PARABOLIC PROBLEMS WITH DYNAMIC BOUNDARY CONDITIONS IN L^p SPACES
PROBLEMI PARABOLICI CON CONDIZIONI AL CONTORNO DINAMICHE NEGLI SPAZI L^p

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Abstract. We illustrate a maximal regularity result for parabolic problems with dynamic boundary conditions in L^p spaces.

Sunto. Illustriamo un risultato di regolarità massimale per problemi parabolici con condizioni al contorno dinamiche, negli spazi L^p.

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We shall consider mixed parabolic problems in the form

$$
\begin{cases} D_t u(t,\xi) - A(\xi, D_\xi)u(t,\xi) = f(t,\xi), \quad t \in (0,T), \xi \in \Omega, \\ D_t u(t,\xi') + B(\xi', D_\xi)u(t,\xi') = h(t,\xi'), \quad t \in (0,T), \xi' \in \partial\Omega, \\ u(0,\xi) = u_0(\xi), \quad \xi \in \Omega. \end{cases} \tag{1}
$$

$A(\xi, D_\xi)$ is a linear strongly elliptic operator of second order in the open bounded subset Ω of $\mathbb{R}^n$ and $B(\xi', D_\xi)$ is a first order differential operator. Concerning these assumptions, we shall be more precise in the sequel.

Systems of the form (1) are strictly connected (at least formally) with systems of the form

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$$(2)\qquad \begin{cases} D_t u(t,\xi) - A(\xi, D_\xi)u(t,\xi) = f(t,\xi), \quad t \in (0,T), \xi \in \Omega, \\ A(\xi', D_\xi)u(t,\xi') + B(\xi', D_\xi)u(t,\xi') = k(t,\xi'), \quad t \in (0,T), \xi' \in \partial\Omega, \\ u(0,\xi) = u_0(\xi), \quad \xi \in \Omega. \end{cases}$$

The boundary condition in (2) is known in literature with the name of "generalized Wentzell boundary condition". Consider the second order operator $A(\xi, D_\xi)$ in Ω. Given the boundary conditions L of order less or equal to two, we can define the operator A, with domain $\{u \in C^2(\overline{\Omega}) : Lu = 0\}$, $Au = A(\xi, D_\xi)u$. Consider the operator $\overline{A}$, defined as the closure of A in the space $C(\overline{\Omega})$ (if existing). Then, in the paper [10] A.D. Wentzell characterized all boundary conditions L such that $\overline{A}$ is the infinitesimal generator of a strongly continuous positive contraction semigroup $(T(t))_{t\geq 0}$ in $C(\overline{\Omega})$. Wentzell's work generalized previous results by W. Feller (see [4]) in space dimension one. In the class of operators found by Wentzell there appear also systems in the described form.

Another motivation for the study of parabolic problems with dynamic boundary conditions is given by G. Ruiz Goldstein (see [6]), who discusses a model of the heat equation with a heat source (or a sink) on $\partial\Omega$, taking to a dynamic boundary condition as in (1).

We pass to illustrate some previous work, just considering L^p settings.

(1) naturally leads to elliptic problems depending on a parameter in the form

$$(3)\qquad \begin{cases} \lambda u(\xi) - A(\xi, D_\xi)u(\xi) = f(\xi), \quad \xi \in \Omega, \\ \\ \lambda u(\xi') + B(\xi', D_\xi)u(\xi') = h(\xi'), \quad \xi' \in \partial\Omega. \end{cases}$$

It was proved in [7] (see, in particular, estimate (2.10)) and in a more detailed form in [5], that, under suitable assumptions, if $p \in (1,\infty)$, $|Arg(\lambda)| \leq \frac{\pi}{2}$ and $|\lambda|$ is properly large, if $f \in L^p(\Omega)$ and $h \in W^{1-1/p,p}(\partial\Omega)$, (3) has a unique solution u in $W^{2,p}(\Omega)$. Moreover, there exists C in $\mathbb{R}^+$ such that

$$|\lambda|(\|u\|_{L^p(\Omega)} + \|u_{|\partial\Omega}\|_{W^{1-1/p,p}(\partial\Omega)}) + \|u\|_{W^{2,p}(\Omega)} + \|u_{|\partial\Omega}\|_{W^{2-1/p,p}(\partial\Omega)} \\ \leq C(\|f\|_{L^p(\Omega)} + \|h\|_{W^{1-1/p,p}(\partial\Omega)}).$$

