ON OPTIMAL HARDY INEQUALITIES IN CONES
DISUGUAGLIANZE DI HARDY OTTIMALI SUI CONI

BAPTISTE DEVYVER, YEHUDA PINCHOVER, AND GEORGIOS PSARADAKIS

Abstract. In this Note we consider a cone Ω in $\mathbb{R}^n$ with a vertex at the origin. We assume that the operator

$$P_\mu := -\Delta - \frac{\mu}{\delta_\Omega^2(x)}$$

is *subcritical* in Ω, where δ_Ω is the distance function to the boundary of Ω and $\mu \leq 1/4$. Under some smoothness assumption of Ω, we show that the following improved Hardy-type inequality

$$\int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x - \mu \int_\Omega \frac{|\varphi|^2}{\delta_\Omega^2} \,\mathrm{d}x \geq \lambda(\mu) \int_\Omega \frac{|\varphi|^2}{|x|^2} \,\mathrm{d}x \qquad \forall \varphi \in C_0^\infty(\Omega),$$

holds true, and that the above inequality is optimal in some definite sense. The constant $\lambda(\mu) > 0$ is given explicitly.

Sunto. In questa nota, consideriamo un cono Ω nello spazio Euclideo $\mathbb{R}^n$, con vertex all'origine. Sia

$$P_\mu := -\Delta - \frac{\mu}{\delta_\Omega^2(x)}$$

un operatore sottocritico in Ω, dovè δ_Ω è la funzione distanza al bordo di Ω, e $\mu \leq 1/4$. Sotto qualche ipotesi di regularità su Ω, dimostriamo che la seguente disuguaglianza di Hardy migliorata

$$\int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x - \mu \int_\Omega \frac{|\varphi|^2}{\delta_\Omega^2} \,\mathrm{d}x \geq \lambda(\mu) \int_\Omega \frac{|\varphi|^2}{|x|^2} \,\mathrm{d}x \qquad \forall \varphi \in C_0^\infty(\Omega),$$

vale, ed è ottimale in un senso preciso. La costante $\lambda(\mu) > 0$ è data esplicitamente.

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Dipartimento di Matematica, Università di Bologna

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1. Introduction

Let P be a *symmetric* and *nonnegative* second-order linear elliptic operator with real coefficients which is defined on a domain $\Omega \subset \mathbb{R}^n$ or on a noncompact manifold Ω, and let q be the associated quadratic form defined on $C_0^\infty(\Omega)$. A *Hardy-type inequality* with a weight $W \gneqq 0$ has the form

$$q(\varphi) \geq \lambda \int_\Omega W(x)|\varphi(x)|^2 \,\mathrm{d}x \qquad \forall \varphi \in C_0^\infty(\Omega), \tag{1.1}$$

where $\lambda > 0$ is a constant. Such an inequality aims to quantify the positivity of P: for instance, if (1.1) holds with $W \equiv \mathbf{1}$ it means that the bottom of the spectrum of the Friedrichs extension of P is positive, and that the equation $(P-\lambda)u = 0$ admits a positive solution in Ω. A nonnegative operator P is called *critical* in Ω if the inequality $P \geq 0$ cannot be improved, meaning that (1.1) holds true if and only if $W \equiv 0$. On the other hand, when (1.1) holds with a nontrivial nonnegative W, then P is said to be *subcritical* in Ω.

Given a subcritical operator P in Ω, there is a huge convex set of weights $W \gneqq 0$ satisfying the inequality (1.1); We will call these weights, *Hardy-weights*. A natural question is to find “large” Hardy-weights. The search for Hardy-type inequalities with “as large as possible” weight function W was proposed by Agmon [1, Page 6].

In a recent paper [6], the authors studied a general (not necessarily symmetric) subcritical second-order *linear* elliptic operator P in a domain $\Omega \subset \mathbb{R}^n$ (or a noncompact manifold), and constructed a Hardy-weight W which is *optimal*. In the case of *symmetric* operator P the main result of [6] reads as follows.

Theorem 1.1 ([6, Theorem 2.2])**.** *Consider a* symmetric *second-order linear elliptic operator P defined in a domain $\Omega \subset \mathbb{R}^n$, and let q be the associated quadratic form. Assume that P is subcritical in Ω. Fix a reference point $x_0 \in \Omega$, and denote $\Omega^\star := \Omega \setminus \{x_0\}$.*

There exists a nonzero nonnegative weight W satisfying the following properties:

(a) *Denote by $\lambda_0 = \lambda_0(P, W, \Omega^\star)$ the largest constant λ satisfying*

$$q(\varphi) \geq \lambda \int_{\Omega^\star} W(x)|\varphi(x)|^2 \,\mathrm{d}x \qquad \forall \varphi \in C_0^\infty(\Omega^\star). \tag{1.2}$$

Then $\lambda_0 > 0$ and the operator $P - \lambda_0 W$ is critical in $\Omega^\star$; that is, the inequality

$$q(\varphi) \geq \int_{\Omega^\star} V(x)|\varphi(x)|^2 \,\mathrm{d}x \qquad \forall \varphi \in C_0^\infty(\Omega^\star)$$

is not valid for any $V \gneqq \lambda_0 W$.

