A SMALE TYPE RESULT AND APPLICATIONS TO CONTACT HOMOLOGY.

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Abstract. In this note we will show that the injection of a suitable subspace of the space of Legendrian loops into the full loop space is an S^1-equivariant homotopy equivalence. Moreover, since the smaller space is the space of variations of a given action functional, we will compute the relative Contact Homology of a family of tight contact forms on the three-dimensional torus.

Sunto. In questa nota mostreremo che l'inclusione di un opportuno sottospazio dello spazio dei cappi Legendriani nello spazio totale dei cappi è un'equivalenza omotopica S^1-equivariante. Inoltre, poichè il primo sottospazio è lo spazio delle variazioni di un dato funzionale azione, calcoleremo l'omologia di contatto per una famiglia di forme di contatto tight sul toro tridimensionale.

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1. Introduction and statement of the result

Let M be a 3-dimensional smooth, orientable, and compact manifold without boundary, and let α be a 1-form on it. The couple (M, α) is called a contact manifold if the form $\alpha \wedge d\alpha$ is a volume form on M. We denote by $\mathcal{H}^1(S^1, M)$ the Sobolev space of the maps from S^1 to M; a curve $x \in \mathcal{H}^1(S^1, M)$ is called legendrian if its tangent vector is in the kernel of α, that is $\alpha(\dot{x}) = 0$. We denote by $\mathcal{L}_\alpha$ the space of legendrian closed curves on M: this space is a subset of the free loop space of M denoted by $\Lambda(S^1, M)$. Let us recall a result of S.Smale [21]:

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Theorem (Smale). *Let (M,α) be a contact manifold. Then the injection*

$$j : \mathcal{L}_\alpha \hookrightarrow \Lambda(S^1, M)$$

is an S^1-equivariant homotopy equivalence.

In a joint paper with A.Maalaoui [18], we proved a result similar to the above theorem: the framework is slightly different and the space $\mathcal{L}_\alpha$ is replaced by a smaller space $\mathcal{C}_\beta$, that appears in some variational problems in contact form geometry (see for instance [3],[4] and [5]). Let us introduce the following assumption:

(A) there exists a smooth vector field $v \in \ker(\alpha)$ such that the dual 1-form $\beta = d\alpha(v,\cdot)$ is a contact form with the same orientation than α.

Under hypothesis (A), we normalize v onto λv so that $\alpha \wedge d\alpha = \beta \wedge d\beta$.
By Smale's theorem, then we know that the injection $\mathcal{L}_\beta$ in $\Lambda(S^1, M)$ is an S^1-equivariant homotopy equivalence. We are interested in a space that is smaller than $\mathcal{L}_\beta$ and it is defined in the following way:

$$\mathcal{C}_\beta = \{x \in \mathcal{L}_\beta ; \alpha_x(\dot{x}) = c > 0\} \tag{1}$$

where c is a constant that depends on the curve x.
As example, let us consider the framework of (S^3, α_0), where

$$S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \ : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$$

and

$$\alpha_0 = x_2 dx_1 - x_1 dx_2 + x_4 dx_3 - x_3 dx_4$$

is the standard contact form on S^3, and let

$$v = -x_4 \partial_{x_1} - x_3 \partial_{x_2} + x_2 \partial_{x_3} + x_1 \partial_{x_4}$$

be a Hopf fibration vector field in $\ker \alpha_0$. The space $\mathcal{C}_\beta$ can be identified as the lift to S^3 (see [3]) of the space $Imm_0(S^1; S^2)$ of immersed curves from S^1 into S^2 of Maslov index zero. Smale's theorem [21] asserts then that the injection $\mathcal{C}_\beta \hookrightarrow \Lambda(S^1, S^3)$ is an S^1-equivariant homotopy equivalence. In [18], we extend this result to a general framework of contact manifold (M,α) under (A) and an additional assumption that we will introduce below. We need, in order to state this second assumption, to define the one-parameter group generated by v, that we will denote by ϕ_s, namely the diffeomorphism generated by the flow

$$\begin{cases} \dfrac{d}{ds}\big(\phi_s(x)\big) = v_{\phi_s(x)} \\ \\ \phi_0(x) = x. \end{cases} \tag{2}$$

By [3] and [5] we know that the kernel of a contact form rotates monotonically in a frame transported by ϕ_s along v. Based on this fact we give the following definition.

