) and (2.11) takes an especially simple form, and the validity of property `(c.)` for such a $\mathcal{C}_0$ is easily verified by direct inspection[9]. On the other hand, if D is strongly pseudoconvex and of class C^3 then at any boundary point it is osculated by a copy of D_0 with small error. Furthermore, one may write the operator $\mathcal{C}$ (for the original domain D) as the sum of $\mathcal{C}_0$ (the corresponding operator for the model domain D_0) plus the "error" operator $\mathcal{C} - \mathcal{C}_0$, and if D is of class C^3 the error operator is easily seen to be bounded, thus concluding the proof of `(c.)`. Finally, a

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[8]Roughly speaking, one wants $C(w,z) \neq \overline{C(z,w)}$, which is the case whenever $D \neq \{|z|^2 < 1\}$.

[9]In fact D_0 is the *Siegel upper half space*: $\{z = (z', x_n + iy_n) \in \mathbb{C}^n \mid |y_n| > |z'|^2\}$, where $z' = (z_1, \ldots, z_{n-1})$.

2nd-order Taylor expansion of ρ in the variable z about the point w shows that the kernel of $\mathcal{A}^{(\sigma)} = \mathcal{C}^* - \mathcal{C}$ has the "smallness" property `(d.)` whenever D is of class C^3 (which allows for good control on the tail of the expansion).

Having constructed an operator $\mathcal{C}$ that satisfies the four properties `(a.)` through `(d.)`, one proceeds as in Section 2.3 to conclude that the $L^p(bD, d\sigma)$-regularity problem for $\mathcal{S}$ is solved with $P = \infty$, whenever D is strongly pseudoconvex and of class C^3, see [48] and [49].

We point out that the methods of proof for items `(c.)` and `(d.)` as described above break down as soon as the regularity of D is below the class C^3. (The "error" operator $\mathcal{C} - \mathcal{C}_0$ that occurred the proof of `(c.)` can no longer be controlled, whereas for `(d.)` there is no control on the size of the tail in the aforementioned Taylor expansion.)

2.6. **Higher dimensional holomorphic kernels for non-smooth domains: kernel construction.** We now describe the results in [63]. As we have seen, a natural requirement for the existence of a holomorphic Cauchy-type kernel (2.11) is that the domain be strongly pseudoconvex, which is a condition that essentially involves two degrees of differentiability of the boundary of the domain. As a result, the threshold of smoothness for a strongly pseudoconvex domain should be the class C^2 (as opposed to the class C^1 for a planar domain): as before, we take ρ to be a strictly plurisubharmonic defining function for D, however now ρ is merely of class C^2. To make up for the lack of differentiability of those second derivatives of ρ that occurred in the definition of the generating form η, see (2.8), we "borrow some regularity" by considering families of functions $\{\tau_{i,j}^{(\epsilon)}\}_\epsilon$ of class C^2 such that

$$\sup_{w\in bD}\left|\frac{\partial^2\rho(w)}{\partial\zeta_i\partial\zeta_j} - \tau_{i,j}^{(\epsilon)}(w)\right| < \epsilon\,, \quad i,j=1,\ldots,n \tag{2.12}$$

for any $0<\epsilon\leq\epsilon_0$, where the size of ϵ_0 is determined by the the strict plurisubharmonicity of ρ, see [82, (2.26)].

One then sets

$$\eta^{(\epsilon)}(w,z) = \sum_{j=1}^n \eta_j^{(\epsilon)}(w,z)\,dw_j$$

with

$$\eta_j^{(\epsilon)}(w,z) = \chi_0(w,z)\bigg(\frac{\partial\rho}{\partial\zeta_j}(w) - \tau_{i,j}^{(\epsilon)}(w)(w_i - z_i)\bigg) + (1-\chi_0(w,z))(\overline{w}_j - \overline{z}_j) \tag{2.13}$$

where χ_0 is again a smooth cutoff function supported in $\{|w-z| < \delta\}$.

It follows from (2.12) and (2.9) that

$$\mathrm{Re}\langle\eta^{(\epsilon)}(w,z), w-z\rangle \geq c|w-z|^2 \tag{2.14}$$

and also

$$\big|\langle\eta^{(\epsilon)}(w,z), w-z\rangle\big| \approx |\langle\eta(w,z), w-z\rangle| \tag{2.15}$$

whenever $z \in \overline{D}$ and $w \in bD$, and for any $0 < \epsilon \leq \epsilon_0$, with the constant c in (2.14) and the implied constants in (2.15) independent of ϵ, see [63, Part I]. It follows from (2.14) that, as was the case for η in the previous section, each $\eta^{(\epsilon)}$ is a generating form (for any $0 < \epsilon < \epsilon_0$). Thus, the resulting Cauchy-Fantappiè kernels

