THE CAUCHY PROBLEM FOR SCHRÖDINGER EQUATIONS WITH TIME-DEPENDENT HAMILTONIAN
IL PROBLEMA DI CAUCHY PER EQUAZIONI DI SCHRÖDINGER CON HAMILTONIANA DIPENDENTE DAL TEMPO

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Abstract. We consider the Cauchy problem for a Schrödinger equation with an Hamiltonian depending also on the time variable and that may vanish at $t = 0$. We find optimal Levi conditions for well-posedness in Sobolev and Gevrey spaces.

Sunto. Si considera il problema di Cauchy per una equazione di Schrödinger con hamiltoniana dipendente anche dal tempo e che puó annullarsi per $t = 0$. Si trovano condizioni di Levi ottimali per la buona posizione in spazi di Sobolev e di Gevrey.

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1. Introduction and main result

Let us consider the Cauchy problem in $[0, T] \times \mathbb{R}^n_x$

$$Su = 0, \quad u(0, x) = u_0(x), \tag{1}$$

for the Schrödinger operator

$$S := \frac{1}{i}\partial_t - a(t)\Delta_x + \sum_{j=1}^{n} b_j(t, x)\partial_{x_j} \tag{2}$$

with a real continuous coefficient $a(t)$ such that

$$ct^\ell \le a(t) \le Ct^\ell \tag{3}$$

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for $\ell \geq 0$ and positive constants c, C. The coefficients b_j in the convection term are continuos with respect to the time variable t and bounded together with all their derivatives with respect to the space variable x. Their behavior for $t \to 0$ and $|x| \to +\infty$ is assumed to be such that

$$|\Re b_j(t,x)| \leq Ct^k \langle x \rangle^{-\sigma},\ 0 \leq k \leq \ell,\ \sigma > 0, \tag{4}$$

$\langle x \rangle = \sqrt{1+|x|^2}$.
We investigate the well-posedness in Sobolev spaces H^m and in Gevrey spaces $H^{\infty,s}$, $s > 1$,

$$H^{\infty,s} := \cap_m H^{m,s},\ H^{m,s} := \cup_{\varrho>0} H^{m,s}_\varrho,\ H^{m,s}_\varrho = e^{-\varrho\langle D_x\rangle^{1/s}} H^m.$$

Gevrey well-posedness can be considered provided that $b_j \in C([0,T];\gamma^s)$, where

$$\gamma^s := \cup_{A>0}\gamma^s_A,\ \gamma^s_A := \left\{ f(x) : \left|\partial_x^\beta f(x)\right| \leq CA^{|\beta|}\beta!^s\ |\beta| \geq 0 \right\}.$$

In the widely studied case of time independent coefficients $a(t) = \tau$, $\tau \neq 0$ a real constant, and $b_j(t,x) = b_j(x)$, we have sharp results of well-posedness in

$$\begin{cases} L^2 \text{ if } \sigma > 1, \\ H^\infty \text{ if } \sigma = 1, \\ H^{\infty,s} \text{ with } s < \frac{1}{1-\sigma} \text{ if } \sigma < 1, \end{cases} \tag{5}$$

see [11] and also [13], [7], [9], [3], [4], [10]. In particular we have the necessity of decay conditions for $|x| \to +\infty$.

Time-depending Hamiltonians occur in applications, for example in the study of quantum boxes. From our results in [1], the same well-posedness as in (5) holds true taking a non degenerating Hamiltonian for $t \to 0$ ($k = \ell = 0$) and even for vanishing coefficients but with the same order $k = \ell$. In the case $k < \ell$ the well-posedness in the usual Sobolev space H^∞ fails and the problem is well-posed only in Gevrey spaces. In fact, we have:

Theorem 1.1. *Let us assume (3) and (4) with* $0 < k < \ell$. *Then, the Cauchy problem (1) for the operator (2) is well-posed in* $H^{\infty,s}$ *if and only if*

$$\begin{cases} s < \frac{\ell+1}{\ell-k} \text{ for } 1-\sigma \leq \frac{\ell-k}{\ell+1}, \\ s < \frac{1}{1-\sigma} \text{ for } \frac{\ell-k}{\ell+1} < 1-\sigma. \end{cases} \tag{6}$$

The necessity of these conditions is proved in [2]. Here we show that they are also sufficient.