Definition 1.1. We say that $\ker\alpha$ *turns well along* v, if starting from any given x_0 in M, the rotation of $\ker\alpha$ along the v-orbit in a transported frame exceeds π.[1]

This last condition can be explicitly checked by using the map ϕ_s (see for instance [8]).

Our second assumption is therefore:

$$(B) \quad \ker\alpha \text{ turns well along } v.$$

We prove the following

Theorem 1.1. *Let (M,α) be a contact compact manifold with no boundary. Then under the assumptions (A) and (B), the injection*

$$\mathcal{C}_\beta \hookrightarrow \Lambda(S^1, M)$$

is an S^1-equivariant homotopy equivalence.

Here we are going to explain the main ideas of the proof and we will give some applications; we refer the reader to [18] for the details.

[1] It is in fact then infinite.

Let us recall some properties that we will use in the sequel. Given the contact form α, we will let ξ be its Reeb vector field. Namely, ξ is the unique vector satisfying

$$\alpha(\xi) = 1, \qquad d\alpha(\xi, \cdot) = 0$$

Under the assumption (A), if w denotes the Reeb vector field of the 1-form β, then there exist two functions τ and $\overline{\mu}$ such that:

$$[\xi, [\xi, v]] = -\tau v, \qquad w = -[\xi, v] + \overline{\mu}\xi,$$

where $\overline{\mu} = d\alpha(v, [v, [\xi, v]])$. We explicitly observe that with the previous notation, the following holds:

$$\dot{x} = a\xi + bv, \quad \forall x \in \mathcal{L}_\beta. \tag{3}$$

Moreover, if x is in $\mathcal{C}_\beta$ then a is a positive constant. One can show (see [3]) that $\mathcal{C}_\beta \setminus M$ has a Hilbert manifold structure. For $x \in \mathcal{C}_\beta$, the tangent space at the curve x is given by the set of vector fields

$$Z = \lambda\xi + \mu v + \eta w,$$

with the coefficients λ, μ and η satisfying the following equations:

$$\begin{cases} \dot{\overbrace{\lambda + \overline{\mu}\eta}} = b\eta - \int_0^1 b\eta, \\ \\ \dot{\eta} = \mu a - \lambda b \end{cases} \tag{4}$$

where λ, μ and η are 1-periodic.

The proof of the main theorem requires several steps. We first apply Smale's theorem to conclude that the injection $\mathcal{L}_\beta \hookrightarrow \Lambda(S^1, M)$ is a homotopy equivalence. Next, we introduce an intermediate space $\mathcal{C}_\beta^+$ defined by

$$\mathcal{C}_\beta^+ = \{x \in \mathcal{L}_\beta; \alpha(\dot{x}) \geq 0\},$$

and we show that we can deform $\mathcal{L}_\beta$ to $\mathcal{C}_\beta^+$: this deformation is not continuous because "Dirac masses" along v are created through this procedure; we will "solve the Dirac masses", showing how they are created along a smooth deformation in $\mathcal{L}_\beta$.

In the last step we "push" the curves of $\mathcal{C}_\beta^+$ into $\mathcal{C}_\beta$: this will be completed by constructing a flow that brings curves with $a \geq 0$ to curves with $a > 0$.

Now we want to make some comments about the assumptions and we give some examples of contact structures for which they hold.