$$\widetilde{C}^{(\epsilon)}(w,z) = \frac{1}{(2\pi i)^n}\frac{\eta^{(\epsilon)}\wedge(\overline{\partial}_w\eta^{(\epsilon)})^{n-1}(w,z)}{\langle\eta^{(\epsilon)}(w,z), w-z\rangle^n} \tag{2.16}$$

have the reproducing property `(a.)` but as before, are only locally holomorphic (for $z \in D$ near $w \in bD$). To achieve global holomorphicity one again has to solve a $\overline{\partial}$-problem to produce suitable correction terms:

$$C^{(\epsilon)}(w,z) = \frac{1}{(2\pi i)^n}\frac{\eta^{(\epsilon)}\wedge(\overline{\partial}_w\eta^{(\epsilon)})^{n-1}(w,z)}{\langle\eta^{(\epsilon)}(w,z), w-z\rangle^n} + C_\rho^{(\epsilon)}(w,z). \tag{2.17}$$

What matters here is that each of the corrections $C_\rho^{(\epsilon)}(w,z)$ satisfies a uniform bound which is independent of ϵ, namely

$$\sup_{(w,z)\in bD\times\overline{D}} |C_\rho^{(\epsilon)}(w,z)| \leq C \quad \text{for any} \quad 0 < \epsilon \leq \epsilon_0.$$

We let $\{\mathcal{C}_\epsilon\}_\epsilon$ denote the resulting family of (globally) holomorphic Cauchy-type integral operators. It is clear from the above that each $\mathcal{C}_\epsilon$ satisfies conditions `(a.)` and `(b.)` in Section 2.3, for any $0 < \epsilon < \epsilon_0$.

2.7. $L^2(bD)$**-regularity of the** $\mathcal{C}_\epsilon$**'s: preliminary observations.** It turns out that the "borrowed regularity" (2.12) is not good enough to prove L^2-boundedness of the $\mathcal{C}_\epsilon$'s by the "osculation by model domain" method that was described in Section 2.5 (there is a problem with controlling the error $\mathcal{C}_\epsilon - \mathcal{C}_0$, so regularity for the $\mathcal{C}_\epsilon$'s cannot be deduced from the corresponding result for $\mathcal{C}_0$) and we must proceed by a different route, namely, by the "$T(1)$-theorem technique". To this end, we make a number of preliminary observations.

• Our first observation [63, Part I] is that there is an ad-hoc measure for bD, which we call *the Leray-Levi measure* λ, that is better suited to study the $\mathcal{C}_\epsilon$'s than the induced Lebesgue measure σ.

More precisely, we set

$$d\lambda(w) = (2\pi i)^{-n} j^*\big(\partial\rho \wedge (\overline{\partial}\partial\rho)^{n-1}\big)(w), \quad w \in bD.$$

Then in fact

$$d\lambda(w) = \Lambda(w) d\sigma(w)$$

with

$$\Lambda(w) = (n-1)!(4\pi)^{-n} |\det \rho(w)|\, |\nabla\rho(w)|, \quad w \in bD, \tag{2.18}$$

where $\det \rho(w)$ is the determinant of the so-called "Levi form for D", which may be identified[10] with the matrix

$$\left\{ \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} \right\} \Bigg|_{z=w}, \quad 1 \leq j,k \leq n-1\,,$$

see [82, Lemma VII.3.9]. Since D is strongly pseudoconvex and of class C^2 it follows that

$$c_1 \leq \Lambda(w) \leq c_2\,, \quad w \in bD,$$

so that $\lambda \approx \sigma$ (the two measures are mutually absolutely continuous) and thus the $\mathcal{C}_\epsilon$'s will be equivalently bounded with respect to either measure.

• Secondly, we have that the function

$$\mathtt{d}(w,z) = |\langle \eta(w,z), w - z \rangle|^{1/2}\,, \quad w, z \in bD$$

[10] here we have chosen a local coordinate system with respect to which the complex tangent space to bD at w is identified with the space $\{(z_1, \ldots, z_{n-1}, 0) \,|\, z_j \in \mathbb{C}\}$.

is a *quasi-distance*, namely, for any $w, z, \zeta \in bD$ we have $\mathtt{d}(w,z) = 0 \iff w = z$; $\mathtt{d}(w,z) \approx \mathtt{d}(z,w)$, and $\mathtt{d}(w,z) \leq C\big(\mathtt{d}(w,\zeta) + \mathtt{d}(\zeta,z)\big)$, see [63, Part I].