For a fast decay, given by $1-\sigma \leq (\ell-k)/(\ell+1)$, the well-posedness is influenced only by the degeneracy but this gives an upper bound for the index s which can not reach the limit value $s = \infty$ corresponding to the usual Sobolev space H^∞. With $(\ell-k)/(\ell+1) < 1-\sigma$ even the degeneracy is overshadowed by the too slow decay.

2. Strategy in the proof

We briefly outline the strategy of the proof that will be given in more details in the following sections. As in [11] and [1] we prove the well-posedness of the Cauchy problem (1) for the operator S in (2) after performing a change of variables

$$v(t,x) = e^\Lambda(t,x,D_x)u(t,x), \tag{7}$$

where $e^\Lambda(t,x,D_x)$, $D = \frac{1}{i}\partial$, is an invertible pseudo-differential operator with symbol $e^{\Lambda(t,x,\xi)}$. With respect to the corresponding case $k < \ell$, $\sigma < 1$ in [1], we get a smaller order for Λ. This leads to larger values for the index s of Gevrey well-posedness.
Here the function $\Lambda(t,x,\xi)$ is real-valued and belongs to $C([0,T]; S^{1/s})$, $s < 1/q$ with

$$q = \max\left\{\frac{\ell-k}{\ell+1}, 1-\sigma\right\}, \tag{8}$$

where S^m denotes the class of symbols of order m. We look for $\Lambda(t,x,\xi)$ in order to establish the energy estimate

$$\|v(t,\cdot)\|_{L^2} \leq C\|v(0,\cdot)\|_{L^2} \tag{9}$$

for any solution of the transformed equation

$$S_\Lambda v = 0, \; S_\Lambda := e^\Lambda S(e^\Lambda)^{-1}. \tag{10}$$

The energy estimate (9) follows by Gronwall's lemma if we find Λ such that

$$iS_\Lambda = \partial_t - ia(t)\Delta_x - A(t,x,D_x). \tag{11}$$

Here $A(t,x,D_x)$ is a pseudo-differential operator of order 1 which is bounded from above in L^2, that is,

$$2\Re\big(A(t,x,D_x)v,v\big) \le C\|v\|^2_{L^2}. \tag{12}$$

In view of the sharp Gårding inequality, in order to get this property for A we seek for a function Λ in (7) that solves

$$\partial_t\Lambda(t,x,\xi) + 2a(t)\sum_{j=1}^{n}\xi_j\partial_{x_j}\Lambda(t,x,\xi) + \Re\sum_{j=1}^{n} b_j(t,x)\xi_j \le 0 \text{ for all } |\xi|\ge h, \tag{13}$$

and such that $\partial_t\Lambda(t,x,\xi)$ has the order 1 and $a(t)\partial_{x_j}\Lambda$ has the order zero.

As it is well-known, the estimate (9) gives the well-posedness in L^2 of the Cauchy problem for the operator S_Λ. Since

$$e^{\Lambda(t)} : H^{m,s} \to H^m,\ \ s < 1/q,$$

is continuous and invertible, then we have a unique solution $u \in C([0,T];H^{\infty,s})$ of (1) for any given initial data $u_0 \in H^{\infty,s}$, $s < \min\{(\ell+1)/(\ell-k), 1/(1-\sigma)\}$.

3. Degeneracy

In this section we construct the solution Λ to the inequality (13) and we estimate it only in the case

$$1-\sigma \le \frac{\ell-k}{\ell+1},$$

that gives

$$q = \frac{\ell-k}{\ell+1}$$

in (8). Few changes appearing in the estimates of Λ in the case $1-\sigma > (\ell-k)/(\ell+1)$ are collected in next section.

For readers' convenience and in order to have a more self-contained paper, we repeat some parts of the construction which are conducted in a similar way as in [1]. The improvement in the case under consideration comes from a sharper analysis in the extended phase-space $\{(t,x,\xi) \in [0,T] \times \mathbb{R}^{2n}_{x,\xi}\}$. First, as in [1], we split it into two zones. Defining the separation line between both zones by

$$t_\xi = \langle \xi \rangle_h^{-\frac{1}{\ell+1}}, \tag{14}$$

where

$$\langle \xi \rangle_h = \sqrt{h^2 + |\xi|^2},\ h \geq 1,$$

we introduce the

$$\text{pseudo-differential zone:} \quad Z_{pd} = \{(t,x,\xi) \in [0,T] \times \mathbb{R}^{2n}_{x,\xi} : t \leq t_\xi\},$$

$$\text{evolution zone:} \quad Z_{ev} = \{(t,x,\xi) \in [0,T] \times \mathbb{R}^{2n}_{x,\xi} : t \geq t_\xi\}.$$

