L^p -LIOUVILLE THEOREMS FOR INVARIANT EVOLUTION EQUATIONS

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ABSTRACT. Some L^p-Liouville theorems for several classes of evolution equations are presented. The involved operators are left invariant with respect to Lie group composition laws in $\mathbb{R}^{N+1}$. Results for both solutions and sub-solutions are given.

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1. INTRODUCTION

We present some L^p-Liouville Theorems for solutions and sub-solutions to a class of evolution equations containing:

heat-type equations on stratified Lie groups

$$\mathcal{L} = \mathcal{L}_0 - \partial_t := \sum_{j=1}^{m} X_j^2 - \partial_t,$$

where $X_1, \ldots, X_m$ are smooth first order linear Partial Differential Operators generating the Lie algebra of a stratified Lie group in $\mathbb{R}^N$;

heat-type equations on stratified Lie groups of the kind

$$\mathcal{L} = \mathcal{L}_0 - \partial_t := \operatorname{div}(A\nabla) + \langle Bx, \nabla \rangle - \partial_t,$$

where A and B are $N \times N$ matrices, A is nonnegative definite possibly degenerate;

Fokker-Planck equations of Mumford type:

$$\mathcal{L} = \mathcal{L}_0 - \partial_t := \partial_{x_1}^2 + \operatorname{sen} x_1 \partial_{x_2} + \cos x_1 \partial_{x_3} - \partial_t.$$

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All these operators have in common that are hypoelliptic and left translation invariant w.r.t. a suitable Lie group composition law in $\mathbb{R}^{N+1}$.

They appear in several contexts, both theoretical and applied: Kinetic Fokker-Planck equations [HN05], Kolmogorov operators of stochastic equations [DP04], PDEs model in finance [Pas05], computer and human vision [Mum94], curvature Brownian motion [WZMC06], phase-noise Fokker-Planck equations [AZ03].

For operators of this kind we prove the following *L^p-Liouville theorem*:

$$\mathcal{L}u = 0 \text{ in } \mathbb{R}^{N+1},\ u \in L^p(\mathbb{R}^{N+1}) \implies u \equiv 0.$$

We also show L^p-Liouville type theorems for solutions (in the weak sense of the distributions) to

$$\mathcal{L}u \geq 0 \text{ in } \mathbb{R}^{N+1}.$$

These last results seem to be useful to give necessary conditions for semilinear equations like

$$\mathcal{L}u = f(u) \text{ in } \mathbb{R}^{N+1}$$

have *non-trivial solutions.* We plan to address this issue by using ideas by Caristi, D'Ambrosio and Mitidieri as presented in the paper [CDM08].

2. The heat equation setting

For simplicity, we would like to show our results in the case of the classical heat operator:

$$\mathcal{L} = \mathcal{H} := \Delta - \partial_t,$$

where $\Delta = \sum_{j=1}^{N} \partial_{x_j}^2$ is the classical Laplace operator in $\mathbb{R}^N$. The points of $\mathbb{R}^{N+1}$ will be denoted by

$$z = (x,t) = (x_1, \ldots, x_N, t).$$

As far as we know, even in the classical setting, several of our results are new.
They can be seen as the evolution counterpart of some results related to the classical Laplacian contained in [CDM08]. We would like to stress, however, a crucial difference between the heat and the Laplace operators:

the lack of the *positive Liouville Theorem.*

More precisely, *nonnegative caloric functions in* $\mathbb{R}^{N+1}$ are *not* necessarily constant:

$$\mathcal{H}u = 0 \text{ in } \mathbb{R}^{N+1}, \ u \geq 0 \ \not\Longrightarrow \ u \equiv const.$$

This is proved, e.g., by the following function

$$e^{x_1+\cdots+x_N+Nt},$$

which is strictly positive, caloric and non constant in $\mathbb{R}^{N+1}$. Nevertheless, L^p-Liouville theorems, in a *suitable form*, hold true for caloric and sub-caloric functions.
To show our results we use:

the mean value characterization of caloric functions;

a Poisson-Jensen formula for sub-caloric functions;

some results and devices from Parabolic Potential Theory.