• Moreover we have that the ensemble $X = \{bD;\, \mathtt{d};\, \lambda\}$ defines a *space of homogeneous type* with *homogeneous dimension* $2n$, see [63, Part I]. That is, we have that λ is a doubling measure for the boundary balls $\mathbf{B}_r(w) = \{z \in bD \,|\, \mathtt{d}(z,w) \leq r\}$ and in fact

$$\lambda\big(\mathbf{B}_r(w)\big) \approx r^{2n} \quad \text{for any } w \in bD \quad \text{and } r > 0.$$

2.8. $L^2(bD, d\lambda)$**-regularity of the** $\mathcal{C}_\epsilon$**'s: ad-hoc decompositions; the role of the Leray-Levi measure; application of** $T(1)$**.** In order to take full advantage of the measure λ we make the following decomposition of each of the $\mathcal{C}_\epsilon$'s, see [63, Part I]:

$$\mathcal{C}_\epsilon = \mathcal{C}^\sharp_\epsilon + \mathcal{R}_\epsilon\,.$$

Here the "essential part" $\mathcal{C}^\sharp_\epsilon$ has kernel

$$C^\sharp_\epsilon(w,z) \;=\; \frac{d\lambda(w)}{\langle \eta^{(\epsilon)}(w,z), w - z\rangle^n} \;=\; \frac{1}{(2\pi i)^n}\frac{j^*(\partial\rho \wedge (\overline{\partial}\partial\rho)^{n-1}(w)}{\langle \eta^{(\epsilon)}(w,z), w - z\rangle^n}$$

and captures the full singularity of $\mathcal{C}_\epsilon$ in the sense that the "remainders" $\mathcal{R}_\epsilon$'s are smoothing operators that map: $L^2(bD) \to C(\overline{D})$, so in particular proving $L^2(bD)$-regularity for $\mathcal{C}_\epsilon$ is equivalent to proving the corresponding result for $\mathcal{C}^\sharp_\epsilon$ (and we will henceforth ignore the $\mathcal{R}_\epsilon$'s).

Now there is a further decomposition of $\mathcal{C}^\sharp_\epsilon$, and a corresponding one for its formal adjoint on $L^2(bD, d\lambda)$ that will play an important role in the application of the $T(1)$-theorem. The basic idea is that one may express the kernels of each of $\mathcal{C}^\sharp_\epsilon$ and its $L^2(bD, d\lambda)$-adjoint $(\mathcal{C}^\sharp_\epsilon)^*$ as "appropriate derivatives" (plus acceptable remainders) that is, as the differentials of $(2n-2)$-forms whose coefficients have better homogeneity than the kernels of each of $\mathcal{C}^\sharp_\epsilon$ and $(\mathcal{C}^\sharp_\epsilon)^*$; the desired decompositions will then result by an application of Stokes' theorem. To put matters more precisely, given $f \in C^1(bD)$ we have [63, Part I]

$$\mathcal{C}^\sharp_\epsilon(f)(z) = \mathcal{E}_\epsilon(df)(z) + \mathcal{R}^\sharp_\epsilon(f)(z), \quad \text{for } z \in bD$$

and

$$(\mathcal{C}^\sharp_\epsilon)^*(f)(z) = \widetilde{\mathcal{E}}_\epsilon(df)(z) + \widetilde{\mathcal{R}^\sharp_\epsilon}(f)(z), \quad \text{for } z \in bD\,.$$

Here the "essential parts" $\mathcal{E}_\epsilon$ and $\widetilde{\mathcal{E}}_\epsilon$ act on continuous 1-forms ω on bD as follows:

$$\mathcal{E}_\epsilon(\omega)(z) = c_n \int_{w\in bD} \frac{\omega \wedge j^*(\overline{\partial}\partial\rho)^{n-1}(w)}{\langle \eta^{(\epsilon)}(w,z), w-z\rangle^{n-1}}, \quad z \in bD,$$

with $c_n = 1/[(n-1)(2\pi i)^n]$. Comparing $\mathcal{E}_\epsilon$ with $\mathcal{C}^\sharp_\epsilon$ we see that the kernel of $\mathcal{E}_\epsilon$ has the improved homogeneity $\langle \eta^{(\epsilon)}(w,z), w-z\rangle^{-n+1}$ (as opposed to $\langle \eta^{(\epsilon)}(w,z), w-z\rangle^{-n}$). Similarly, we have

$$\widetilde{\mathcal{E}}_\epsilon(\omega)(z) = c_n \int_{w\in bD} \frac{\omega \wedge j^*(\overline{\partial}\partial\rho)^{n-1}(w)}{\langle \overline{\eta}^{(\epsilon)}(z,w), \overline{z}-\overline{w}\rangle^{\,n-1}}, \quad z \in bD.$$

In both decompositions, the remainders $\mathcal{R}^\sharp_\epsilon$ and $\widetilde{\mathcal{R}^\sharp_\epsilon}$ are once again smoothing operators: $L^2(bD, d\lambda) \to C(\overline{D})$. We point out that the decomposition for $(\mathcal{C}^\sharp_\epsilon)^*$ takes full advantage of the measure $d\lambda$, in the sense that the decomposition for the adjoint of $\mathcal{C}^\sharp_\epsilon$ is valid only if the adjoint is computed with respect to the duality for $L^2(bD, d\lambda)$ (there is no such decomposition for the adjoint of $\mathcal{C}^\sharp_\epsilon$ in $L^2(bD, d\sigma)$).