Localizing to the pseudo-differential zone a solution of (13) in Z_{pd} is simply given by

$$\Lambda_{pd}(h,t,\xi) = -M\langle \xi \rangle_h \int_0^t \tau^k \chi(\tau/t_\xi) d\tau, \tag{15}$$

where $\chi(y)$ is a cut-off function in $\gamma^s(\mathbb{R})$, $0 \leq \chi(y) \leq 1$, $\chi(y) = 1$ for $|y| \leq 1/2$, $\chi(y) = 0$ for $|y| \geq 1$, $y\chi'(y) \leq 0$, and $M \geq M_0$ is a large constant.

The symbol $\Lambda_{pd}(h,t,\xi)$ is of order $(\ell - k)/(\ell + 1)$ by the above definition (14) of t_ξ. Taking a sufficiently large M it follows

$$\partial_t \Lambda_{pd}(h,t,\xi) + \chi(t/t_\xi) \Re \sum_{j=1}^n b_j(t,x)\xi_j \leq 0 \tag{16}$$

since

$$\sum_{j=1}^n |\Re b_j(t,x)\xi_j| \leq M_0 t^k \langle x \rangle^{-\sigma} |\xi| \leq M_0 t^k |\xi|.$$

Moreover, we have

$$|\partial_\xi^\alpha \Lambda_{pd}(h,t,\xi)| \leq C_0 M A^{|\alpha|} \alpha!^s \langle \xi \rangle_h^{\frac{\ell-k}{\ell+1} - |\alpha|}, \tag{17}$$

$$|\partial_\xi^\alpha \partial_t \Lambda_{pd}(h,t,\xi)| \leq C_0 M A^{|\alpha|} \alpha!^s \langle \xi \rangle_h^{1 - \frac{k}{\ell+1} - |\alpha|} \tag{18}$$

with constants C_0 and A which are independent of h. This large parameter h will be used for many estimates and, in particular, it is used also to get the invertibility of the operator e^Λ in the transformed equation (10).

Coming to the evolution zone, we split it into two sub-zones:

$$Z_{ev}^1 = \{(t,x,\xi) \in [0,T] \times \mathbb{R}^{2n}_{x,\xi} : t \geq t_\xi, \langle x \rangle \leq t^{\ell+1}\langle \xi \rangle\},$$

$$Z_{ev}^2 = \{(t,x,\xi) \in [0,T] \times \mathbb{R}^{2n}_{x,\xi} : t \geq t_\xi, \langle x \rangle \geq t^{\ell+1}\langle \xi \rangle\}.$$

A solution of (13) in Z_{ev}^2 is given by

$$\Lambda_{ev}^2(h,t,\xi) = -KM\langle \xi \rangle_h^{1-\sigma} \int_0^t \tau^{k-(\ell+1)\sigma}\big(1-\chi(2\tau/t_\xi)\big)d\tau. \tag{19}$$

Taking a sufficiently large M (the constant $K > 1$ will be fixed later independently of all other parameters) we have

$$\partial_t \Lambda_{ev}^2(h,t,\xi) + \big(1-\chi(t/t_\xi)\big)\big(1-\chi(\langle x \rangle / t^{\ell+1}\langle \xi \rangle)\big)\Re \sum_{j=1}^n b_j(t,x)\xi_j \leq 0 \tag{20}$$

in view of

$$\sum_{j=1}^n |\Re b_j(t,x)\xi_j| \leq M_0 t^k \langle x \rangle^{-\sigma}\langle \xi \rangle \leq M_0 t^{k-(\ell+1)\sigma}\langle \xi \rangle^{1-\sigma}, \ \langle x \rangle \geq t^{\ell+1}\langle \xi \rangle/2.$$