2.1. **Some recalls from Parabolic Potential Theory.**

We start recalling some results from Parabolic Potential Theory that we will use in our proofs:

The existence of the *fundamental solution* for the heat operator $\mathcal{H}$

$$\Gamma : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}, \quad \Gamma(x,t) = \begin{cases} 0 & \text{if } t \leq 0 \\ (4\pi t)^{\frac{-N}{2}} e^{-\frac{|x|^2}{4t}} & \text{if } t > 0 \end{cases},$$

that allows to define the *heat ball* or *caloric ball* of center $z = (x,t)$ and radius r

$$\Omega_r(z) = \left\{ z \in \mathbb{R}^{N+1} \ : \ \Gamma(z-\zeta) > \frac{1}{r} \right\}$$

and, over this ball, the *Watson kernel*

$$K_r(z) = K_r(x,t) = \frac{1}{r} \frac{|\nabla_x \Gamma(x,t)|^2}{\Gamma^2(x,t)} = \frac{1}{r} \left(\frac{|x|}{2t} \right)^2.$$

Now, we say that a function $u \in C^\infty(O, \mathbb{R})$ is *caloric* in an open set $O \subseteq \mathbb{R}^{N+1}$ if

$$\mathcal{H}u = 0 \text{ in } O$$

and we recall the following theorem characterizing the caloric functions.

Pini-Watson Theorem. *If u is caloric in O then*

$$(*) \qquad u(z) = M_r(u)(z) := \int_{\Omega_r(z)} u(\zeta) K_r(z-\zeta)\, d\zeta \qquad \forall\ \overline{\Omega_r(z)} \subseteq O.$$

Viceversa: if $u \in C(O, \mathbb{R})$ satisfies $(*)$, then

$$u \in C^\infty(O, \mathbb{R}) \quad \textit{and} \quad \mathcal{H}(u) = 0 \ \textit{in } O.$$

We need to recall as well the definition of *sub-caloric function* and some results related to sub caloric functions:

A function $u : O \longrightarrow [-\infty, \infty[$ is *sub-caloric* if

(i) u is upper semicontinuous;

(ii) $u > -\infty$ in a dense subset in O;

(iii) $u(z) \le M_r(u)(z) \qquad \forall\ \overline{\Omega_r(z)} \subseteq O.$

Proposition 2.1. *Let $u : O \longrightarrow [-\infty, \infty[$ be u.s.c. Then u is sub-caloric in O if and only if*

$$(**) \qquad u \in L^1_{\mathrm{loc}}(O), \quad \mathcal{H}u \ge 0 \ \textit{in } \mathcal{D}'(O), \quad u(z) = \lim_{r \searrow 0} M_r(u)(z).$$

Moreover, if $u \in L^1_{\mathrm{loc}}(O)$ is a weak solution to

$$\mathcal{H}u \ge 0,$$

there exists a sub-caloric function $\hat{u}$ in O s.t.

$$u(z) = \hat{u}(z) \qquad \textit{a.e. in } O.$$

The function $\hat{u}$ is given by

$$\hat{u}(z) := \lim_{r \searrow 0} M_r(u)(z), \qquad z \in O.$$

Remark 2.2. *By Riesz-Schwartz Theorem, if u is sub-caloric there exists a nonnegative Radon measure μ in $\mathbb{R}^{N+1}$ such that*

$$\mathcal{H}u = \mu.$$

For the sub-caloric functions a *caloric Poisson-Jensen formula* holds.

If u is *sub-caloric* in $\mathbb{R}^{N+1}$ and $\mu = \mathcal{H}u$, then

$$\text{(PJ)} \qquad u(z) = M_r(u)(z) - N_r(\mu)(z) \qquad \forall\, z \in \mathbb{R}^{N+1},$$

where M_r is the Pini-Watson average operator $(*)$ and

$$N_r(\mu)(z) := \frac{1}{r}\int_0^r \left(\int_{\Omega_\rho(z)} \left(\Gamma(z-\zeta) - \frac{1}{\rho} \right) \, d\mu(\zeta) \right) \, d\rho.$$

Next proposition will play a crucial role in what follows.

Proposition 2.3. *Let μ be a nonnegative Radon measure in $\mathbb{R}^{N+1}$ such that, for every r,*

$$N_r(\mu)(z) = 0 \quad \forall\, z \in T,\ \overline{T} = \mathbb{R}^{N+1}.$$

Then $\mu \equiv 0$.

The last formula we would like to recall is a global *representation formula for bounded above sub-caloric functions.*

Let u be sub-caloric in $\mathbb{R}^{N+1}$ such that

$$U := \sup_{\mathbb{R}^{N+1}} u < \infty.$$

Then, if $\mu = \mathcal{H}u$,

$$u(z) = U - \int_{\mathbb{R}^{N+1}} \Gamma(z-\zeta)\, d\mu(\zeta) + h(z) \qquad \forall z \in \mathbb{R}^{N+1},$$

where h is a caloric function in $\mathbb{R}^{N+1}$, $h \le 0$.