(b) *The constant λ_0 is also the best constant for (1.2) with test functions supported in $\Omega' \subset \Omega$, where Ω' is either the complement of any fixed compact set in Ω containing x_0 or any fixed punctured neighborhood of x_0.*

(c) *The operator $P - \lambda_0 W$ is null-critical in $\Omega^\star$; that is, the corresponding Rayleigh-Ritz variational problem*

$$\inf_{\varphi \in \mathcal{D}_P^{1,2}(\Omega^\star)} \left\{ \frac{q(\varphi)}{\int_{\Omega^\star} W(x)|\varphi(x)|^2 \,\mathrm{d}x} \right\} \tag{1.3}$$

admits no minimizer. Here $\mathcal{D}_P^{1,2}(\Omega^\star)$ is the completion of $C_0^\infty(\Omega^\star)$ with respect to the norm $u \mapsto \sqrt{q(u)}$.

(d) *If furthermore $W > 0$ in $\Omega^\star$, then the spectrum and the essential spectrum of the Friedrichs extension of the operator $W^{-1}P$ on $L^2(\Omega^\star, W\,\mathrm{d}x)$ are both equal to $[\lambda_0, \infty)$.*

Definition 1.1. A weight function that satisfies properties (a)–(d) is called an *optimal Hardy weight* for the symmetric operator P in Ω.

Denote by $\bar{\infty}$ ideal point in the one-point compactification of Ω. The optimal Hardy weight W in Theorem 1.1 is obtained by applying the so-called *supersolution construction*: It turns out that there are two positive solutions u_i, $i = 0, 1$ of $Pu = 0$ in $\Omega^\star$ satisfying

$$\lim_{\substack{x \to x_0 \\ x \in \Omega}} \frac{u_1(x)}{u_0(x)} = \lim_{\substack{x \to \bar{\infty} \\ x \in \Omega}} \frac{u_0(x)}{u_1(x)} = 0;$$

the optimal Hardy weight W is then given by

$$W := \frac{P u_{1/2}}{u_{1/2}},$$

where $u_{1/2} := (u_0 u_1)^{1/2}$.

In [6, Theorem 11.6], the authors extend Theorem 1.1 and get an optimal Hardy-weight W in the *entire* domain Ω, in the case of *boundary singularities*, where the two singular points of the Hardy-weight are located at $\partial\Omega \cup \{\bar\infty\}$ and not at $\bar\infty$ and at an isolated interior point of Ω as in Theorem 1.1. Roughly speaking, we assume that the coefficients of P are regular up to the (Martin) boundary of Ω outside two Martin boundary points $\{\zeta_0, \zeta_1\}$, and that the Martin functions u_i at ζ_i, $i = 0, 1$, satisfy

$$\lim_{\substack{x\to\zeta_0 \\ x\in\Omega}} \frac{u_1(x)}{u_0(x)} = \lim_{\substack{x\to\zeta_1 \\ x\in\Omega}} \frac{u_0(x)}{u_1(x)} = 0. \tag{1.4}$$

Then the supersolution construction produces an *optimal* Hardy weight $W = \frac{Pu_{1/2}}{u_{1/2}}$, where $u_{1/2} := (u_0u_1)^{1/2}$.

The following example illustrates [6, Theorem 11.6] and motivates our present study.

Example 1.1 ([6, Example 11.1])**.** Let $P_0 = -\Delta$, and consider the cone Ω with vertex at the origin, and given by

$$\Omega := \{x \in \mathbb{R}^n \mid r(x) > 0, \omega(x) \in \Sigma\}, \tag{1.5}$$

where Σ is a Lipschitz domain in the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, $n \geq 2$, and (r, ω) denotes the spherical coordinates of x (i.e., $r = |x|$, and $\omega = x/|x|$).

Let θ be the principal eigenfunction of the (Dirichlet) Laplace-Beltrami operator $-\Delta_S$ on Σ with principal eigenvalue $\sigma = \lambda_0(-\Delta_S, \mathbf{1}, \Sigma)$ (for the definition of λ_0 see (2.1)), and set

$$\gamma_\pm := \frac{2-n \pm \sqrt{(2-n)^2 + 4\sigma}}{2}.$$

Then the positive harmonic function

$$u_\pm(r, w) := r^{\gamma_\pm}\theta(\omega)$$

are the Martin kernels at ∞ and 0 [17] (see also [4]).