From the definition (14) of t_ξ, the function Λ_{ev}^2 satisfies

$$|\partial_\xi^\alpha \Lambda_{ev}^2(h,t,\xi)| \leq \begin{cases} C_0 M A^{|\alpha|}\alpha!^s \langle \xi \rangle_h^{\frac{\ell-k}{\ell+1}-|\alpha|}, \ 1-\sigma < \frac{\ell-k}{\ell+1}, \\ C_0 M A^{|\alpha|}\alpha!^s \langle \xi \rangle_h^{\frac{\ell-k}{\ell+1}-|\alpha|} \log\langle \xi \rangle_h, \ 1-\sigma = \frac{\ell-k}{\ell+1}, \end{cases} \tag{21}$$

$$|\partial_\xi^\alpha \partial_t \Lambda_{ev}^2(h,t,\xi)| \leq C_0 M A^{|\alpha|}\alpha!^s \langle \xi \rangle_h^{1-\frac{k}{\ell+1}-|\alpha|} \tag{22}$$

since here we have $k+1-(\ell+1)\sigma \leq 0$, $1/t \leq 4\langle \xi \rangle_h^{1/(\ell+1)}$. The constants C_0 and A are independent of h.

The support of the function Λ_{ev}^2 contains the whole evolution zone Z_{ev} and not only Z_{ev}^2 because it will be also used to control the derivative $\partial_t \Lambda_{ev}^1$ of the term Λ_{ev}^1, localized to Z_{ev}^1, of the solution $\Lambda_{ev} = \Lambda_{ev}^1 + \Lambda_{ev}^2$ of (13) in Z_{ev}. At this point we will fix the constant K in (19).

In order to construct such a function Λ_{ev}^1 we consider the solution $\lambda(t,x,\xi)$ of the equation

$$\sum_{j=1}^n \xi_j \partial_{x_j}\lambda(t,x,\xi) + |\xi| g(t,x,\xi) = 0 \tag{23}$$

that is given for $\xi \neq 0$ by

$$\lambda(t,x,\xi) = -\int_0^{x\cdot\omega} g(t, x-\tau\omega, \xi)d\tau \text{ with } \omega = \xi/|\xi|.$$

We take

$$\lambda_{0,1}(t,x,\xi) = -\int_0^{x\cdot\omega} g_1(t,x-\tau\omega,\xi)d\tau \text{ with } g_1(t,x,\xi) = M\langle x\rangle^{-\sigma}\chi\big(\langle x\rangle/t^{\ell+1}\langle\xi\rangle\big) \tag{24}$$

and

$$\lambda_{0,2}(t,x,\xi) = -\int_0^{x\cdot\omega} g_2(t,x-\tau\omega,\xi)d\tau \text{ with } g_2(t,x,\xi) = M\langle x\cdot\omega\rangle^{-\sigma}\chi\big(\langle x\rangle/t^{\ell+1}\langle\xi\rangle\big). \tag{25}$$

Then we define

$$\begin{aligned}\lambda_0(h,t,x,\xi) &= \\ &\Big(\chi\big(2x\cdot\omega/\langle x\rangle\big)\lambda_{0,1}(t,x,\xi) + \big(1-\chi\big(2x\cdot\omega/\langle x\rangle\big)\big)\lambda_{0,2}(t,x,\xi)\Big)\big(1-\chi(|\xi|/h)\big)\end{aligned} \tag{26}$$

since we need to solve (13) only for large $|\xi| \geq h$.

The function λ_0 solves

$$\sum_{j=1}^n \xi_j\partial_{x_j}\lambda_0(h,t,x,\xi) + M|\xi|\langle x\rangle^{-\sigma}\chi\big(\langle x\rangle/t^{\ell+1}\langle\xi\rangle\big) \leq 0,\ |\xi| \geq h, \tag{27}$$

and for multi-indices α, β, $\beta \neq 0$, it satisfies the estimates

$$|\partial_\xi^\alpha\lambda_0(h,t,x,\xi)| \leq C_0MA^{|\alpha|}\alpha!^s t^{(\ell+1)(1-\sigma)}\langle\xi\rangle_h^{1-\sigma-|\alpha|}, \tag{28}$$

$$|\partial_x^\beta\partial_\xi^\alpha\lambda_0(h,t,x,\xi)| \leq C_0MA^{|\alpha+\beta|}(\alpha+\beta)!^s\langle\xi\rangle_h^{-|\alpha|}, \tag{29}$$

where the constant C_0 and A are independent of large h.