3. L^p-Liouville Theorems for caloric functions

We begin with proving the following theorem:

Theorem 3.1. *Let $u \in C^\infty(\mathbb{R}^{N+1})$ be a caloric function*

$$\mathcal{H}u = 0 \quad \text{in } \mathbb{R}^{N+1}.$$

Suppose $u \in L^p(\mathbb{R}^{N+1})$ for a suitable $p \in [1, \infty]$.

Then $u \equiv 0$.

Remark 3.2. *The analogous result for harmonic functions is an easy consequence of the Gauss mean value property. Indeed, let $\Delta u = 0$ in $\mathbb{R}^N$.*

If $u \in L^p(\mathbb{R}^N)$ and $1 \le p < \infty$:

$$|u(x)| = \left| \fint u(y)\ dy \right| \le \left(\frac{1}{|B_r(x)|} \right)^{\frac{1}{p}} \|u\|_{L^p(\mathbb{R}^N)} \longrightarrow 0 \qquad \text{as } r \longrightarrow \infty, \quad \forall\ x \in \mathbb{R}^N.$$

This argument does *not* work for the heat equation because the kernel K_r in the Pini-Watson mean value Theorem for caloric function is *unbounded.*

Our approach, for the heat equation, is based on the caloric Poisson-Jensen formula (PJ)

$$u = M_r(u) - N_r(\mathcal{H}u).$$

Here is a detailed sketch of the proof of Theorem 3.1.

I step

Lemma 3.3. *Let $u \in C^2(\mathbb{R}^{N+1}, \mathbb{R})$ be such that*

$$\mathcal{H}u \ge 0 \quad (or\ \le 0) \qquad in\ \mathbb{R}^{N+1},$$

$$u \in L^1(\mathbb{R}^{N+1}).$$

Then,

$$\mathcal{H}u \equiv 0\ \ in\ \mathbb{R}^{N+1}.$$

Proof. An easy exchange of integrals shows that

$$\int_{\mathbb{R}^{N+1}} M_r(u)(z)\ dz = \int_{\mathbb{R}^{N+1}} u(z)\ dz \qquad \forall r > 0.$$

Then, from (PJ) we get

$$\int_{\mathbb{R}^{N+1}} N_r(\mathcal{H}u)(z)\ dz = 0 \qquad \forall r > 0.$$

Since $\mathcal{H}u \ge 0$ (≤ 0) everywhere, this gives

$$N_r(\mathcal{H}u)(z) = 0 \text{ a.e. in } \mathbb{R}^{N+1},$$

so that, keeping in mind Proposition 3.4,

$$\mathcal{H}u \equiv 0 \text{ in } \mathbb{R}^{N+1}.$$

□

II step

A simple direct computation shows that if $u \in C^2(\mathbb{R}^{N+1}, \mathbb{R})$ and $F \in C^2(\mathbb{R}, \mathbb{R})$, then

$$\mathcal{H}(F(u)) = F'(u)\mathcal{H}(u) + F''(u)|\nabla_x u|^2.$$

III step

Let $\mathcal{H}u = 0$ in $\mathbb{R}^{N+1}$ and

$$u \in L^p(\mathbb{R}^{N+1}),\ 1 \le p < \infty.$$

Define

$$v := F(u),$$

where

$$F : \mathbb{R} \longrightarrow \mathbb{R}, \quad F(t) = (\sqrt{1+t^2} - 1)^p = \left(\frac{t^2}{\sqrt{1+t^2}+1} \right)^p.$$

Since

$$0 \le F(t) \le |t|^p \quad \text{and} \quad F''(t) > 0 \quad \forall t \neq 0,$$

we have

$$0 \le v \le |u|^p \quad \Longrightarrow \quad v \in L^p(\mathbb{R}^{N+1}),$$

$$\mathcal{H}(v) = F''(u)|\nabla_x u|^2 \ge 0 \quad \text{in } \mathbb{R}^{N+1}.$$

Then, by Lemma 3.3, $\mathcal{H}(v) \equiv 0$, i.e.,

$$F''(u)|\nabla_x u|^2 = 0 \ \text{ in } \mathbb{R}^{N+1}$$

$$\Downarrow$$

$$|\nabla_x u|^2 = 0 \ \text{ in } U_0 = \{u \neq 0\}$$

$$\Downarrow$$

$$\Delta u = 0 \ \text{ in } U_0 \quad \Longrightarrow (\mathcal{H}u = 0)\ \partial_t u = 0 \ \text{ in } U_0$$

$$\Downarrow$$

$$|\nabla_z u| = 0 \ \text{ in } U_0$$

$$\Downarrow$$

$$u \equiv 0 \ \text{ in } \mathbb{R}^{N+1}.$$

This proves our L^p-Liouville theorem for $1 \le p < \infty$.