Using these decompositions one then shows that the functions

$$\mathfrak{h} = \mathcal{C}^\sharp_\epsilon(1) \quad \text{and} \quad \mathfrak{h}^* = (\mathcal{C}^\sharp_\epsilon)^*(1) \tag{2.19}$$

are continuous on bD, see [63, Part I].

The properties of the homogenous space $\{bD, \mathtt{d}, \lambda\}$ along with the above decompositions for $\mathcal{C}^\sharp_\epsilon$ and its $L^2(bD, d\lambda)$-adjoint $(\mathcal{C}^\sharp_\epsilon)^*$ now ensure that for any $0 < \epsilon \le \epsilon_0$, the operators $T := \mathcal{C}^\sharp_\epsilon$ satisfy all the hypotheses of the $T(1)$-theorem, namely: the kernel of $\mathcal{C}^\sharp_\epsilon$ satisfies appropriate size and regularity estimates; the operator $\mathcal{C}^\sharp_\epsilon$ is weakly bounded and satisfies the cancellation conditions[11] (2.19), see [63, Part I]. This concludes the proof of the $L^p(bD, d\lambda)$-regularity of $\mathcal{C}^\sharp_\epsilon$ and therefore the proof of item `(c.)` (see Section 2.3) for each of the operators $\mathcal{C}_\epsilon$'s.

[11]In fact $\mathfrak{h}$ and $\mathfrak{h}^*$ can be proved to be be Hölder continuous in the sense that e.g., $|\mathfrak{h}(w) - \mathfrak{h}(z)| \le C\mathtt{d}(w,z)^\alpha$, $w, z \in bD$ for any $0 < \alpha < 1$, and this in turn allows to reduce the application of the $T(1)$ theorem to the simpler setting: $T_0(1) = 0 = T_0^*(1)$ for a suitable auxiliary operator T_0, see [62, Section 6.3] and also [63, Part I].

2.9. $L^p(bD, d\lambda)$**-regularity of the Szegő projection (Leray-Levi measure).** The notion of orthogonal projection (in particular the definition of the Szegő projection) relies on the specific measure that is being used in the definition of $L^2(bD)$: different measures give rise to different orthogonal projections and there no simple way of deducing regularity for one projection from the corresponding result for the other. In this section we highlight the procedure carried out in [63, Part II] to solve the $L^p(bD)$-regularity problem for the Szegő projection defined with the respect to the Leray-Levi measure and its corresponding duality on $L^2(bD)$:

$$(f, g) = \int\limits_{w \in bD} f(w)\overline{g}(w)\, d\lambda(w). \tag{2.20}$$

We will denote such projection $\mathcal{S}_\lambda$. What is still missing from the procedure summarized in Section 2.3 is item `(d.)`, namely the "smallness" of the operators $\mathcal{A}_\epsilon^{(\lambda)} = \mathcal{C}_\epsilon^* - \mathcal{C}_\epsilon$, where the adjoint $\mathcal{C}_\epsilon^*$ is computed with respect to the duality (2.20). Going back to the setting of [49] (D of class C^3) such smallness resulted from the following estimate for the kernel of $\mathcal{A}^{(\sigma)}$ (denoted below by $A^{(\sigma)}(w, z)$)

$$|A^{(\sigma)}(w, z)| \le C|w - z|^{2+\beta} \quad \text{whenever} \quad \mathtt{d}(w, z) \le \delta \tag{2.21}$$

for some $\beta > 0$ (in fact for $\beta = 1$), which ultimately gave the compactness of $\mathcal{A}^{(\sigma)}$ in $L^p(bD, d\sigma)$: one considered the operators $\{\mathcal{A}_\delta^{(\sigma)}\}_\delta$ with kernels

$$\big(1 - \chi_\delta(|w - z|^2)\big)\, A^{(\sigma)}(w, z)$$

for a smooth cutoff function $\chi_\delta(t)$. Such operators are obviously compact in $L^p(bD, d\sigma)$ for $1 < p < \infty$. Now the estimate (2.21) grants $\|\mathcal{A}_\delta^{(\sigma)} - \mathcal{A}^{(\sigma)}\|_{L^p \to L^p} \le C\delta^\beta$, and the compactness of $\mathcal{A}^{(\sigma)}$ then follows by letting $\delta \to 0$.