This implies that the operator

$$
(4) \quad \begin{cases} \mathcal{A} : \{(u, g) \in W^{2,p}(\Omega) \times W^{2-1/p,p}(\partial\Omega) : g = u_{|\partial\Omega}\} \to L^p(\Omega) \times W^{1-1/p,p}(\partial\Omega), \\ \\ \mathcal{A}(u, g) = (A(\xi, D_\xi)u, -B(\cdot, D_\xi)u_{|\partial\Omega}) \end{cases}
$$

is the infinitesimal generator of an analytic semigroup in the space $L^p(\Omega) \times W^{1-1/p,p}(\partial\Omega)$. Quasilinear developments of this result were given in [2].

More recently, these problems have been considered in weaker forms, allowing less regular solutions, starting from variational formulations and generalized Wentzell boundary conditions. The space $L^p(\Omega) \times W^{1-1/p,p}(\partial\Omega)$ has been replaced by $L^p(\Omega) \times L^p(\partial\Omega)$ and the operator $\mathcal{A}$ by some extension of G, defined as

$$
\begin{cases} D(G) = \{(u, u_{|\partial\Omega}) : u \in C^2(\overline{\Omega}), Au + \beta\frac{\partial u}{\partial \nu} + \gamma u = 0 \text{ in } \partial\Omega\}, \\ Gu = (Au, -\beta\frac{\partial u}{\partial \nu} - \gamma u) \end{cases}
$$

The solution is usually intended in a weak sense, A is in divergence form and B is the corresponding conormal derivative. Then it was shown in [1] and in [3] that these extensions give rise to infinitesimal generators of analytic semigroups of contraction in $L^p(\Omega) \times L^p(\partial\Omega)$ (properly normed) if $1 < p < \infty$ (in fact, some degeneracy in $\partial\Omega$ is allowed in [3]). A different proof of some of the results of [3] (L^p setting) has been recently given in [8], where only bounded coefficients and Lipschitz boundary are required. This paper contains also the important remark that the fact that the semigroups are of contraction for $p \in (1, \infty)$ and analytic for $p = 2$ implies, as a consequence of a deep result by D. Lamberton (see [9]), maximal regularity for every $p \in (1, \infty)$.

I pass to illustrate some results taken from my work [5]. The main result is the following

Theorem 1. *Suppose that the following assumptions are satisfied:*

(D1) Ω *is an open bounded subset of* $\mathbb{R}^n$, *lying on one side of its boundary* $\partial\Omega$, *which is a submanifold of class* C^2 *of* $\mathbb{R}^n$;

(D2) $A(\xi, D_\xi) = \sum_{|\alpha|\le 2} a_\alpha(\xi) D_\xi^\alpha$, $a_\alpha \in C(\overline{\Omega})$ $\forall \alpha$ *with* $|\alpha| \le 2$; *if* $|\alpha| = 2$, a_α *is real valued and* $\sum_{|\alpha|=2} a_\alpha(\xi)\eta^\alpha \ge N|\eta|^2$ *for some* $N \in \mathbb{R}^+$, $\forall \xi \in \overline{\Omega}$, $\forall \eta \in \mathbb{R}^n$;

(D3) $B(\xi', D_\xi) = \sum_{|\alpha|\le 1} b_\alpha(\xi) D_\xi^\alpha$, $b_\alpha \in C^1(\partial\Omega)$ $\forall \alpha$ *with* $|\alpha| \le 1$; *if* $|\alpha| = 1$, b_α *is real valued and* $\sum_{|\alpha|=1} b_\alpha(\xi')\nu(\xi')^\alpha < 0$ $\forall \xi' \in \partial\Omega$, *where we have indicated with* $\nu(\xi')$ *the unit normal vector to* $\partial\Omega$ *in* ξ' *pointing inside* Ω.

Let $p \in (1, \infty) \setminus \{\frac{3}{2}\}$. *Then the following conditions are necessary and sufficient in order that (1) have a unique solution* u *in* $W^{1,p}((0,T); L^p(\Omega)) \cap L^p((0,T); W^{2,p}(\Omega))$, *with* $u_{|(0,T)\times\partial\Omega} \in W^{1,p}((0,T); W^{1-1/p,p}(\partial\Omega)) \cap L^p((0,T); W^{2-1/p,p}(\partial\Omega))$:

(a) $f \in L^p((0,T) \times \Omega)$;

(b) $h \in L^p((0,T); W^{1-1/p,p}(\partial\Omega))$;

(c) $u_0 \in W^{2-2/p,p}(\Omega)$, *and, in case* $p > \frac{3}{2}$, $u_{0|\partial\Omega} \in W^{2-2/p,p}(\partial\Omega)$.