Using the supersolution construction it follows that the function

$$u_{1/2} := (u_0u_1)^{1/2} = r^{(2-n)/2}\theta(\omega)$$

is a positive supersolution of the equation $Pu = 0$ in Ω. The obtained Hardy-weight is given by

$$W(x) := \frac{Pu_{1/2}}{u_{1/2}} = \frac{(n-2)^2 + 4\sigma}{4|x|^2}$$

and the corresponding Hardy-type inequality reads as

$$\int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x \geq \frac{(n-2)^2+4\sigma}{4}\int_\Omega \frac{|\varphi|^2}{|x|^2}\,\mathrm{d}x \qquad \forall\varphi\in C_0^\infty(\Omega). \tag{1.6}$$

Moreover, it follows from [6, Theorem 11.6] that W is an optimal Hardy-weight, and that the spectrum and the essential spectrum of $W^{-1}(-\Delta)$ is $[1,\infty)$. Note that for $\Sigma = \mathbb{S}^{n-1}$, $n \geq 3$, we obtain the classical Hardy inequality in the punctured space. We also remark that the Hardy-type inequality (1.6) and the *global* optimality of the constant $\frac{(n-2)^2+4\sigma}{4}$ are not new (cf. [8, 14]), however even the fact that (1.6) cannot be improved was not known before.

Let

$$\delta(x) = \delta_\Omega(x) := \mathrm{dist}\,(x, \partial\Omega)$$

be the distance function to the boundary of a domain Ω.

The aim of this Note is to present an extension of the result in Example 1.1 to the case of the Hardy operator

$$P_\mu := -\Delta - \frac{\mu}{\delta_\Omega^2(x)} \qquad \text{in } \Omega,$$

where Ω is the cone defined by (1.5), and $\mu \leq \mu_0 := \lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega)$ (for the definition of λ_0 see (2.1)). In particular, we present an explicit expression for the optimal Hardy weight W corresponding to the singular points 0 and ∞, for the associate best Hardy constant, and for the corresponding ground state. Note that since the potential $\delta_\Omega^{-2}(x)$ is singular on $\partial\Omega$, [6, Theorem 11.6] is not applicable for P_μ with $\mu \neq 0$, and we had to come up with new techniques and ideas to treat this case. For some recent results concerning sharp Hardy inequalities with boundary singularities see [5, 10, 11] and references therein.

The outline of the present paper is as follows. In Section 2 we fix the setting and notations, and introduce some basic definitions. In Section 3 we use an approximation argument to obtain two positive multiplicative solutions of the equation $P_\mu u = 0$ in Ω of the form $u_\pm(r, w) := r^{\gamma_\pm}\theta(\omega)$, while in Section 4 we use the boundary Harnack principle of A. Ancona [3] and the methods in [13, 17] to get an explicit representation theorem for the positive solutions of the equation $P_\mu u = 0$ in Ω that vanish (in the potential theory sense) on $\partial\Omega \setminus \{0\}$. Section 5 is devoted to the presentation of our main result. For the sake of brevity of this Note, we omit all proofs; they appear in an upcoming paper that contains further related results, see [7].

2. Preliminaries

In this section we fix our setting and notations, and introduce some basic definitions. We denote $\mathbb{R}_+ := (0, \infty)$, and

$$\mathbb{R}^n_+ := \mathbb{R}_+ \times \mathbb{R}^{n-1} = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_1 > 0\}.$$

Throughout the paper $\Omega \subset \mathbb{R}^n$ is a domain, where $n \geq 2$. The distance function to the boundary of Ω is denoted by δ_Ω. We write $\Omega_1 \Subset \Omega$ if Ω is open, $\overline{\Omega}_1$ is compact and $\overline{\Omega}_1 \subset \Omega$.

Let $f, g : \Omega \to [0, \infty)$. We denote $f \asymp g$ if there exists a positive constant C such that $C^{-1}g \leq f \leq Cg$ in Ω. Also, we write $f \gneqq 0$ in Ω if $f \geq 0$ in Ω but $f \neq 0$ in Ω. We denote by $\mathbf{1}$ the constant function taking the value 1 in Ω.

In the present paper we consider a second-order linear elliptic operator P defined on a domain $\Omega \subset \mathbb{R}^n$, and let $W \gneqq 0$ be a given function. It is assumed throughout the paper that the operator P is *symmetric* and locally uniformly elliptic. Moreover, we assume that the coefficients of P and the function W are locally sufficiently regular in Ω (see [6]). For such an operator P and $\lambda \in \mathbb{R}$, we denote $P_\lambda := P - \lambda W$.

Definition 2.1. The operator P is said to be *nonnegative in* Ω, and we write $P \geq 0$ in Ω, if the equation $Pu = 0$ in Ω admits a positive (super)solution.

By the well-known Allegretto-Piepenbrink theorem for symmetric second-order elliptic operators P (see for example [2]), $P \geq 0$ in Ω is equivalent to the quadratic form of P being nonnegative on $C_0^\infty(\Omega)$. Unless otherwise stated, it is assumed that $P \geq 0$ in Ω. For such a nonnegative operator P, we have (see [18]):

Theorem 2.1. *Suppose that P is a nonnegative symmetric operator in Ω.*

1. The operator P in subcritical in Ω if and only if P admits a positive minimal Green function in Ω.

2. The operator P in subcritical in Ω if and only if P admits a unique (up to a multiplicative constant) positive supersolution of the equation $Pu = 0$ in Ω.

