INTRINSIC STRATIFICATIONS OF ANALYTIC VARIETIES

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Abstract. By associating a Lie algebra of analytic vector fields to every point of an analytic variety and using the associated Nagano foliation, this work presents a coordinate-free stratification of analytic sets.

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Contents

1. Notations and Basics

Let $\mathcal{M}$ be an analytic (ie, $\mathcal{C}^\omega$) manifold and $\boldsymbol{V}$ an **analytic subvariety** of $\mathcal{M}$, ie, every $x \in \mathcal{M}$ belongs to an arbitrary open subset $\mathcal{U}$ of $\mathcal{M}$ with the following property: $\exists$ a

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family $\mathcal{F}(\boldsymbol{V},\mathcal{U})$ of $\mathbb{R}$-valued, analytic functions in $\mathcal{U}$ s. t.

$$\boldsymbol{V}\cap\mathcal{U}=$$
$$\{x\in\mathcal{U};\ \forall f\in\mathcal{F}(\boldsymbol{V},\mathcal{U}),\ f(x)=0\};$$

$\boldsymbol{V}$ is always *closed*. $\mathcal{M}$. We can take $\mathcal{F}(\boldsymbol{V},\mathcal{U})=\mathfrak{I}(\boldsymbol{V},\mathcal{U})$ the set of all real, analytic functions in $\mathcal{U}$ s. t. $f\equiv 0$ in $\mathcal{U}$; $\mathfrak{I}(\boldsymbol{V},\mathcal{U})$ is an ideal in $\mathcal{C}^\omega(\mathcal{U})$, finitely generated if $\mathcal{U}\subset\subset\mathcal{M}$ is small.

We shall denote by $\overline{S}$ the *closure* of S in $\mathcal{M}$, $\partial S=\overline{S}\backslash S$ its *boundary*.

Definition 1. *An* ***analytic stratification*** *of* $\boldsymbol{V}$ *is a locally finite partition*

$$\boldsymbol{V}=\bigcup_\alpha\mathcal{S}_\alpha \tag{1.1}$$

s. t., for every index α*:*

(1) $\mathcal{S}_\alpha$ *is a connected, immersed analytic submanifold of* $\mathcal{M}$*;*
(2) $\forall\beta\neq\alpha$, $\mathcal{S}_\alpha\cap\overline{\mathcal{S}_\beta}\neq\varnothing\Longrightarrow\dim\mathcal{S}_\alpha<\dim\mathcal{S}_\beta$ *and* $\mathcal{S}_\alpha\subset\partial\mathcal{S}_\beta$.

Every $\mathcal{S}_\alpha$ *is an* ***analytic stratum*** *of* $\boldsymbol{V}$.

Thus $\alpha\neq\beta\Longleftrightarrow\mathcal{S}_\alpha\cap\mathcal{S}_\beta=\varnothing$; then $\mathcal{S}_\alpha\subset\partial\mathcal{S}_\beta\Longleftrightarrow\mathcal{S}_\alpha\subset\overline{\mathcal{S}_\beta}$. That (1.1) is locally finite means that each compact set $K\subset\mathcal{M}$ intersects at most finitely many $\mathcal{S}_\alpha$.

1.1. **Complex Whitney's umbrella.** This is the complex hypersurface

$$\boldsymbol{W}^{\mathbb{C}}=\left\{z\in\mathbb{C}^3;\ z_1^2=z_3z_2^2\right\};$$

we have

$$d\left(z_1^2-z_3z_2^2\right)=2z_1dz_1-2z_2z_3dz_2-z_2^2dz_3$$

The *regular part* of $\boldsymbol{W}^{\mathbb{C}}$ is the complex-analytic submanifold of $\mathbb{C}^3$,

$$\mathfrak{R}\left(\boldsymbol{W}^{\mathbb{C}}\right)=$$
$$\left\{z\in\mathbb{R}^3;\ z_1^2=z_3z_2^2,\ |z_1|+|z_2|\neq 0\right\};$$

$\mathfrak{R}\left(\boldsymbol{W}^{\mathbb{C}}\right)$ is connected ($\Longleftrightarrow\boldsymbol{W}^{\mathbb{C}}$ irreducible).

The *singular part:*

$$\mathfrak{S}\left(\boldsymbol{W}^{\mathbb{C}}\right) = z_3\text{-“axis”} = z_3\text{-plane},$$

is a complex-analytic subvariety of $\mathbb{C}^3$.