(II) If $p > \frac{3}{2}$ *the solution is unique.*

(III) In case $1 < p < \frac{3}{2}$, *the solution is not unique: more precisely, for each* g_0 *in* $W^{2-2/p,p}(\partial\Omega)$, *(1) has a unique solution* u *such that* $u_{|(0,T)\times\partial\Omega}(0, \cdot) = g_0$.

In the remaining part of this note, we shall try to give some hint about the meaning of Theorem 1. We begin by considering the simpler situation that $\Omega = \mathbb{R}^n_+$, $A(\xi, D_\xi) = \Delta_\xi$ and $B(\xi', D_\xi)$ is a first order differential operator with constant coefficients. Explicitly, we shall consider the problem

$$(5) \quad \begin{cases} D_t u(t,x,y) - D_x^2 u(t,x,y) - \Delta_y u(t,x,y) = f(t,x,y), \\ (t,x,y) \in (0,T) \times \mathbb{R}^+ \times \mathbb{R}^{n-1}, \\ D_t u(t,0,y) - \gamma D_x u(t,0,y) + v \cdot \nabla_y u(t,0,y) = g(t,y), \quad (t,y) \in (0,T) \times \mathbb{R}^{n-1}, \\ u(0,x,y) = u_0(x,y), \quad (x,y) \in \mathbb{R}^+ \times \mathbb{R}^{n-1}, \end{cases}$$

with $\gamma \in \mathbb{R}^+$ and $v \in \mathbb{R}^{n-1}$.

The first step is an analysis of the elliptic problem depending on the complex parameter λ

$$(6) \quad \begin{cases} \lambda u(x,y) - D_x^2 u(x,y) - \Delta_y u(x,y) = f(x,y), \\ (x,y) \in \mathbb{R}^+ \times \mathbb{R}^{n-1}, \\ \lambda u(0,y) - \gamma D_x u(0,y) + v \cdot \nabla_y u(0,y) = h(y), \quad y \in \mathbb{R}^{n-1}. \end{cases}$$

We begin by considering the Dirichlet problem depending on λ

$$\begin{cases} \lambda u(x,y) - D_x^2 u(x,y) - \Delta_y u(x,y) = f(x,y), \\ (x,y) \in \mathbb{R}^+ \times \mathbb{R}^{n-1}, \\ u(0,y) = g(y), \quad y \in \mathbb{R}^{n-1}. \end{cases} \tag{7}$$

The following result is well known:

Lemma 1. *Consider system (7). Let $p \in (1,\infty)$. Then, if $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, $f \in L^p(\mathbb{R}^n_+)$, $g \in W^{2-1/p,p}(\mathbb{R}^{n-1})$, (7) has a unique solution u in $W^{2,p}(\mathbb{R}^n_+)$. Moreover, $\forall \phi \in [0,\pi)$, there exists $C(\phi) \in \mathbb{R}^+$ such that, if $|\lambda| \geq 1$ and $|Arg(\lambda)| \leq \phi$,*

$$|\lambda| \|u\|_{L^p(\mathbb{R}^n_+)} + \|u\|_{W^{2,p}(\mathbb{R}^n_+)} \leq C(\phi)(\|f\|_{L^p(\mathbb{R}^n_+)} + \|g\|_{W^{2-1/p,p}(\mathbb{R}^{n-1})} + |\lambda|^{1-1/(2p)} \|g\|_{L^p(\mathbb{R}^{n-1})}).$$

In case $f = 0$, u can be represented in the form

$$u(x,\cdot) = \exp(-x(\lambda + A)^{1/2})g = \mathcal{F}^{-1}(\exp(-x(\lambda + |\cdot|^2)^{1/2}\mathcal{F}g) \tag{8}$$