Definition 2.2. Suppose that P is critical in Ω, then the unique positive (super)solution of the equation $Pu = 0$ in Ω is called the *(Agmon) ground state* of P in Ω.

We emphasize that the notion of subcriticality above is closely related to the improvement of Hardy inequalities. Let P and $W \gneqq 0$ be as above, the *generalized principal eigenvalue* is defined by

$$\lambda_0 := \lambda_0(P, W, \Omega) := \sup \Big\{ \lambda \in \mathbb{R} \mid P_\lambda = P - \lambda W \geq 0 \ \text{ in } \Omega \Big\}. \tag{2.1}$$

We also define

$$\lambda_\infty = \lambda_\infty(P, W, \Omega) := \sup \Big\{ \lambda \in \mathbb{R} \mid \exists K \subset\subset \Omega \text{ s.t. } P_\lambda \geq 0 \ \text{ in } \Omega \setminus K \Big\}.$$

Recall that if the operator P is symmetric in $L^2(\Omega,\, \mathrm{d}x)$, and $W > 0$, then λ_0 (resp. λ_∞) is the infimum of the $L^2(\Omega,\, W\mathrm{d}x)$-spectrum (resp. $L^2(\Omega,\, W\mathrm{d}x)$-essential spectrum) of the Friedrichs extension of $\tilde{P} := W^{-1}P$ (see for example [2] and references therein). Note that $\tilde{P}$ is symmetric on $L^2(\Omega,\, W\mathrm{d}x)$, and has the same quadratic form as P.

Throughout the paper we fix a cone

$$\Omega := \{ x \in \mathbb{R}^n \mid r(x) > 0, \omega(x) \in \Sigma \}, \tag{2.2}$$

where Σ is a Lipschitz domain in the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, $n \geq 2$. For $x \in \Sigma$, we will denote by $\mathrm{d}_\Sigma(x)$ the (spherical) distance from x to the boundary of Σ. Note that δ_Ω is clearly a homogeneous function of degree 1, that is,

$$\delta_\Omega(x) = |x| \delta_\Omega\left(\frac{x}{|x|}\right) = r\delta_\Omega(\omega), \quad \text{and} \ \ \delta_\Omega(x) = \sin\big(\mathrm{d}_\Sigma(x)\big) \ \ \text{near the boundary.} \tag{2.3}$$

For spectral results and Hardy inequalities with homogeneous weights on $\mathbb{R}^n$ see [12].

Since the distance function to the boundary of any domain is Lipschitz continuous, Euler's homogeneous function theorem implies that

$$x \cdot \nabla \delta_\Omega(x) = \delta_\Omega(x) \quad \text{a.e. in } \Omega. \tag{2.4}$$

In fact, Euler's theorem characterizes all sufficiently smooth positive homogeneous functions. Hence, (2.4) characterizes the cones in $\mathbb{R}^n$.

Let Δ_S be the Laplace-Beltrami operator on the unit sphere $S := \mathbb{S}^{n-1}$. Then in spherical coordinates, the operator

$$P_\mu := -\Delta - \frac{\mu}{\delta_\Omega(x)^2}$$

has the following skew symmetric form

$$P_\mu u(r,\omega) = -\frac{\partial^2 u}{\partial r^2} - \frac{n-1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left(-\Delta_S u - \mu \frac{u}{\delta_\Omega^2(\omega)}\right) \qquad r>0,\ \omega\in\Sigma. \tag{2.5}$$

It turns out that for any Lipschitz cone the Hardy inequality holds true (as in the case of sufficiently smooth *bounded* domain [16]).

Lemma 2.1. *Let Ω be a Lipschitz cone, and let $\mu_0 := \lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega)$. Then*

$$0 < \mu_0 \le \frac{1}{4}\,. \tag{2.6}$$

In other words, the following Hardy inequality holds true.

$$\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x \ge \mu_0 \int_\Omega \frac{|\varphi|^2}{\delta_\Omega^2}\,\mathrm{d}x \qquad \forall \varphi\in C_0^\infty(\Omega), \tag{2.7}$$

where $0<\mu_0\le\frac{1}{4}$ is the best constant.

Moreover, $\mu_0 = 1/4$ if Ω is convex, and in this case $P_{1/4}$ is subcritical.

Remark 2.1. Clearly, P_μ is subcritical in Ω for all $\mu<\mu_0$. We show in Theorem 5.2 that if $\mu_0<1/4$, then the operator P_{μ_0} is critical in Ω (cf. [16, Theorem II]).