Note that the positivity of β in (2.21) is crucial: the estimate $|A^{(\sigma)}(w, z)| \le C\mathtt{d}(w, z)^2$ would only yield the inconclusive inequality $\|\mathcal{A}_\delta^{(\sigma)} - \mathcal{A}^{(\sigma)}\|_{L^p \to L^p} \le C$. However in our less regular setting there is no analog of (2.21) with $\beta > 0$. In fact the operator $\mathcal{A}_\epsilon^{(\lambda)}$ will in general fail to be compact in $L^p(bD, \lambda)$, see [6, Corollary 5], and one must proceed by a different analysis. What holds in place of (2.21) is the following, "weaker" smallness for the kernels of the operators $\mathcal{A}_\epsilon^{(\lambda)}$ (denoted $A_\epsilon^{(\lambda)}(w,z)$), namely:

$$|A_\epsilon^{(\lambda)}(w,z)| \leq C\,\epsilon\,\mathtt{d}(w,z)^2 \quad \text{whenever} \quad \mathtt{d}(w,z) \leq \delta\,, \tag{2.22}$$

To use (2.22) we consider the operators $\mathcal{A}_{\epsilon,\delta}^{(\lambda)}$ with kernel

$$\chi_\delta(\mathtt{d}(w,z))\, A_\epsilon^{(\lambda)}(w,z)$$

where $A_\epsilon^{(\lambda)}(w,z)$ is the kernel of $\mathcal{A}_\epsilon^{(\lambda)}$ and $\chi_\delta(t)$ is a smooth cutoff function. Then we have that

$$\mathcal{A}_\epsilon^{(\lambda)} = \mathcal{A}_{\epsilon,\delta}^{(\lambda)} + \mathcal{R}_{\epsilon,\delta}^{(\lambda)}\,. \tag{2.23}$$

Now one may apply the $T(1)$-theorem (as in the previous section) to prove that the "essential part" $\mathcal{A}_{\epsilon,\delta}^{(\lambda)}$ is bounded: $L^p(bD,d\lambda)\to L^p(bD,d\lambda)$ for $1<p<\infty$. In fact (2.22) yields the improved estimate [63, Part II]

$$\|\mathcal{A}_{\epsilon,\delta}^{(\lambda)}\|_{L^p(bD,d\lambda)\to L^p(bD,d\lambda)} \leq \epsilon^{1/2}M_p, \quad 1<p<\infty \tag{2.24}$$

for any $0<\delta<\delta_0(\epsilon)$ and for any $0<\epsilon\leq\epsilon_0$, where the bound M_p is symmetric in p, i.e. $M_p = M_{p'}$ whenever $1/p+1/p'=1$.

On the other hand, the "remainder" operators $\mathcal{R}_{\epsilon,\delta}^{(\lambda)}$ (whose kernel are supported outside of the critical balls $\{\mathtt{d}(w,z)<\delta\}$) are readily seen to map: $L^1(bD,d\lambda)\to L^\infty(bD)$ (although their $L^p\to L^p$-norms may be very large).

We now proceed to compare $\mathcal{S}_\lambda$ with the Cauchy-type integrals $\mathcal{C}_\epsilon$. By items `(a.)` and `(b.)` (proved in Section 2.6) and proceeding as in Section 2.3, we recover the identity

$$\mathcal{S}_\lambda\,[I-\mathcal{A}_\epsilon^{(\lambda)}] = \mathcal{C}_\epsilon \quad \text{in } L^2(bD,d\lambda) \quad \text{for any} \;\; 0<\epsilon<\epsilon_0. \tag{2.25}$$

Combining the above with (2.23) we get

$$\mathcal{S}_\lambda\,[I-\mathcal{A}_{\epsilon,\delta}^{(\lambda)}] = \mathcal{C}_\epsilon - \mathcal{S}_\lambda\,\mathcal{R}_{\epsilon,\delta}^{(\lambda)} \quad \text{in } L^2(bD,d\lambda) \quad \text{for any} \;\; 0<\epsilon<\epsilon_0.$$

We now fix $1<p<\infty$ and prove $L^p(bD,d\lambda)$-regularity of $\mathcal{S}_\lambda$ for such p; for the time being we take $1<p<2$, so that the two inclusions: $L^p(bD,d\lambda)\hookrightarrow L^1(bD,d\lambda)$ and