Assumption (A) holds for a number of contact structures with suitable vector fields v in their kernels. For instance it is satisfied for the standard contact form α_0 on S^3, and also for the family of contact structures on T^3 given by

$$\alpha_n = \cos(2n\pi z)dx + \sin(2n\pi z)dy$$

All the contact forms in the previous examples are tight: there are also overtwisted contact forms satisfying (A) (see for instance [11] about the definition of tight and overtwisted). This is the case of the first non-standard 1-form on S^3, given by Gonzalo-Varela in [15]:

$$\alpha_1 = -(\cos(\frac{\pi}{4} + \pi(x_3^2 + x_4^2))(x_2dx_1 - x_1dx_2) + \sin(\frac{\pi}{4} + \pi(x_3^2 + x_4^2))(x_4dx_3 - x_3dx_4))$$

where the (explicit) existence of a suitable v satisfying (A) is proved in [20].

The assumption (B) holds also for the previous mentioned examples; moreover this assumption has a deeper meaning. In fact, it was proved in a paper of Gonzalo [14], that (B) holds if and only if α extends to a contact circle, namely there exists another contact form α_2 transverse to α (their kernels intersect transversally) with intersection the line spanned by v, such that

$$\cos(s)\alpha + \sin(s)\alpha_2$$

is a contact form for every $s \in \mathbb{R}$.

Let us observe that α_1 defined above represents the first example of an overtwisted contact circle on a compact manifold. In fact, in [13] Geigs and Gonzalo give an example of an overtwisted contact circle on $\mathbb{R}^3$ and they point out that they do not know an explicit example of overtwisted contact circle on a compact closed manifold: α_1 with the v found in [20] is such an example.

Moreover, using this criteria we can give some conditions under which (B) holds:

Lemma 1.1. *Assume that (A) holds, then (B) holds if one of the following conditions is satisfied:*

$$(i) \quad |\overline{\mu}| < 2$$

$$(ii) \quad \text{there exists a map } u \text{ on } M \text{ such that } \overline{\mu} = u_v.$$

where we denoted by $u_v := du(v)$ *the derivative of* u *along the vector field* v. *Moreover, if* $\overline{\mu} = 0$ *then* α *is tight.*

Proof. We use the characterization stated above for contact circles. Let s be a real number, and consider the 1-form

$$\alpha_s = \cos(s)\alpha + \sin(s)\beta;$$

then

$$\alpha_s \wedge d\alpha_s = \cos^2(s)\alpha \wedge d\alpha + \sin^2(s)\beta \wedge d\beta + \cos(s)\sin(s)(\alpha \wedge d\beta + \beta \wedge d\alpha)$$

Notice now that $\alpha \wedge d\beta(\xi, v, w) = -\overline{\mu}$, thus we have

$$\alpha_s \wedge d\alpha_s(\xi, v, w) = 1 - \frac{\sin(2s)}{2}\overline{\mu}$$

and the conclusion follows for (i).
For (ii) we consider

$$\alpha_s = \cos(s)\alpha + \sin(s)e^u\beta$$

and same computation as before yields

$$\alpha_s \wedge d\alpha_s = \cos^2(s)\alpha \wedge d\alpha + e^{2u}\sin^2(s)\beta \wedge d\beta + \sin(s)\cos(s)e^u(\alpha \wedge d\beta + \alpha \wedge du \wedge \beta).$$

Evaluating at $(\xi, v, [\xi, v])$ we get:

$$\alpha_s \wedge d\alpha_s = \cos^2(s) + e^{2u}\sin^2(s) + e^u\sin(s)\cos(s)(u_v - \overline{\mu})$$

and therefore (ii) follows.
Now notice that if $\overline{\mu} = 0$ then we have what it is called a "taut" contact circle, therefore based on the result of Geigs-Gonzalo [13], we have that α and β are tight. $\square$

2. Applications

Here we will consider a family of contact structures on the torus T^3 and we will compute their relative Contact Homology. We will set the problem in a suitable variational framework and we will use the techniques developed by A.Bahri in his works [3], [1], [2] and with Y.Xu in [8].