3. Positive multiplicative solutions

As above, let Ω be a Lipschitz cone. By Lemma 2.1 the generalized principal eigenvalue $\mu_0 := \lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega)$ satisfies $0<\mu_0\le 1/4$. The following theorem shows that for $\mu\le\mu_0$ the equation $P_\mu u = 0$ in Ω admits positive multiplicative (separated) solutions.

Theorem 3.1. *Let $\mu\le\mu_0$. Then the equation $P_\mu u=0$ in Ω admits positive solutions of the form*

$$u_\pm(x) = |x|^{\gamma_\pm}\phi_\mu\left(\frac{x}{|x|}\right), \tag{3.1}$$

where ϕ_μ is a positive solution of the equation

$$\left(-\Delta_S - \frac{\mu}{\delta_\Omega^2(\omega)}\right)u = \sigma(\mu)u \qquad \text{in } \Sigma, \tag{3.2}$$

$$-\frac{(n-2)^2}{4} \leq \sigma(\mu) := \lambda_0\left(-\Delta_S - \frac{\mu}{\delta_\Omega^2}, \mathbf{1}, \Sigma\right), \tag{3.3}$$

and

$$\gamma_\pm := \frac{2-n \pm \sqrt{(2-n)^2 + 4\sigma(\mu)}}{2}. \tag{3.4}$$

Moreover, if $\sigma(\mu) > -\frac{(n-2)^2}{4}$*, then there are two linearly independent positive solutions of the equation* $P_\mu u = 0$ *in* Ω *of the form* (3.1)*, and* P_μ *is subcritical in* Ω*.*

In particular, for any $\mu \leq \mu_0$ *we have* $\sigma(\mu) > -\infty$*.*

Remark 3.1. Note that for $n = 2$, $\Sigma = \mathbb{S}^1$, and $\mu = \mu_0 = 0$, we obtain $\sigma(0) = 0$, $\gamma_\pm = 0$, and $P_0 = -\Delta$ is critical in the cone $\mathbb{R}^2 \setminus \{0\}$.

Remark 3.2. Let Σ be a bounded domain in a smooth Riemannian manifold M, and let d_Σ be the Riemannian distance function to the boundary $\partial\Sigma$. If $\partial\Sigma$ is sufficiently smooth, then the Hardy inequality with respect to the weight $(\mathrm{d}_\Sigma)^{-2}$ holds in Σ with a positive constant C_H [19]. A sufficient condition for the validity of a such Hardy inequality is that Σ is *boundary distance regular*, and this condition holds true if Σ satisfies either the *uniform interior cone condition* or the *uniform exterior ball condition* (see the definitions in [19]). For other sufficient conditions for the validity of the Hardy inequality on Riemannian manifolds see for example [15].

Hence, if the cone $\Omega \subset \mathbb{R}^n \setminus \{0\}$ is smooth enough, then $\Sigma \subset \mathbb{S}^{n-1}$ is boundary distance regular. So, for such $\Sigma \subset \mathbb{S}^{n-1}$ there exists $C > 0$ such that $-\Delta_S - \frac{C}{\mathrm{d}_\Sigma^2} \geq 0$ in Σ. Note that $\mathrm{d}_\Sigma(\omega) \asymp \delta_\Omega(\omega)|_\Sigma$ in Σ, therefore, $-\Delta_S - \frac{C_1}{\delta_\Omega^2} \geq 0$ in Σ for some $C_1 > 0$.

In the next proposition we study the possible values of the generalized principal eigenvalue of the operator $-\Delta_S - \mu\delta_\Omega^{-2}$ in Σ.

Proposition 3.1. *Let* $\sigma(\mu) = \lambda_0(-\Delta_S - \mu\delta_\Omega^{-2}, \mathbf{1}, \Sigma)$*. Then*

1. $\sigma(\mu) \geq -\frac{(n-2)^2}{4}$ *for any* $\mu \leq \mu_0$*, and if* $\Sigma \in C^2$*, and* $\mu_0 < \frac{1}{4}$*, then* $\sigma(\mu_0) = -\frac{(n-2)^2}{4}$*.*
2. $\sigma(\mu) = -\infty$ *for any* $\mu > 1/4$*.*
3. *If* $\Sigma \in C^2$*, then* $\sigma(\mu) > -\infty$ *for all* $\mu < 1/4$*.*

4. The structure of $\mathcal{K}^0_{P_\mu}(\Omega)$

As above, let Ω be a Lipschitz cone. By Lemma 2.1 the generalized principal eigenvalue $\mu_0 := \lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega)$ satisfies $0 < \mu_0 \leq 1/4$.

Let x_1 be a fixed reference point in Ω. Denote by $\mathcal{K}^0_{P_\mu}(\Omega)$ the convex set of all positive solutions u of the equation $P_\mu u = 0$ in Ω satisfying the normalization condition $u(x_1) = 1$, and the Dirichlet boundary condition $u = 0$ on $\partial\Omega \setminus \{0\}$ in the sense of the Martin boundary, that is, any $u \in \mathcal{K}^0_{P_\mu}(\Omega)$ has minimal growth on $\partial\Omega \setminus \{0\}$. For the definition of minimal growth on a portion Γ of $\partial\Omega$, see [17].