$L^2(bD, d\lambda) \hookrightarrow L^p(bD, d\lambda)$ are bounded[12]. It follows that the operator

$$\mathcal{S}_\lambda \, \mathcal{R}^{(\lambda)}_{\epsilon,\delta} : L^p \hookrightarrow L^1 \to L^\infty \hookrightarrow L^2 \to L^2 \hookrightarrow L^p$$

is bounded for any $0 < \delta < \delta_0(\epsilon)$ and any $0 < \epsilon \leq \epsilon_0$ (here we have used the $L^1 \to L^\infty$-regularity of $\mathcal{R}^{(\lambda)}_{\epsilon,\delta}$ and the $L^2 \to L^2$-regularity of $\mathcal{S}_\lambda$). Moreover $\mathcal{C}_\epsilon$ is bounded: $L^p \to L^p$ for any $0 < \epsilon \leq \epsilon_0$ by item `(c.)` (which was proved in sections 2.7 and 2.8). We now fix $\epsilon = \epsilon(p) << 1$ so that

$$\epsilon^{1/2} M_p < 1\,,$$

where M_p is as in (2.24). Then by (2.24) we have that for any $\delta \leq \delta_0(\epsilon)$ the operator

$$I - \mathcal{A}^{(\lambda)}_{\epsilon,\delta}$$

is invertible in $L^p(bD, d\lambda)$ by a partial Neumann series, and has bounded inverse.

We conclude from the above that

$$\mathcal{S}_\lambda \, = \, [\mathcal{C}_\epsilon - \mathcal{S}_\lambda \, \mathcal{R}^{(\lambda)}] \, [I - \mathcal{A}^{(\lambda)}_{\epsilon,\delta}]^{-1} \quad \text{in} \quad L^2(bD, d\lambda). \tag{2.26}$$

However, by what has been said, the right-hand side of this identity is a bounded operator in $L^p(bD, d\lambda)$, thus showing that $\mathcal{S}_\lambda$ extends to a bounded operator in $L^p(bD, d\lambda)$ for any $1 < p \leq 2$. By duality (and the fact that $(\mathcal{S}_\lambda)^* = \mathcal{S}_\lambda$) it follows that $\mathcal{S}_\lambda$ is also bounded in $L^p(bD, d\lambda)$ for any $2 \leq p < \infty$. The L^p-regularity problem for $\mathcal{S}_\lambda$ is therefore solved with $P = \infty$, whenever D is a bounded, strongly pseudoconvex domain of class C^2.

2.10. **$L^p(bD, d\lambda)$-regularity of the Szegő projection: other measures.** We recall from Section 2.7 that the Leray-Levi measure λ and the induced Lebesgue measure σ are mutually absolutely continuous, see (2.18) and comments thereafter. It follows that the holomorphic Cauchy-type integrals $\{\mathcal{C}_\epsilon\}_\epsilon$ are equivalently bounded in $L^p(bD, d\lambda)$ and $L^p(bD, d\sigma)$ and, more generally in $L^p(bD, \varphi d\lambda)$ where φ is any continuous function on bD with uniform upper and lower bounds. On the other hand, if we denote the Szegő projection for $L^2(bD, \varphi d\lambda)$ by $\mathcal{S}_\varphi$, there is no direct way to compare $\mathcal{S}_\lambda$ with $\mathcal{S}_\varphi$ that would allow to deduce $L^p(bD, \varphi d\lambda)$-regularity for $\mathcal{S}_\varphi$ from $L^p(bD, d\lambda)$-regularity of $\mathcal{S}_\lambda$.

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[12]here we are using the hypothesis that the domain D is bounded.

Instead, what we are able to compare the two adjoints of the operator $\mathcal{C}_\epsilon$ with respect to these different measures. Letting $\mathcal{A}^{(\varphi)}_{\epsilon,\delta}$ denote the operator analogous to (2.23) (however now defined for the measure $\varphi d\lambda$), we have the following inequality [63, Part II]

- $\|\mathcal{A}^{(\varphi)}_{\epsilon,\delta}\|_{L^p\to L^p} \leq \|\mathcal{A}^{(\lambda)}_{\epsilon,\delta}\|_{L^p\to L^p} + \sup\limits_{w\in bD} |\varphi^{-1}(w)|\, \|[\mathcal{C}_{\epsilon,\delta},\varphi]\|_{L^p\to L^p}$

where $\mathcal{C}_{\epsilon,\delta}$ is the operator with kernel $\chi_\delta(\mathtt{d}(w,z))\, C^{(\epsilon)}(w,z)$, see (2.17), and $[T,\varphi] = T\,M_\varphi - M_\varphi T$ denotes the commutator with the multiplication operator M_φ (multiplication by φ).

Furthermore, we have that [63, Part II]

- $\|[\mathcal{C}_{\epsilon,\delta},\varphi]\|_{L^p\to L^p} \leq \epsilon M_p$ for any $\delta \leq \delta_0(\epsilon)$ and for any $0 < \epsilon \leq \epsilon_0$.

Taking these two facts into account, the proof of the $L^p(bD, \varphi d\lambda)$-regularity of $\mathcal{S}_\varphi$ may now be obtained by following the same steps as in the proof of the corresponding result for $\mathcal{S}_\lambda$ on $L^p(bD, d\lambda)$, see Section 2.9.