In (8) we have indicated with A the operator $-\Delta_y$. It is well known that A is a positive operator in every space $W^{\theta,p}(\mathbb{R}^{n-1})$ and $-(\lambda + A)^{1/2}$ is the infinitesimal generator of an analytic semigroup in each of these spaces. Observe that formula (8) can be easily deduced (at least, formally) as an application of the Fourier transform. Employing it and setting $B := -v \cdot \nabla_y$, we obtain that, in order that u satisfies (6), g must be a solution to

$$\lambda g + \gamma(\lambda + A)^{1/2} g + Bg = h. \tag{9}$$

Employing the Fourier transform, (9) can be written in the form

$$\lambda g + \mathcal{F}^{-1}[\gamma(\lambda + |\eta|^2)^{1/2} + iv \cdot \eta]\mathcal{F}g = h. \tag{10}$$

(10) suggests that (9) is, in fact, a perturbation of

$$\lambda g + \gamma A^{1/2} g = h. \tag{11}$$

To this aim, observe that it is crucial the assumption that the coefficients of B are real. So we obtain the following

Lemma 2. *Let $\theta \in [0,\infty)$, $p \in (1,\infty)$. Then there exists $R \in \mathbb{R}^+$ such that, if $Re(\lambda) \geq 0$ and $|\lambda| \geq R$, (9) has a unique solution g in $W^{\theta+1,p}(\mathbb{R}^{n-1})$ $\forall h \in W^{\theta,p}(\mathbb{R}^{n-1})$.*

Of course, we employ Lemma 2 with $\theta = 1 - \frac{1}{p}$. So we have:

Proposition 1. *Consider system (6). Then there exists $R \in \mathbb{R}^+$ such that, if $Re(\lambda) \geq 0$, $|\lambda| \geq R$, $f \in L^p(\mathbb{R}^n_+)$, $h \in W^{1-1/p,p}(\mathbb{R}^{n-1})$, there is a unique solution u in $W^{2,p}(\mathbb{R}^n_+)$. Moreover, there is $C \in \mathbb{R}^+$ such that*

$$|\lambda| \|u\|_{L^p(\mathbb{R}^n_+)} + \|u\|_{W^{2,p}(\mathbb{R}^n_+)} \leq C(\|f\|_{L^p(\mathbb{R}^n_+)} + \|h\|_{W^{1-1/p,p}(\mathbb{R}^{n-1})}).$$

Proof. Let R be as in statement of Lemma 2, $Re(\lambda) \geq 0$, $|\lambda| \geq R$. By Lemma 1, there is a unique v in $W^{2,p}(\mathbb{R}^n_+)$ such that

$$\begin{cases} \lambda v(x,y) - D_x^2 v(x,y) - \Delta_y v(x,y) = f(x,y), \\ (x,y) \in \mathbb{R}^+ \times \mathbb{R}^{n-1}, \\ v(0,y) = 0, \quad y \in \mathbb{R}^{n-1}. \end{cases}$$

and

$$|\lambda| \|v\|_{L^p(\mathbb{R}^n_+)} + \|v\|_{W^{2,p}(\mathbb{R}^n_+)} \leq C_0 \|f\|_{L^p(\mathbb{R}^n_+)}.$$

Setting $z := u - v$, z should solve the system

$$\begin{cases} \lambda z(x,y) - D_x^2 z(x,y) - \Delta_y z(x,y) = 0, \\ (x,y) \in \mathbb{R}^+ \times \mathbb{R}^{n-1}, \\ \lambda z(0,y) - \gamma D_x z(0,y) + v \cdot \nabla_y z(0,y) = h(y) + \gamma D_x v(0,y), \quad y \in \mathbb{R}^{n-1}. \end{cases}$$

Setting $g := z_{|x=0}$, from Lemmata 1-2, we deduce

$$|\lambda| \|z\|_{L^p(\mathbb{R}^n_+)} + \|z\|_{W^{2,p}(\mathbb{R}^n_+)} \leq C_1(\|g\|_{W^{2-1/p,p}(\mathbb{R}^{n-1})} + |\lambda|^{1-1/(2p)} \|g\|_{L^p(\mathbb{R}^{n-1})}),$$

$$\|g\|_{W^{2-1/p,p}(\mathbb{R}^{n-1})} \leq C_2(\|h\|_{W^{1-1/p,p}(\mathbb{R}^{n-1})} + \|D_x v(0,\cdot)\|_{W^{1-1/p,p}(\mathbb{R}^{n-1})})$$

$$\leq C_3(\|h\|_{W^{1-1/p,p}(\mathbb{R}^{n-1})} + \|v\|_{W^{2,p}(\mathbb{R}^n_+)})$$