3.1. L^p-Liouville theorems for $0 < p < 1$.

In a similar way to the proof of the previous theorem we can prove that nonnegative solutions to the heat equation satisfy also a L^p-Liouville property for $0 < p < 1$.

Theorem 3.4. *Let u be a smooth solution to $\mathcal{H}u = 0$ in $\mathbb{R}^{N+1}$, $u \geq 0$ and*

$$u^p \in L^1(\mathbb{R}^{N+1}) \quad \textit{for} \quad p \in]0,1[.$$

Then $u \equiv 0$.

3.2. Some applications.

The devices used in the proofs of our L^p-Liouville theorems allow to get Liouville-type theorems for semilinear equations.

Theorem 3.5. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing C^1-function such that $f^{-1}(\{0\}) = 0$. Define*

$$F : \mathbb{R} \longrightarrow \mathbb{R}, \quad F(t) = \int_0^t f(s)\, ds.$$

Let $u \in C^2(\mathbb{R}^{N+1}, \mathbb{R})$ be a classical solution to

$$\mathcal{H}u = f(u) \quad \textit{in } \mathbb{R}^{N+1}.$$

If $F(u) \in L^1(\mathbb{R}^{N+1})$ then

$$u \equiv 0.$$

Proof. Define

$$v := F(u).$$

Then $v \in C^2(\mathbb{R}^{N+1}, \mathbb{R})$ and

$$\begin{aligned} \mathcal{H}v &= F'(u)\mathcal{H}u + F''(u)|\nabla_x|^2 \\ &= (f(u))^2 + f'(u)|\nabla_x|^2 \geq 0 \qquad (f \nearrow) \end{aligned}$$

Since $v \in L^1(\mathbb{R}^{N+1})$ from Lemma 3.3 we obtain

$$\mathcal{H}v \equiv 0 \iff (f(u))^2 + f'(u)|\nabla_x u|^2 \equiv 0 \implies f(u) \equiv 0 \implies u \equiv 0.$$

□

Corollary 3.6. $\mathcal{H}u = \lambda u, \quad \lambda \geq 0, \quad u \in L^2(\mathbb{R}^{N+1}) \implies u \equiv 0.$

Corollary 3.7. $\mathcal{H}u = |u|^{p-1}u, \quad 1 \leq p < \infty, \quad u \in L^{p+1}(\mathbb{R}^{N+1}) \implies u \equiv 0.$

4. L^p-Liouville Theorem for sub-caloric functions

We begin proving the following theorem that we will extend later to all the weak solutions to $\mathcal{H}u \geq 0$ in $L^p(\mathbb{R}^{N+1})$.

Theorem 4.1. *Let $u \in L^1(\mathbb{R}^{N+1})$ be a weak solution to*

$$\mathcal{H}u \geq 0 \quad \textit{in } \mathbb{R}^{N+1}.$$

Then

$$u(z) = 0 \quad \textit{a.e.} \quad \textit{in } \mathbb{R}^{N+1}.$$

Proof. Let $\hat{u}$ be a sub-caloric representative of u and let $\mu \in \mathcal{M}^+(\mathbb{R}^{N+1}),\ \mu = \mathcal{H}u$. By caloric Poisson-Jensen formula we have

$$\hat{u} = M_r(\hat{u}) - N_r(\mu).$$

Since $\hat{u} = u$ a.e., we have $\hat{u} \in L^1(\mathbb{R}^{N+1})$. On the other hand,

$$\int_{\mathbb{R}^{N+1}} \hat{u}\ dz = \int_{\mathbb{R}^{N+1}} M_r(\hat{u})\ dz.$$