3. Further results

3.1. **The Bergman projection.** One may also state the L^p-regularity problem for the *Bergman projection*, that is the orthogonal projection of $L^2(D, dV)$ onto the Bergman space $\vartheta(D) \cap L^2(D, dV)$ (namely, the space of functions that are holomorphic in D and square-integrable on D with respect to the measure on D induced by the Lebesgue measure for $\mathbb{C}^n$ via the inclusion: $D \subset \mathbb{C}^n$). The L^p-regularity problem for the Bergman projection was studied by Ligocka [66] in the setting of bounded, strongly pseudconvex domains of class C^4, and was recently extended [60] to the class C^2, with $L^p(D, dV)$-regularity holding for $1 < p < \infty$. This problem can be approached in a fashion similar to the L^p-regularity problem for the Szegő projection, but is in fact simpler than that problem, in several respects:

• There is no advantage in considering "ad-hoc" volume measures for the domain D (some "solid" version of the Leray-Levi measure) and one may work directly with the induced Lebesgue measure dV.

• In this context, the role of the "holomorphic Cauchy integrals" $\mathcal{C}_\epsilon$'s is played by "solid" integral operators $\mathbf{B}_\epsilon$ acting on $L^p(D, dV)$, whose kernel is essentially the "derivative" of the kernels of the $\mathcal{C}_\epsilon$'s, specifically, it is the $2n$-form $\overline{\partial}_w \widetilde{C}^{(\epsilon)}(w,z)$, see (2.16) (then corrected to achieve global holomorphicity). Such operators will still produce (and reproduce) holomorphic functions from merely L^1 data, see [61, Propositions 3.1 and 3.2]. By their nature, these operators are less singular than the $\mathcal{C}_\epsilon$'s and their $L^p(D, dV)$-regularity can be established by direct means (with no need to invoke the $T(1)$-theorem, [60, Section 4]).

• The solution of the L^p-regularity problem for the Bergman projection of a strongly pseudoconvex domain of class C^2 now follows a parallel argument to the corresponding result for the Szegő projection, by proving "ϵ-smallness" for the kernels of $\mathbf{B}^*_\epsilon - \mathbf{B}_\epsilon$ with ϵ again tailored to the size of the Lebesgue exponent p, see [60, Sections 5. and 6.].

• In fact one also proves L^p-regularity for the "absolute Bergman projection", that is the operator whose kernel is the *absolute value* of the Bergman kernel, see [60, Section 6] and [22, Section I.1, Example 1]. (We point out that the corresponding statement for the "absolute Szegő projection" is known to be false by the very nature of the Szegő kernel, whose treatment requires cancellation conditions that would be lost by considering its absolute value.)

3.2. **Holomorphic Cauchy integrals below the C^2-threshold.** A theory of holomorphic Cauchy integrals can also be developed for so-called *strongly $\mathbb{C}$-linearly convex* domains. While $\mathbb{C}$-linear convexity is a stronger notion than pseudoconvexity (in the sense that any strongly $\mathbb{C}$-linearly convex domain of class C^2 is strongly pseudoconvex but the converse is not true [62, Proposition 3.2 and Example pg.797]), it is a notion that rests on only *one* derivative of the defining function and is therefore naturally meaningful for domains of class C^1. See [3], [45] and [62, Section 3] for the definition and main properties of $\mathbb{C}$-linear convexity.

In [62] we study existence and regularity of holomorphic Cauchy integrals for bounded, strongly $\mathbb{C}$-linear convex domains of class $C^{1,1}$. In this context the $C^{1,1}$ category plays a role analogous to the Lipschitz category for a planar domain; the relevant kernel is the *Cauchy-Leray kernel*

$$K(w,z) = \frac{1}{(2\pi)^n} j^* \left(\frac{\partial\rho \wedge (\overline{\partial}\partial\rho)^{n-1}(w)}{\langle \partial\rho(w), w-z\rangle^n} \right) = \frac{d\lambda(w)}{\langle \partial\rho(w), w-z\rangle^n}\,. \tag{3.1}$$

• This kernel was first identified by Leray [65] in the context of (strongly convex) domains of class C^2. A new and substantial obstacle that arises in the $C^{1,1}$ setting is the fact that the (familiar) numerator of $K(w,z)$ may not make sense: while the Rademacher Theorem ensures that the $C^{1,1}$ function ρ be twice differentiable almost everywhere in $\mathbb{C}^n$, here we are taking its restriction to the boundary bD which is in fact a zero-measure subset of $\mathbb{C}^n$, and the coefficients of $\overline{\partial}\partial\rho$ may indeed be undefined on bD (explicit examples can be given); however, the pullback by the inclusion $j^*(\overline{\partial}\partial\rho)$ only pertains the *tangential components* of such coefficients, which are indeed well-defined, see [62, Proposition 23]. As a result, one has that the Leray-Levi measure $d\lambda$ is well defined also in this less regular context, and it is again equivalent to the induced Lebesgue measure $d\sigma$, see [62, Section 3.4].