Taking into consideration the function Λ_{ev}^2 that was already introduced in (19) and the above defined λ_0 we complete the solution Λ_{ev} of (13) in the evolution zone after taking

$$\begin{cases}\Lambda_{ev}(h,t,x,\xi) = \Lambda_{ev}^1(h,t,x,\xi) + \Lambda_{ev}^2(h,t,\xi),\\ \Lambda_{ev}^1(h,t,x,\xi) = \big(1-\chi(t/t_\xi)\big)t^{k-\ell}\lambda_0(h,t,x,\xi).\end{cases} \tag{30}$$

From (27), (3) and (4) we have

$$2a(t)\sum_{j=1}^n \xi_j\partial_{x_j}\Lambda_{ev}^1(h,t,x,\xi) + \big(1-\chi(t/t_\xi)\big)\chi\big(\langle x\rangle/t^{\ell+1}\langle\xi\rangle\big)\Re\sum_{j=1}^n b_j(t,x)\xi_j \leq 0 \tag{31}$$

for $|\xi| \geq h$ after taking a sufficiently large $M \geq M_0$. Then, from

$$\left|\partial_t \Lambda^1_{ev}(h,t,x,\xi)\right| \leq K_0 M t^{k-\sigma(\ell+1)} \langle\xi\rangle_h^{1-\sigma}$$

and $1 - \chi(t/2t_\xi) = 1$ on the support of $\partial_t \Lambda^1_{ev}$, we still have a solution to the inequality (20) by taking the sum $\Lambda_{ev} = \Lambda^1_{ev} + \Lambda^2_{ev}$ in place of the single term Λ^2_{ev} after having fixed $K \geq K_0 + 1$ in the definition (19). This, together with (31) gives

$$\partial_t \Lambda_{ev}(h,t,x,\xi) + 2a(t)\sum_{j=1}^n \xi_j \partial_{x_j} \Lambda_{ev}(h,t,x,\xi) + \big(1 - \chi(t/t_\xi)\big)\Re \sum_{j=1}^n b_j(t,x)\xi_j \leq 0 \tag{32}$$

for $|\xi| \geq h$ since Λ^2_{ev} does not depend on x.

Using (16) and (32) we have solutions Λ to (13) which are defined by

$$\Lambda(h,t,x,\xi) = \varrho(t)\langle\xi\rangle_h^{\frac{1}{s}} + \Lambda_{pd}(h,t,\xi) + \Lambda_{ev}(h,t,x,\xi) \tag{33}$$

with $\varrho'(t) < 0$ and $1/s > q$ with q from (8), here $1/s > (\ell-k)/(\ell+1)$. The weight function $\varrho(t)\langle\xi\rangle_h^{1/s}$ will be used to absorb the terms of order q in the asymptotic expansion of the transformed operator S_Λ in (10).

We summarize all the properties of $\Lambda(h,t,x,\xi)$ that we need in the following proposition:

Proposition 3.1. *Let us assume (3) and (4) with $1-\sigma \leq (\ell-k)/(\ell+1)$, and let us consider the symbol $\Lambda(h,t,x,\xi)$ which is defined by (33) with $1/s > (\ell-k)/(\ell+1)$. Let $N > 0$, $\varrho_0 > 0$, $\delta \in \big[0, 1/s - (\ell-k)/(\ell+1)\big)$ be given constants.*
Then we can choose the parameters $M \geq M_0$, $h \geq h_0$, M_0 is independent of all other parameters, $h_0 = h_0(\delta, \varrho_0, N)$, and the function $\varrho(t)$ such that

$$\partial_t \Lambda + 2a(t)\sum_{j=1}^n \xi_j \partial_{x_j}\Lambda + \Re\sum_{j=1}^n b_j(t,x)\xi_j \leq C_h - N\left(\varrho(t)\langle\xi\rangle_h^{1/s} + \langle\xi\rangle_h^{(\ell-k)/(\ell+1)+\delta}\right) \tag{34}$$

with

$$0 < \varrho(t) \leq \varrho_0, \;\; 0 \leq t \leq T. \tag{35}$$

Furthermore, Λ satisfies for all multi-indices α the estimates

$$|\partial_\xi^\alpha \Lambda(h,t,x,\xi)| \le \begin{cases} C_0 A^{|\alpha|}\alpha!^s\Big(\varrho(t)\langle\xi\rangle_h^{\frac{1}{s}} + \langle\xi\rangle_h^{\frac{\ell-k}{\ell+1}}\Big)\langle\xi\rangle_h^{-|\alpha|}, \ 1-\sigma < \frac{\ell-k}{\ell+1}, \\ C_0 A^{|\alpha|}\alpha!^s\Big(\varrho(t)\langle\xi\rangle_h^{\frac{1}{s}} + \langle\xi\rangle_h^{\frac{\ell-k}{\ell+1}}\log\langle\xi\rangle_h\Big)\langle\xi\rangle_h^{-|\alpha|}, \ 1-\sigma = \frac{\ell-k}{\ell+1}, \end{cases} \tag{36}$$