Therefore $N_r(\mu) \in L^1(\mathbb{R}^{N+1})$ and

$$\int_{\mathbb{R}^{N+1}} N_r(\mu)\ dz = 0.$$

Since $N_r(\mu) \geq 0$, this implies $N_r(\mu) = 0$ a.e. in $\mathbb{R}^{N+1}$ and, by Proposition 2.3, $\mu = 0$. Thus $\mathcal{H}\hat{u} = 0$. By Theorem 3.1, $\hat{u} \equiv 0$ hence

$$u(z) = 0 \quad \text{a.e.} \quad \text{in } \mathbb{R}^{N+1}.$$

□

From this Theorem we obtain,

Theorem 4.2. *Let $u \in L^1_{\text{loc}}(\mathbb{R}^{N+1})$ be a weak solution*

$$\mathcal{H}u \geq 0 \quad in\ \mathcal{D}'(\mathbb{R}^{N+1}).$$

Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be a convex increasing function.
If $F(u) \in L^1(\mathbb{R}^{N+1})$, then

$$F(u) = 0 \quad a.e. \quad in\ \mathbb{R}^{N+1}.$$

Proof. Let $\hat{u}$ be the sub-caloric representation of u. Define

$$v := F(\hat{u}).$$

Then, $v : \mathbb{R}^{N+1} \longrightarrow [-\infty, \infty[$ is u.s.c., $v(z) > -\infty$ a.e. in $\mathbb{R}^{N+1}$ and, for every $z \in \mathbb{R}^{N+1}$ and $r > 0$,

$$\begin{aligned} v(z) =& F(\hat{u}(z)) \leq F(M_r(\hat{u}(z))) \\ & (\text{since } \hat{u} \leq M_r(\hat{u}) \text{ and } F \nearrow) \\ & \leq M_r(F(\hat{u}))(z) \\ & (\text{by the convexity of } F \text{ and the Jensen inequality}) \\ & = M_r(v)(z) \end{aligned}$$

Then, v is sub-caloric. Moreover, since

$$v(z) = F(\hat{u}(z)) = F(u(z)) \text{ a.e.} \quad \text{in } \mathbb{R}^{N+1},$$

we have $v \in L^1(\mathbb{R}^{N+1})$. By Theorem 4.1, $v \equiv 0$ so that

$$F(u)(z) = 0 \text{ a.e.} \quad \text{in } \mathbb{R}^{N+1}.$$

□

The following theorem can be proved exactly as the previous one.

Theorem 4.3. *Let $u \in L^1_{\text{loc}}(\mathbb{R}^{N+1})$ be a nonnegative weak solution of*

$$\mathcal{H}u \geq 0 \quad in\ \mathbb{R}^{N+1}.$$

Let $F : [0, \infty[\to \mathbb{R}$ be a convex increasing function. If $F(u) \in L^1(\mathbb{R}^{N+1})$, then

$$F(u) = 0 \quad a.e. \quad in\ \mathbb{R}^{N+1}.$$

Corollary 4.4. *Let* $u \in L^p(\mathbb{R}^{N+1})$, $1 \leq p < \infty$, *be a nonnegative solution of*

$$\mathcal{H}u \geq 0 \quad \text{in } \mathbb{R}^{N+1}.$$

Then

$$u = 0 \quad a.e. \quad \text{in } \mathbb{R}^{N+1}.$$

Proof. Just apply Theorem 4.3 by taking

$$F(t) = t^p.$$

□

5. Operators to which our results extend

As we wrote in the introduction, our results hold for *heat-type equations on stratified Lie groups, heat-type equations on stratified Lie groups, Fokker-Planck equations of Mumford type.* Actually, these techniques work for a general class of operators

$$\mathcal{L} = \sum_{i,j=1}^{N} a_{ij}(z)\partial_{x_i}\partial_{x_j} + \sum_{j=1}^{N} b_j(z)\partial_{x_j} - \partial_t \qquad \text{in } \mathbb{R}^{N+1},$$

where $A = (a_{ij})$ is a $N \times N$ symmetric and positive semidefinite matrix and the coefficients a_{ij}, b_j are smooth functions. If the operator is hypoelliptic and not totally degenerate, the results we need from parabolic potential theory apply to $\mathcal{L}$ (see [LP99]). If we require that there exists a a Lie group composition law $\circ$ in $\mathbb{R}^{N+1}$ s.t. $\mathcal{L}$ is left translation invariant w.r.t. $\circ$, all the Liouville-type theorems that we stated for the heat operator until here still hold.