If $\mu_0 < 1/4$ and $\Sigma \in C^2$ outside 0, then in Theorem 5.2 we will show that the operator P_{μ_0} is critical in Ω, and therefore, the equation $P_{\mu_0} u = 0$ in Ω admits (up to a multiplicative constant) a unique positive supersolution. Moreover, by Theorem 3.1, the unique positive solution is a multiplicative solution of the form (3.1).

The following Theorem characterizes the structure of $u \in \mathcal{K}^0_{P_\mu}(\Omega)$ for any $\mu < \mu_0$.

Theorem 4.1. *Let Ω be a Lipschitz cone, and let $\mu < \mu_0 \leq 1/4$. Then $\mathcal{K}^0_{P_\mu}(\Omega)$ is the convex hull of two linearly independent positive solutions of the equation $P_\mu u = 0$ in Ω of the form*

$$u_\pm(x) = |x|^{\gamma_\pm} \phi_\mu\left(\frac{x}{|x|}\right), \tag{4.1}$$

where ϕ_μ is the unique positive solution of the equation

$$\left(-\Delta_S - \frac{\mu}{\delta_\Omega^2(\omega)}\right) u = \sigma(\mu) u \qquad \text{in } \Sigma, \tag{4.2}$$

$$\sigma(\mu) := \lambda_0\left(-\Delta_S - \frac{\mu}{\delta_\Omega^2}, \mathbf{1}, \Sigma\right) > -\frac{(n-2)^2}{4}, \tag{4.3}$$

and

$$\gamma_\pm := \frac{2 - n \pm \sqrt{(2-n)^2 + 4\sigma(\mu)}}{2}. \tag{4.4}$$

5. The main result

Recall that by Theorem 3.1, if $\mu \leq \mu_0$, then

$$\sigma(\mu) := \lambda_0\left(-\Delta - \frac{\mu}{\delta_\Omega^2}, \mathbf{1}, \Sigma\right) \geq -\frac{(n-2)^2}{4},$$

and there exists a positive solution ϕ_μ of the equation

$$\left(-\Delta_S - \frac{\mu}{\delta_\Omega^2} - \sigma(\mu)\right) u = 0 \qquad \text{in } \Sigma.$$

Proposition 5.1. *Let Ω be a Lipschitz cone. Let $\mu \leq \mu_0$, and let*

$$\lambda(\mu) := \frac{(2-n)^2 + 4\sigma(\mu)}{4}\,. \tag{5.1}$$

Then $\lambda(\mu) \geq 0$, and the following Hardy inequality holds true in Ω:

$$\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x - \mu \int_\Omega \frac{|\varphi|^2}{\delta_\Omega^2}\,\mathrm{d}x \geq \lambda(\mu) \int_\Omega \frac{|\varphi|^2}{|x|^2}\,\mathrm{d}x \qquad \forall\varphi \in C_0^\infty(\Omega). \tag{5.2}$$

Proof. The fact that $\lambda(\mu) \geq 0$ follows from $\sigma(\mu) \geq -\frac{(n-2)^2}{4}$, which has been proved in Theorem 3.1. Define

$$\psi(x) = |x|^{\frac{2-n}{2}} \phi_\mu\left(\frac{x}{|x|}\right).$$

Then, taking into account that

$$\left(-\Delta_S - \sigma(\mu) - \frac{\mu}{\delta_\Omega^2}\right) \phi_\mu = 0 \qquad \text{in } \Sigma,$$

and writing P_μ in spherical coordinates (2.5), it is immediate to check that ψ is a positive solution of the equation

$$\left(P_\mu - \frac{\lambda(\mu)}{|x|^2}\right) u = 0 \qquad \text{in } \Omega.$$

By the Allegretto-Piepenbrink theorem, it follows that the Hardy inequality (5.2) holds true. ■

Remark 5.1. In the case $\mu < \mu_0$, the Hardy inequality (5.2) can be obtained using the *supersolution construction* of [6]: indeed, by Theorem 4.1, the equation $P_\mu u = 0$ has two linearly independent, positive solutions in Ω, of the form

$$u_\pm(x) = |x|^{\gamma_\pm} \phi_\mu\left(\frac{x}{|x|}\right).$$

By the *supersolution construction* ([6, Lemma 5.1]), the function

$$\psi := \sqrt{u_+ u_-} = |x|^{\frac{2-n}{2}} \phi_\mu\left(\frac{x}{|x|}\right)$$

is a positive solution of

$$\left(P_\mu - \frac{|\nabla(u_+/u_-)|^2}{4(u_+/u_-)^2}\right) u = 0 \qquad \text{in } \Omega.$$

It is easy to check that

$$\frac{|\nabla(u_+/u_-)|^2}{4(u_+/u_-)^2} = \frac{\lambda(\mu)}{|x|^2},$$

and by the Allegretto-Piepenbrink theorem, the Hardy inequality (5.2) holds.