Let us then define the torus $T^3 = S^1 \times S^1 \times S^1$, parameterized with coordinates

$$(x, y, z) \in (0, 2\pi) \times (0, 2\pi) \times (0, 2\pi)$$

and by identifying 0 and 2π. On the torus we consider the family of infinitely many differential one-forms defined by

$$\alpha_n = \cos(nz)dx + \sin(nz)dy, \qquad n \in \mathbb{N}.$$

A direct computation shows that

$$d\alpha_n = n\sin(nz)dx \wedge dz - n\cos(nz)dy \wedge dz$$

and consequently

$$\alpha_n \wedge d\alpha_n = -ndx \wedge dy \wedge dz.$$

Therefore, for every $n \in \mathbb{N}$, (T^3, α_n) is a contact manifold, with contact structure given by $\sigma_n = \ker(\alpha_n)$. In particular by a classification result due to Y.Kanda [16], we have that every tight contact structure on T^3 is contactomorphic [2] to one of the α_n; moreover for $n \neq m$, the contact structures σ_n and σ_m are not contactomorphic.

Our main result is the following:

Theorem 2.1. *Let g be a homotopy class of the two-dimensional torus T^2, then for every $n \in \mathbb{N}$, we have*

$$H_k(\alpha_n, g) = \begin{cases} \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \ n \ \textit{times}, & \textit{if } k = 0, 1 \\ 0, & \textit{if } k > 1 \end{cases} \tag{5}$$

[2]Two contact forms α_1, α_2 on M are contactomorphic if there exists a diffeomorphism $\varphi : M \to M$, that preserves the kernels, namely: $\varphi^*(\alpha_1) = \lambda\alpha_2$, for some non-zero function λ on M.

Moreover we prove that the homology is locally stable, namely we consider small perturbations of the forms in the family $\{\alpha_n\}$ and we show that our computations still hold.
We also show some additional algebraic relations between the contact homologies of the family $\{\alpha_n\}$: in particular we exhibit an equivariant homology reduction under the action of $\mathbb{Z}_k$, that is, for every integer k, we prove the existence of a morphism

$$f_* : H_*(\alpha_{kn}, g) \longrightarrow H_*(\alpha_n, g)$$

that corresponds to an equivariant homology reduction under the action of the group $\mathbb{Z}_k$, namely

$$H_*(\alpha_n, g) = H_*^{\mathbb{Z}_k}(\alpha_{kn}, g).$$

Finally, we consider the case of a more general 2-torus bundles over S^1

$$T^2 \times \mathbb{R}/(x, y, z) = (A(x, y), z + 2\pi)$$

where A is a given matrix in $SL_2(\mathbb{Z})$, with the contact forms introduced by Giroux [12]

$$\alpha_h = \cos(h(z))dx + \sin(h(z))dy$$

where h is a strictly increasing function. We prove that for the related contact structures Theorem 2.1 still holds.
Again, here we will discuss these computations and we will explain the basic strategy to get the results: we refer the reader to [19] for a detailed proof.
First, we notice that there are other results on contact homology computations, see F.Bourgeois [9] and F.Bourgeois-V.Colin [10], where the authors compute the homology using the cylindrical contact homology and also E.Lebow [17] which computed the embedded contact homology for 2-torus bundles.
Now, let us briefly introduce the general framework. In order to apply the theory developed by A.Bahri, we first need to show that condition (A) holds true in our setting. Next, we define the action functional

$$J(x) = \int_0^1 \alpha(\dot{x}) \tag{6}$$

on the subspace C_β of $\mathcal{H}^1(S^1, M)$ defined by (1). Then the following result by A.Bahri-D.Bennequin holds [3]:

Theorem 2.2. *J is a C^2 functional on C_β whose critical points are of finite Morse index and are periodic orbits of ξ.*