5.1. L^p-Liouville Theorems for homogeneous operators.

If we suppose, moreover, that there exists a group of dilations

$$\delta_\lambda(x_1, \dots, x_N, t) = (\lambda^{\sigma_1}, \dots, \lambda^{\sigma_N}, \lambda^2 t), \ \lambda > 0, \quad Q := \sigma_1 + \cdots \sigma_N + 2.$$

such that operators to which our results extend are homogeneous of degree two, we can improve our results.

5.1.1. *L^p-Liouville Theorems for subsolutions.*

In the corollary 4.4 we can drop the sign restriction on u and prove the caloric analogue of Theorem 4.5 in the quoted paper by Caristi, D'Ambrosio and Mitidieri [CDM08].

Theorem 5.1. *Let $u \in L^1_{\text{loc}}(\mathbb{R}^{N+1})$ be a weak solution of*

$$\mathcal{L}u \geq 0 \quad in\ \mathbb{R}^{N+1}.$$

If $u \in L^p(\mathbb{R}^{N+1})$ for a suitable $p \in [1, \frac{Q}{Q-2}]$, then

$$u = 0 \quad a.e. \quad in\ \mathbb{R}^{N+1}.$$

Proof. Let $\hat{u}$ be the sub-caloric representative of u.
Then

$$\hat{u}^+ = \min\{\hat{u}, 0\} \text{ is sub-caloric}$$

and, being $\hat{u}^+ = u^+$ a.e.,

$$\hat{u}^+ \in L^p(\mathbb{R}^{N+1}) \text{ for some } p \in \left[1, \frac{Q}{Q-2}\right].$$

By Corollary 4.4, $u^+ \equiv 0$. Then, $\hat{u} \leq 0$.
Then,

$$\hat{u} = \hat{U} - \Gamma * \mu + h,$$

where $\hat{U} = \sup \hat{u}$ (≤ 0), $\mu = \mathcal{H}u$, h is caloric and ≤ 0 in $\mathbb{R}^{N+1}$. Since $\hat{u} = u$ a.e. and $u \in L^p(\mathbb{R}^{N+1})$, $\hat{u} \in L^p(\mathbb{R}^{N+1})$ for some $p \in [1, \frac{Q}{Q-2}]$. As a consequence, being

$$\hat{u} \leq \hat{U} \leq 0, \quad \hat{u} \leq -\Gamma * \mu \leq 0, \quad \hat{u} \leq h \leq 0,$$

we have

$$\hat{U} = 0, \quad \Gamma * \mu \in L^p(\mathbb{R}^{N+1}), \quad h \in L^p(\mathbb{R}^{N+1}).$$

By Theorem 3.1, $h \equiv 0$. Moreover, by next lemma,

$$\mu = 0.$$

Summing up

$$\hat{u} \equiv 0 \text{ and}$$

$$u = 0 \text{ a.e.} \quad \text{in } \mathbb{R}^{N+1}.$$

□

Lemma 5.2. *Let μ be a nonnegative Radon measure such that*

$$\Gamma * \mu \in L^p(\mathbb{R}^{N+1})$$

for some $p \in [1, \frac{2}{Q-2}]$. Then $\mu = 0$.

Remark 5.3. *For every fixed $p > \frac{2}{Q-2}$ there exists $\mu \not\equiv 0$ such that*

$$\Gamma * \mu \in L^p(\mathbb{R}^{N+1}).$$

5.1.2. *L^∞-Liouville Theorems.* In the case that $\mathcal{L}$ is also homogeneous Theorem 3.1 can be extended to $p = \infty$:

Theorem 5.4. *Let u be a solution to*

$$\mathcal{L}u = 0 \quad \text{in } \mathbb{R}^{N+1}.$$

If $u \in L^\infty(\mathbb{R}^{N+1})$ then $u \equiv$ const.

We remark that if $\mathcal{L}$ is not homogeneous in general this last result does not hold. In fact, for example, consider the Kolmogorov-type operator in $\mathbb{R}^3 = \mathbb{R}^2_x \times \mathbb{R}_t$

$$\mathcal{L} = \partial^2_{x_1} + \left(x_1 - \frac{1}{2}x_2\right)\partial_{x_1} + \left(\frac{1}{2}x_1 - x_2\right)\partial_{x_2} - \partial_t.$$

This operator belongs to our class of operators and satisfies all the results of the previous sections but is not homogeneous and, by a result by Priola and Zabczyk, has a bounded solution in $\mathbb{R}^3$ which is not constant (see [PZ04, Theorem 3.1]).

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