We first investigate the optimality of the Hardy inequality (5.2) when $\mu < \mu_0$:

Theorem 5.1. *Let Ω be a Lipschitz cone, and let $\mu < \mu_0$. Then $\lambda(\mu) > 0$. Furthermore the weight $W := \frac{\lambda(\mu)}{|x|^2}$ is an optimal Hardy weight for the operator P_μ in Ω in the following sense:*

(1) *The operator $P_\mu - \frac{\lambda(\mu)}{|x|^2}$ is critical in Ω, i.e.* (5.2) *holds true, but the Hardy inequality*

$$\int_\Omega |\nabla\varphi|^2 \, \mathrm{d}x - \mu \int_\Omega \frac{|\varphi|^2}{\delta_\Omega^2} \, \mathrm{d}x \geq \int_\Omega V(x)|\varphi|^2 \, \mathrm{d}x \qquad \forall \varphi \in C_0^\infty(\Omega)$$

does not hold for any $V \gneqq W$. In particular,

$$\lambda_0\left(P_\mu, \frac{1}{|x|^2}, \Omega\right) = \lambda(\mu).$$

(2) *The constant $\lambda(\mu)$ is also the best constant for* (5.2) *with test functions supported either in Ω_R or in $\Omega \setminus \overline{\Omega_R}$, where Ω_R is a fixed truncated cone. In particular,*

$$\lambda_\infty\left(P_\mu, \frac{1}{|x|^2}, \Omega\right) = \lambda(\mu).$$

(3) *The operator $P_\mu - \frac{\lambda(\mu)}{|x|^2}$ is* null-critical *at 0 and at infinity in the following sense: For any $R > 0$ the (Agmon) ground state of the operator $P_\mu - \frac{\lambda(\mu)}{|x|^2}$ given by*

$$v(x) := |x|^{(2-n)/2} \phi_\mu\left(\frac{x}{|x|}\right)$$

satisfies

$$\int_{\Omega_R} \left(|\nabla v|^2 - \mu \frac{|v|^2}{\delta_\Omega^2}\right) \mathrm{d}x = \int_{\Omega \setminus \Omega_R} \left(|\nabla v|^2 - \mu \frac{|v|^2}{\delta_\Omega^2}\right) \mathrm{d}x = \infty.$$

In particular, the variational problem

$$\inf_{\varphi \in \mathcal{D}^{1,2}_{P_\mu}(\Omega)} \left\{ \frac{\int_\Omega |\nabla \varphi|^2 \,\mathrm{d}x - \mu \int_\Omega \frac{|\varphi|^2}{\delta_\Omega^2} \,\mathrm{d}x}{\displaystyle\int_\Omega \frac{|\varphi|^2}{|x|^2} \,\mathrm{d}x} \right\}$$

does not admit a minimizer.

(4) *The spectrum and the essential spectrum of the Friedrichs extension of the operator* $W^{-1}P_\mu = \lambda(\mu)^{-1}|x|^2 P_\mu$ *on* $L^2(\Omega, W\,\mathrm{d}x)$ *are both equal to* $[1,\infty)$.

Remark 5.2. As we have noticed in Remark 5.1, if $\mu < \mu_0$, then the Hardy inequality (5.2) can be obtained by applying the *supersolution construction* from [6]. Thus, Theorem 5.1 extends Theorem 1.1 to the particular singular case, where Ω is a cone and P_μ is the Hardy operator (which is singular on $\partial\Omega$).

We now turn to the case $\mu = \mu_0$, for which we need to assume more regularity on Ω:

Theorem 5.2. *Let* Ω *be a cone such that* $\Sigma \in C^2$. *Then*

1. If $\mu_0 < \frac{1}{4}$, *then* $\lambda(\mu_0) = 0$, *and the operator* P_{μ_0} *is critical in* Ω, *and null-critical around* 0 *and* ∞. *In particular, the Hardy inequality*

$$\int_\Omega |\nabla \varphi|^2 \,\mathrm{d}x \geq \mu_0 \int_\Omega \frac{\varphi^2}{\delta_\Omega^2} \,\mathrm{d}x \qquad \forall \varphi \in C_0^\infty(\Omega),$$

cannot be improved.

2. If $\mu_0 = \frac{1}{4}$ *and* $\lambda(\frac{1}{4}) = 0$, *then the operator* $P_{1/4}$ *is critical in* Ω, *and null-critical around* 0 *and* ∞. *In particular, the Hardy inequality*

$$\int_\Omega |\nabla \varphi|^2 \,\mathrm{d}x \geq \frac{1}{4} \int_\Omega \frac{\varphi^2}{\delta_\Omega^2} \,\mathrm{d}x \qquad \forall \varphi \in C_0^\infty(\Omega),$$

cannot be improved.