$$|\partial_\xi^\alpha \partial_t \Lambda(h,t,x,\xi)| \le C_0 A^{|\alpha|}\alpha!^s\Big(\varrho'(t)\langle\xi\rangle_h^{\frac{1}{s}} + \langle\xi\rangle_h^{\frac{\ell-k+1}{\ell+1}}\Big)\langle\xi\rangle_h^{-|\alpha|}, \tag{37}$$

and for all multi-indices α, β with $|\beta|>0$, $j=0,1$, the estimates

$$|\partial_x^\beta \partial_\xi^\alpha \partial_t^j \Lambda(h,t,x,\xi)| \le C_0 M A^{|\alpha+\beta|}(\alpha+\beta)!^s \langle\xi\rangle_h^{\frac{\ell-k+j}{\ell+1}-|\alpha|}, \tag{38}$$

$$|a(t)\partial_x^\beta \partial_\xi^\alpha \Lambda(h,t,x,\xi)| \le C_0 M A^{|\alpha+\beta|}(\alpha+\beta)!^s \langle\xi\rangle_h^{-|\alpha|}. \tag{39}$$

The constants C_0 and A are independent of the parameters $h \ge h_0$ and $M \ge M_0$. In particular, Λ has the order $1/s$, $\Lambda - \varrho(t)\langle\xi\rangle_h^{1/s}$ the order $(\ell-k)/(\ell+1)$ (with an extra factor $\log\langle\xi\rangle_h$ for $1-\sigma = (\ell-k)/(\ell+1)$), $\partial_t\Lambda$ has the order at most 1, $a(t)\partial_{x_j}\Lambda$, $j=1,\dots,n$, the order 0.

Proof. The function $\Lambda_{pd}(h,t,\xi)+\Lambda_{ev}(h,t,x,\xi)$ is a solution to (13) for $|\xi|\ge h$. Therefore we have (34) after taking in (33) the solution of

$$\varrho'(t) + N\varrho(t) + N h^{\frac{\ell-k}{\ell+1}+\delta-\frac{1}{s}} = 0, \quad \varrho(0)=\varrho_0, \tag{40}$$

for the weight function $\varrho(t)\langle\xi\rangle_h$. Since $(\ell-k)/(\ell+1)+\delta-1/s<0$ we can make $Nh^{(\ell-k)/(\ell+1)+\delta-1/s}$ so small for $h\ge h_0$ such that (35) is satisfied.

The estimates (36) and (37) for the term $\Lambda_{pd}+\Lambda_{ev}$ in (33) follow from (17), (18), (21) and (22). For $\partial_t^j\Lambda_{ev}^1$, $j\in\{0,1\}$, we use (28) and

$$t^{k-\ell+(\ell+1)(1-\sigma)-j}\langle\xi\rangle_h^{1-\sigma} \le C_0\langle\xi\rangle_h^{\frac{\ell-k+j}{\ell+1}} \text{on the support of } 1-\chi(t/t_\xi),$$

by the definition (14) of t_ξ and the present assumption $1-\sigma\le(\ell-k)/(\ell+1)$. In the same way, the estimates (38) for $\partial_x^\beta\partial_\xi^\alpha\partial_t^j\Lambda = \partial_x^\beta\partial_\xi^\alpha\partial_t^j\Lambda_{ev}^1$ follow from (29) and the definition of t_ξ (Λ_{ev}^1 is the only term depending on x in (33)).

Finally, from (3) and the definitions (30), (33) we have

$$|a(t)\partial_x^\beta\partial_\xi^\alpha\Lambda(h,t,x,\xi)| \le Ct^k|\partial_x^\beta\partial_\xi^\alpha\lambda_0(h,t,x,\xi)|,$$

hence (39) is a direct consequence of (29). □

4. Slow decay

In this section we estimate the solution Λ to (13) which is given by (33) in the case

$$1-\sigma > \frac{\ell-k}{\ell+1},$$

that is,

$$q = 1-\sigma$$

in (8). Some estimates are modified because now we are not always dealing with singular powers of t. We only need the splitting into pseudo-differential and evolution zones to control $\partial_x^\beta\Lambda$.