Therefore, we compute the Morse homology related to C_β, but due to our main result (1.1), we compute the Morse homology of the full loop space indeed.
The second derivative of J at a critical point x (i.e. $b = 0$, see (3)) reads as:

$$J''(x) \cdot z \cdot z = \int_0^1 \dot{\eta}^2 - a^2\eta^2\tau \tag{7}$$

The major difficulties in this variational analysis are the lack of compactness (that is the Palais-Smale condition does not hold) and the loss of the Fredholm condition. In fact, the linearized operator is not Fredholm in general and this means that in the Morse theoretical methods one cannot apply the implicit function theorem anymore and therefore the Morse lemma does not hold. For instance, we know that the Fredholm assumption is violated for the standard contact structure on S^3 and the first exotic structure of Gonzalo and Varela [15]. Anyway, there is a simple criteria to check whether a violation occurs or not based on some properties of the transport map ϕ of the special legendrian vector field v. First, by looking at the functional J in the larger space $\mathcal{C}_\beta^+$, we notice that it is not affected by the introduction of a "back and forth" v piece (that is, J does not change if $\pm v$ pieces are added to the curves). So, let us take a curve that is transverse to v, and at a point $x(t_0)$ we introduce a "back and forth" v piece of length s and let us call x_ϵ the curve obtained by introducing a small "opening" piece of length ϵ between the two v pieces. Then we have

$$J(x_\epsilon) = J(x) - \epsilon(\alpha_{x(t_0)}(d\phi_{-s}(\xi)) - 1) + o(\epsilon).$$

Thus if there exists $s > 0$ such that $\alpha(\phi_{-s}(\xi)) > 1$, then we would have a decreasing direction from the level $J(x)$ and we would be able to bypass a critical point without changing the topology even though it has a finite Morse index, and this is due exactly to the loss of the Fredholm condition. Hence we can state the following:

Lemma 2.1 (Bahri [7])**.** *If $\phi_{-s}^*(\alpha)(\xi) < 1$, for every $s \neq 0$, then J satisfies the Fredholm condition.*

We will show that in our framework the Fredholm property does not hold. In fact, we will see that we will have situations for which there will exist $s \neq 0$, such that $\phi_{-s}^{*}(\alpha)(\xi) = 1$.

Now, in order to prove Theorem 2.1, we need first to compute explicitly all the quantities defined in this variational framework for our family of contact forms $\{\alpha_n\}$.

Later, since we show that the second derivative of J has a direction of degeneracy corresponding to the action of $[\xi, v]$, the critical points come in circles. This degeneracy can be removed by a small perturbation of the functional in a neighborhood of the critical points in order to "break the symmetry".

Then, in order to compute explicitly the homology in our framework, we need to worry about the non-compactness due to the presence of asymptotes. To deal with that, we show that the the critical points at infinity have always higher energy so that they cannot interact with our critical points, that is, cancellations cannot occur. Hence we have only to count the number of periodic orbits. The idea is the same as in the theory of critical points at infinity, namely after compactifying the space, by adding the asymptotes, the classical Morse theory tells us that indeed $\partial^2 = 0$: in this situation the boundary operator ∂ has two components $\partial = \partial_{per} + \partial_{\infty}$. The operator ∂_{per} counts the number of pseudo-gradient flow lines between periodic orbits (actual critical points) and ∂_{∞} counts the flow lines between critical points at infinity and periodic orbits. Therefore to show that we have compactness in our homology theory, we need that $\partial_{per}^2 = 0$. Now, since

$$\partial^2 = \partial_{per}^2 + \partial_{\infty}^2 + \partial_{per}\partial_{\infty} + \partial_{\infty}\partial_{per},$$

the proof reduces to showing that

$$\partial_{per}\partial_{\infty} + \partial_{\infty}\partial_{per} = 0$$

when applied to periodic orbits, so that compactness holds.

Finally, even if the Fredholm condition is violated, we are nevertheless able to show that the homology is locally stable along isotopies.

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