$$\leq C_4(\|f\|_{L^p(\mathbb{R}^n_+)} + \|h\|_{W^{1-1/p,p}(\mathbb{R}^{n-1})}),$$

$$|\lambda|^{1-1/(2p)}\|g\|_{L^p(\mathbb{R}^{n-1})} \leq C|\lambda|^{1-1/(2p)}\|g\|_{W^{1-1/p,p}(\mathbb{R}^{n-1})}$$

$$\leq C|\lambda|^{-1/(2p)}(\|h\|_{W^{1-1/p,p}(\mathbb{R}^{n-1})} + \|D_x v(0,\cdot)\|_{W^{1-1/p,p}(\mathbb{R}^{n-1})})$$

$$\leq C|\lambda|^{-1/(2p)}(\|f\|_{L^p(\mathbb{R}^n_+)} + \|h\|_{W^{1-1/p,p}(\mathbb{R}^{n-1})}).$$

From the estimates of v and z we draw the conclusion. □

As a simple consequence, we obtain the following

Theorem 2. *Let $p \in (1,\infty)$. We define the following operator G:*

$$\begin{cases} G: & D(G) = \{(u,g) \in W^{2,p}(\mathbb{R}^n_+) \times W^{2-1/p,p}(\mathbb{R}^{n-1}) : g = u(0,\cdot)\} \\ & \to L^p(\mathbb{R}^n_+) \times W^{1-1/p,p}(\mathbb{R}^{n-1}), \\ & \\ G(u,g) & := (\Delta u, \gamma D_x u(0,\cdot) - v \cdot \nabla_y u(0,\cdot)) \\ & = (\Delta u, \gamma D_x u(0,\cdot) - v \cdot \nabla_y g). \end{cases}$$

Then G is the infinitesimal generator of an analytic semigroup in the space $L^p(\mathbb{R}^n_+) \times W^{1-1/p,p}(\mathbb{R}^{n-1})$.

Proof. Applying a well known characterization of infinitesimal generators of analytic semigroups, we can try to show that $D(G)$ is dense in $L^p(\mathbb{R}^n_+) \times W^{1-1/p,p}(\mathbb{R}^{n-1})$ and there exist R, C positive such that, if $\lambda \in \mathbb{C}$, $Re(\lambda) \geq 0$ and $|\lambda| \geq R$, $\lambda \in \rho(G)$ and

$$\|(\lambda - G)^{-1}\|_{\mathcal{L}(L^p(\mathbb{R}^n_+)\times W^{1-1/p,p}(\mathbb{R}^{n-1}))} \leq C|\lambda|^{-1}.$$

We omit the first item. Concerning the second, it follows from Proposition 1 that, for some $R > 0$, $\{\lambda \in \mathbb{C} : Re(\lambda) \geq 0, |\lambda| \geq R\} \subseteq \rho(G)$. In fact, $(\lambda - G)^{-1}(f,h) = (u,g)$ with $g = u_{|x=0}$. Moreover,

$$\|u\|_{L^p(\mathbb{R}^n_+)} \leq C|\lambda|^{-1}(\|f\|_{L^p(\mathbb{R}^n_+)} + \|h\|_{W^{1-1/p,p}(\mathbb{R}^{n-1})})$$

and

$$\|g\|_{W^{1-1/p,p}(\mathbb{R}^{n-1})} = \|\lambda^{-1}(\gamma D_x u(0,\cdot) - v\cdot\nabla_y u(0,\cdot) + h)\|_{W^{1-1/p,p}(\mathbb{R}^{n-1})}$$

$$\leq C|\lambda|^{-1}(\|u\|_{W^{2,p}(\mathbb{R}^n_+)} + \|h\|_{W^{1-1/p,p}(\mathbb{R}^{n-1})})$$

$$\leq C|\lambda|^{-1}(\|f\|_{L^p(\mathbb{R}^n_+)} + \|h\|_{W^{1-1/p,p}(\mathbb{R}^{n-1})}).$$

□

As a consequence of Theorem 2 and the theory of analytic semigroups we obtain the following