For Λ_{pd} from (15) the inequalities (17) and (18) remain unchanged. We just observe that the order $(\ell-k)/(\ell+1)$ of Λ_{pd} is now smaller than $1-\sigma$.

The estimates (21) for Λ_{ev}^2 from (19) become

$$|\partial_\xi^\alpha\Lambda_{ev}^2(h,t,\xi)| \le C_0MA^{|\alpha|}\alpha!^s\langle\xi\rangle_h^{1-\sigma-|\alpha|} \tag{41}$$

since here we have $k+1-\sigma(\ell+1)>0$ and we can bound $t^{k+1-\sigma(\ell+1)}$ by a constant for $t\in[t_\xi,T]$. No additional effect comes from the localization in the evolution zone. In the same way, still without localizing, the inequality

$$|\partial_\xi^\alpha\partial_t\Lambda_{ev}^2(h,t,\xi)| \le C_0MA^{|\alpha|}\alpha!^s t^{k-\sigma(\ell+1)}\langle\xi\rangle_h^{1-\sigma-|\alpha|}$$

with the L^1 factor $t^{k-\sigma(\ell+1)}$ would be sufficient in dealing with energy estimates. Using the definition of t_ξ in the case $k-\sigma(\ell+1)<0$ we can have bounded semi-norms of the symbol $\partial_t\Lambda_{ev}^2$ in all cases and (22) becomes

$$|\partial_\xi^\alpha\partial_t\Lambda_{ev}^2(h,t,\xi)| \le \begin{cases} C_0MA^{|\alpha|}\alpha!^s\langle\xi\rangle_h^{1-\frac{k}{\ell+1}-|\alpha|}, -1<k-\sigma(\ell+1)<0, \\ C_0MA^{|\alpha|}\alpha!^s\langle\xi\rangle_h^{1-\sigma-|\alpha|}, k-\sigma(\ell+1)\ge 0. \end{cases} \tag{42}$$

Also for $\partial_\xi^\alpha \Lambda_{ev}^1$, Λ_{ev}^1 is defined in (30), we have not any effect from $t > t_\xi/2$ on its support. The inequality (28) and $k+1-\sigma(\ell+1) > 0$ lead to the same estimates (41) for $\partial_\xi^\alpha \Lambda_{ev}^1$ as for $\partial_\xi^\alpha \Lambda_{ev}^2$. Here the order of Λ_{ev}^1 is $1-\sigma$.
In a similar way, (42) holds true for $\partial_\xi^\alpha \partial_t \Lambda_{ev}^1$.

We need the localization in the evolution zone for $\partial_x^\beta \partial_t^j \Lambda = \partial_x^\beta \partial_t^j \Lambda_{ev}^1$, $|\beta| > 0$, $j \in \{0,1\}$, since in (29) we have not any power of t to compensate the singular factor $t^{k-\ell}$ in the definition of Λ_{ev}^1. The estimate (38) remains unchanged, we just observe that for $j=0$ the order $(\ell-k)/(\ell+1)$ of $\partial_x^\beta \Lambda_{ev}^1$ is now smaller than $1-\sigma$.

Finally, we have the same inequality (39) for $a(t)\partial_x^\beta \Lambda$.

Summing up, for $1-\sigma > (\ell-k)/(\ell+1)$ we have the following properties of Λ, similar to those ones collected in Proposition 3.1 for $1-\sigma \le (\ell-k)/(\ell+1)$.

Proposition 4.1. *Let us assume (3) and (4) with $1-\sigma > (\ell-k)/(\ell+1)$, and let us consider the symbol $\Lambda(h,t,x,\xi)$ which is defined by (33) with $1/s > 1-\sigma$. Let $N > 0$, $\varrho_0 > 0$, $\delta \in [0, 1/s - 1 + \sigma)$ be any given constants. Then we can choose the parameters $M \ge M_0$, $h \ge h_0$, M_0 is independent of all other parameters, $h_0 = h_0(\delta, \varrho_0, N)$, and the function $\varrho(t)$ such that*

$$\partial_t \Lambda + 2a(t)\sum_{j=1}^n \xi_j \partial_{x_j}\Lambda + \Re \sum_{j=1}^n b_j(t,x)\xi_j \le C_h - N\left(\varrho(t)\langle\xi\rangle_h^{\frac{1}{s}} + \langle\xi\rangle_h^{1-\sigma+\delta}\right) \tag{43}$$

with $\varrho(t)$ satisfying (35).