3. If $\mu_0 = \frac{1}{4}$ *and* $\lambda(\frac{1}{4}) > 0$, *then the weight* $W := \frac{\lambda(\frac{1}{4})}{|x|^2}$ *is optimal in the sense of Theorem 5.1. In particular, the Hardy inequality* (5.2) *cannot be improved.*

Remark 5.1. If Ω is a convex cone such that $\Sigma \in C^2$, then we have $\lambda(\frac{1}{4}) > 0$.

In the particular case of the half-space we can compute the constants appearing in Theorems 5.1 and 5.2.

Example 5.1 (see [6, Example 11.9] and [9])**.** Let $\Omega=\mathbb{R}^n_+$, and $\mu\le 1/4$, and consider the subcritical operator $P_\mu:=-\Delta-\frac{\mu}{|x_1|^2}$ in Ω. Let α_+ be the largest root of the equation $\alpha(1-\alpha)=\mu$, and let

$$\beta(\mu):=1-n-\sqrt{1-4\mu}=2-n-2\alpha_+$$

be the nonzero root of the equation

$$\beta\big(\beta+n-1+\sqrt{1-4\mu}\big)=0.$$

Then

$$v_0(x):=x_1^{\alpha_+},\qquad v_1(x):=x_1^{\alpha_+}|x|^{\beta(\mu)}$$

are two positive solutions of the equation $P_\mu u=0$ in Ω that vanish on $\partial\Omega\setminus\{0\}$.

Therefore, $\lambda(\mu)=\big(\beta(\mu)\big)^2/4$, and for $\mu\le\mu_0=1/4$ we have the following optimal Hardy inequality

$$\int_{\mathbb{R}^n_+}|\nabla\varphi|^2\,\mathrm{d}x-\mu\int_{\mathbb{R}^n_+}\frac{\varphi^2}{x_1^2}\,\mathrm{d}x\geq\left(\frac{n-1+\sqrt{1-4\mu}}{2}\right)^2\int_{\mathbb{R}^n_+}\frac{\varphi^2}{|x|^2}\,\mathrm{d}x\quad\forall\varphi\in C_0^\infty(\mathbb{R}^n_+).$$

In particular, the operator $-\Delta-\frac{\mu}{|x_1|^2}-\frac{\lambda(\mu)}{|x|^2}$ is critical in $\mathbb{R}^n_+$ with the ground state $\psi(x):=x_1^{\alpha_+}|x|^{\beta(\mu)/2}$. Note that for $\mu=0$ we obtain the well known (optimal) Hardy inequality

$$\int_{\mathbb{R}^n_+}|\nabla\varphi|^2\,\mathrm{d}x\geq\frac{n^2}{4}\int_{\mathbb{R}^n_+}\frac{\varphi^2}{|x|^2}\,\mathrm{d}x\qquad\forall\varphi\in C_0^\infty(\mathbb{R}^n_+),$$

while for $\mu=\mu_0=1/4$ we obtain the optimal double Hardy inequality

$$\int_{\mathbb{R}^n_+}|\nabla\varphi|^2\,\mathrm{d}x+\frac{1}{4}\int_{\mathbb{R}^n_+}\frac{1}{x_1^2}\varphi^2\,\mathrm{d}x\geq\frac{(n-1)^2}{4}\int_{\mathbb{R}^n_+}\frac{\varphi^2}{|x|^2}\,\mathrm{d}x\quad\forall\varphi\in C_0^\infty(\mathbb{R}^n_+).\tag{5.3}$$

We note that if Ω is any domain admitting a supporting hyperplane H at zero, then $\delta_\Omega\le\delta_H$ in Ω. Hence, Example 5.1 implies that for any $\mu\le 1/4$ we have

Corollary 5.1. *Suppose that a domain Ω admits a supporting hyperplane at zero, then*

$$\lambda_0(P_\mu,|x|^{-2},\Omega)\le\frac{\big(n-1+\sqrt{1-4\mu}\big)^2}{4}\,.$$

Assume further that Ω is a cone, then for any $\mu\le\mu_0$ we have

$$-\frac{(n-2)^2}{4}\le\sigma(\mu):=\lambda_0\left(-\Delta_S-\frac{\mu}{\delta_\Omega^2},\mathbf{1},\Sigma\right)\le\frac{2n-2-4\mu+(2n-2)\sqrt{1-4\mu}}{4}\,.$$

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Baptiste Devyver, Department of Mathematics, University of British Columbia, Vancouver, Canada

E-mail address: devyver@math.ubc.ca

Yehuda Pinchover, Department of Mathematics, Technion - Israel Institute of Technology, Haifa 32000, Israel

E-mail address: pincho@techunix.technion.ac.il

Georgios Psaradakis, Department of Mathematics, Technion - Israel Institute of Technology, Haifa 32000, Israel

E-mail address: georgios@techunix.technion.ac.il