Corollary 1. *Consider system (5). Let* $p \in (1,\infty)$, $\epsilon \in \mathbb{R}^+$, $f \in C^\epsilon([0,T];L^p(\mathbb{R}^n_+))$, $h \in C^\epsilon([0,T];W^{1-1/p,p}(\mathbb{R}^{n-1}))$, $u_0 \in W^{2,p}(\mathbb{R}^n_+)$. *Then there exists a unique solution* u *in* $C^1([0,T];L^p(\mathbb{R}^n_+)) \cap C([0,T];W^{2,p}(\mathbb{R}^n_+))$, *with* $g := u_{|x=0} \in C^1([0,T];W^{1-1/p,p}(\mathbb{R}^{n-1}))$. (u,g) *admits the representation*

$$(u(t,\cdot),g(t,\cdot)) = e^{tG}(u_0,u_{0|x_n=0}) + \int_0^t e^{(t-s)G}(f(s,\cdot),g(s,\cdot))ds.$$

Corollary 2. *Consider the system*

$$(12)\quad \begin{cases} D_t u(t,x,y) - D_x^2 u(t,x,y) - \Delta_y u(t,x,y) = 0, \\ (t,x,y) \in (0,T)\times\mathbb{R}^+\times\mathbb{R}^{n-1}, \\ D_t u(t,0,y) - \gamma D_x u(t,0,y) + v\cdot\nabla_y u(t,0,y) = 0, \quad (t,y)\in(0,T)\times\mathbb{R}^{n-1}, \\ u(0,x,y) = u_0(x,y), \quad (x,y)\in\mathbb{R}^+\times\mathbb{R}^{n-1}. \end{cases}$$

Let $u_0 \in L^p(\mathbb{R}^n_+)$. *Then,* $\forall h_0 \in W^{1-1/p,p}(\mathbb{R}^{n-1})$, *there is a unique element* u *of* $C([0,T]; L^p(\mathbb{R}^n_+)) \cap C^1((0,T];W^{2,p}(\mathbb{R}^n_+))$, *with* $u_{|x=0} \in C^1((0,T];W^{1-1/p,p}(\mathbb{R}^{n-1})$, *such that:*

(I) u *satisfies (12);*

(II) $\lim_{t\to 0}\|u(t,0,\cdot) - h_0\|_{W^{1-1/p,p}(\mathbb{R}^{n-1})} = 0.$

Of course, $(u(t,\cdot),u_{|x=0}(t,\cdot)) = e^{tG}(u_0,h_0)$. Corollaries 1-2 suggest that (5) has, at most, one solution in the setting of solutions u which are regular, in the sense that the prescribed regularity implies that the trace at $u_{|t=0}$ has a trace at $x=0$, which must coincide with the trace at $t=0$ of $u_{|x=0}$.

This enlightens the statement of Theorem 1: if $u \in W^{1,p}((0,T);L^p(\Omega)) \cap L^p((0,T);W^{2,p}(\Omega))$, then $u_0 = u_{|t=0} \in W^{2-2/p,p}(\Omega)$. If $p > \frac{3}{2}$, $2 - \frac{2}{p} > \frac{1}{p}$, which implies that $u_{|t=0}$ admits a trace in $x = 0$. Moreover, the belonging of u to $W^{1,p}((0,T);L^p(\Omega)) \cap L^p((0,T);W^{2,p}(\Omega))$ implies that $u_{|(0,T)\times\partial\Omega} \in W^{1-1/(2p),2-1/p}((0,T)\times\partial\Omega)) = W^{1-1/(2p),p}((0,T);L^p(\partial\Omega)) \cap L^p((0,T),W^{2-1/p,p}(\partial\Omega))$. However, we are requiring a stronger regularity of u in $(0,T)\times\partial\Omega$, namely that $g := u_{|(0,T)\times\partial\Omega} \in W^{1,p}((0,T);W^{1-1/p,p}(\partial\Omega)) \cap L^p((0,T),W^{2-1/p,p}(\partial\Omega))$. This implies that $g_{|t=0}$ is well defined and belongs to $W^{2-2/p,p}(\partial\Omega)$. So, in case $p > \frac{3}{2}$, $g_{|t=0}$ must coincide with $u_{0|\partial\Omega} = u(0,\cdot)_{|\partial\Omega}$. In this case, we have to take $u_0 \in W^{2-2/p,p}(\Omega)$, such that $u_{0|\partial\Omega} = g_{|t=0} \in W^{2-2/p,p}(\partial\Omega)$. In case $p < \frac{3}{2}$, u_0 does not admit a trace at $t = 0$. In this case, we can choose $g_{|t=0}$ arbitrarily in $W^{2-2/p,p}(\partial\Omega)$ and we can prove that, for any choice, we obtain a solution with the prescribed regularity.

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