Furthermore, Λ satisfies for all multi-indices α the estimates

$$|\partial_\xi^\alpha \Lambda(h,t,x,\xi)| \le C_0 A^{|\alpha|}\alpha!^s \Big(\varrho(t)\langle\xi\rangle_h^{\frac{1}{s}} + \langle\xi\rangle_h^{1-\sigma}\Big)\langle\xi\rangle_h^{-|\alpha|}, \tag{44}$$

$$|\partial_\xi^\alpha \partial_t \Lambda(h,t,x,\xi)| \le C_0 A^{|\alpha|}\alpha!^s \Big(\varrho'(t)\langle\xi\rangle_h^{\frac{1}{s}} + \langle\xi\rangle_h^{1-\sigma'}\Big)\langle\xi\rangle_h^{-|\alpha|}, \tag{45}$$

where $\sigma' \ge 0$ is given by

$$\sigma' = \min\Big\{\sigma, \frac{k}{\ell+1}\Big\},$$

and for all multi-indices α, β with $|\beta| > 0$, $j = 0,1$, the estimates (38), (39).

The constants C_0 and A are independent of the parameters $h \ge h_0$ and $M \ge M_0$. In particular, Λ has the order $1/s$, $\Lambda - \varrho(t)\langle\xi\rangle_h^{1/s}$ the order $1-\sigma$, $\partial_t\Lambda$ the order at most 1, $a(t)\partial_{x_j}\Lambda$, $j = 1,\dots,n$, the order 0.

5. Verification

We can now conclude the proof of the results of Theorem 1.1 in the sufficient direction using the calculus for pseudo-differential operators of infinite order in [12]. We refer to [1] for the fully detailed computation.

For $h \geq h_0$ the operator e^{Λ} with symbol $e^{\Lambda(h,t,x,\xi)}$, $\Lambda(h,t,x,\xi)$ is defined by (33), is continuous and invertible from the space $H^{m,s}_{\varrho}$ to H^m for $\varrho < \varrho_0$ and we have the asymptotic expansion:

$$e^{\Lambda}(h,t,x,D_x)(iS)\big(e^{\Lambda}(h,t,x,D_x)\big)^{-1} = \partial_t - ia(t)\Delta_x - A(h,t,x,D_x) \tag{46}$$

with

$$A(h,t,x,\xi) = \partial_t\Lambda(h,t,x,\xi) + 2a(t)\sum_{j=1}^{n}\xi_j\partial_{x_j}\Lambda(h,t,x,\xi) + \sum_{j=1}^{n} b_j(t,x)\xi_j + R(h,t,x,\xi), \tag{47}$$

where $R(h,t,x,\xi)$ denotes a symbol that satisfies

$$|\partial_\xi^\alpha\partial_x^\beta R(h,t,x,\xi)| \leq C_{\alpha\beta}\Big(\varrho(t)\langle\xi\rangle_h^{\frac{1}{s}} + \langle\xi\rangle_h^{q+\delta}\Big)\langle\xi\rangle_h^{-|\alpha|} \tag{48}$$

with constants $C_{\alpha\beta}$ which are independent of $h \geq h_0$ and a suitable $\delta \in [0, 1/s - q)$. Here q is defined by (8), $\delta = \delta(s,\ell,k,\sigma)$.

Now we can fix the large parameter N and then $h = h_0$ in (34) and (43) to conclude the inequality

$$2\Re A(h_0,t,x,\xi) \leq C \tag{49}$$

which gives immediately by sharp Gårding inequality the desired estimate to above in L^2

$$2\Re\big(A(h_0,t,x,D_x)v,v\big) \leq C\|v\|_{L^2}^2,\ \ v \in L^2,$$

since $A(h_0,t,x,\xi)$ is a symbol of order 1.

The well-posedness in L^2 of the Cauchy problem

$$e^{\Lambda_{\ell-k}} iS\big(e^{\Lambda_{\ell-k}}\big)^{-1}v = 0, \quad v(0,x) = v_0(x) \tag{50}$$

follows after application of the energy method. This gives the well-posedness of the Cauchy problem (1) in $H^{\infty,s}$ for $s < 1/q$, q defined by (8). This completes the proof of the sufficiency of the conditions given in Theorem 1.1.

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