• The strong $\mathbb{C}$-linear convexity of D ensures that the global bound: $|\langle\partial\rho(w), w-z\rangle| \geq c|w-z|^2$ holds for any $w \in bD$ and any $z \in \overline{D}$, see [62, (3.4)]. Thus, the Cauchy-Leray kernel is globally holomorphic (no need for correction). While not universal in the sense of Section 2.4, The Cauchy-Leray kernel is *canonical* in the sense that it does not depend on the choice of defining function (while each of the numerator and denominator in (3.1) depend on the choice of defining function ρ, their quotient does not, see [62, Proposition 4.1]). This is in great contrast with the situation for the kernels of the operators $\mathcal{C}$ and $\mathcal{C}_\epsilon$ considered in sections 2.5 – 2.8, which do depend on the choice of defining function and are thus non-canonical.

• Letting $\mathcal{K}$ denote the Cauchy-Leray operator with kernel (3.1), we prove that $\mathcal{K}$ is bounded in $L^p(bD, d\lambda)$ for any $1 < p < \infty$ (and thus $L^p(bD, d\sigma)$), by a $T(1)$-theorem for a space of homogeneous type again informed by the geometry and regularity of bD (in a spirit that is similar to the situation described in Sections 2.7 and 2.8), and again under the simpler cancellation conditions $T(1) = 0 = T^*(1)$, see [62, Section 6].

• Our hypotheses on the domain D are optimal in the sense that for any $0 < \alpha < 1$ there are strongly $\mathbb{C}$-linearly convex domains D_α of class $C^{1,\alpha}$, for which the Cauchy-Leray operator $\mathcal{K}$ is well-defined but unbounded on each of $L^2(bD, d\lambda)$ and $L^2(bD, d\sigma)$, see [6, Section 6, Example 2]; similarly, there is a smooth *weakly* $\mathbb{C}$-linearly convex domain D

for which $\mathcal{K}$ is well-defined but unbounded on each of $L^2(bD, d\lambda)$ and $L^2(bD, d\sigma)$, see [6, Section 6, Example 1].

• The difference $\mathcal{K}^* - \mathcal{K}$ has no inherent "smallness": not even the weaker "ϵ-smallness" (2.22) (if, say, one were to approximate ρ with smoother functions $\tau^{(\epsilon)}$)[13]. Thus the study of the Bergman and Szegő projections for strongly $\mathbb{C}$-linearly convex domains requires a different approach and is the object of current investigation.

3.3. **Representations for the Hardy and Bergman spaces of holomorphic functions.** As an application of the L^p-regularity of the holomorphic Cauchy-type integrals and of the Szegő and Bergman projections we obtain various representations for the Hardy and Bergman spaces of holomorphic functions. Specifically, for a strongly pseudoconvex domain of class C^2, we have the following, see [64] (see also [58]):

• The space of functions holomorphic in a neighborhood of $\overline{D}$ is dense in $\mathcal{H}^p(bD, \varphi d\lambda)$ (a consequence of the L^p-regularity of the $\mathcal{C}_\epsilon$'s).

• $\mathcal{C}_\epsilon : L^p(bD, \varphi d\lambda) \to \mathcal{H}^p(bD, \varphi d\lambda)$ for any $1 < p < \infty$. Furthermore, $f \in \mathcal{H}^p(bD, \varphi d\lambda)$ if, and only if $\mathcal{C}_\epsilon f = f$ (again a consequence of the L^p-regularity of the $\mathcal{C}_\epsilon$'s).

Corresponding statements hold for the situation when D is strongly $\mathbb{C}$-linearly convex and of class $C^{1,1}$ (with the $\mathcal{C}_\epsilon$'s replaced by the Cauchy-Leray operator $\mathcal{K}$).

Furthermore, for a strongly pseudoconvex domain of class C^2 we have, see [61, Proposition 7.1] and [64]:

• The space of functions holomorphic in a neighborhood of $\overline{D}$ is dense in $\vartheta(D) \cap L^p(D, dV)$ (a consequence of the L^p-regularity of the $\mathbf{B}_\epsilon$'s).

• $\mathcal{S}_\varphi : L^p(bD, \varphi d\lambda) \to \mathcal{H}^p(bD, \varphi d\lambda)$ for any $1 < p < \infty$. Moreover, $f \in \mathcal{H}^p(bD, \varphi d\lambda)$ if, and only if $\mathcal{S}_\varphi f = f$ (a consequence of the L^p-regularity of $\mathcal{S}_\varphi$).

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