1.2. **Real Whitney's umbrella.** The intersection $\boldsymbol{W}^{\mathbb{C}} \cap \mathbb{R}^3$ is the hypersurface,

$$\boldsymbol{W} = \left\{x \in \mathbb{R}^3;\ x_1^2 = x_3 x_2^2\right\}.$$

The regular part of $\boldsymbol{W}$,

$$\mathfrak{R}(\boldsymbol{W}) =$$

$$\{x \in \boldsymbol{W};\ |x_1| + |x_2| \neq 0,\ x_3 > 0\} \cup$$

$$\{x \in \boldsymbol{W};\ x_1 = x_2 = 0,\ x_3 < 0\}.$$

is the union of two disjoint $\mathcal{C}^\omega$ submanifolds of dimension 2 and 1.

The singular part of $\boldsymbol{W}$,

$$\mathfrak{S}(\boldsymbol{W}) =$$

$$\{x \in \boldsymbol{W};\ x_1 = x_2 = 0,\ x_3 \geq 0\}.$$

is not a $\mathcal{C}^\omega$ submanifold but a **semi-analytic set**.

1.3. **Nagano foliation of an an analytic subvariety.**

Definition 2. *An* ***analytic foliation*** *of* $\mathcal{M}$ *is a family* $\boldsymbol{\Phi}$ *of* immersed *analytic submanifolds (without self-intersections) s. t.*

(1) *Every submanifold* $\mathcal{L} \in \boldsymbol{\Phi}$ *is connected.*
(2) *Every* $x \in \mathcal{M}$ *lies in a unique* $\mathcal{L} \in \boldsymbol{\Phi}$*.*

In general $\dim \mathcal{L}$ varies with $\mathcal{L}$.

We recall the classical **Nagano theorem** (see [Nagano, 1966]):

Theorem 1. *Let* $\mathfrak{g}$ *be a Lie algebra* (for the Lie bracket) *of analytic, real vector fields in* $\mathcal{M}$*. There is a foliation of* $\mathcal{M}$ *consisting of integral manifolds of* $\mathfrak{g}$*.*

If $\mathcal{L}$ is a leaf of $\mathfrak{g}$ and $x \in \mathcal{L}$ then $T_x\mathcal{L} = \mathfrak{g}_x$. Integral manifolds are *maximal* by definition.

In the sequel $\mathcal{U} \subset \mathcal{M}$ is open, $\mathcal{C}^\omega(\mathcal{U}; T\mathcal{M})$ is the Lie algebra of real $\mathcal{C}^\omega$ vector fields in $\mathcal{U}$.

Definition 3. *We denote by* $\mathfrak{g}(\boldsymbol{V}, \mathcal{U})$ *the Lie subalgebra of* $\mathcal{C}^\omega(\mathcal{U}; T\mathcal{M})$ *consisting of the real* $\mathcal{C}^\omega$ *vector fields* X *in* $\mathcal{U}$ *verifying:*

($\star$): *The restriction of* X *to an arbitrary open set* $\mathcal{U}' \subset \mathcal{U}$ *maps* $\mathfrak{I}(\boldsymbol{V}, \mathcal{U}')$ *into itself.*

If $\mathcal{U} \cap \boldsymbol{V} = \mathcal{U}$ then $\mathfrak{I}(\boldsymbol{V}, \mathcal{U}') = \{0\}$ whatever $\mathcal{U}' \subset \mathcal{U}$ open and $\mathfrak{g}(\boldsymbol{V}, \mathcal{U}) = \mathcal{C}^\omega(\mathcal{U}; T\mathcal{M})$. If $\mathcal{U} \cap \boldsymbol{V} = \varnothing$ then $\mathfrak{I}(\boldsymbol{V}, \mathcal{U}') = \mathcal{C}^\omega(\mathcal{U}')$ whatever $\mathcal{U}' \subset \mathcal{U}$ open and here also $\mathfrak{g}(\boldsymbol{V}, \mathcal{U}) = \mathcal{C}^\omega(\mathcal{U}; T\mathcal{M})$. Possibly $\mathfrak{g}(\boldsymbol{V}, \mathcal{U}) = \{0\}$ if $\mathcal{U}$ is "too large"; $\mathfrak{g}(\boldsymbol{V}, \mathcal{U}) \neq \{0\}$ if $\mathcal{U}$ is the domain of analytic local coordinates $x_1, ..., x_n$ and if $\mathfrak{I}(\boldsymbol{V}, \mathcal{U}) \neq \{0\}$: in this case, $\mathfrak{g}(\boldsymbol{V}, \mathcal{U})$ contains every $X = \sum_{j=1}^n c_j(x) \frac{\partial}{\partial x_j}$ with $c_j \in \mathfrak{I}(\boldsymbol{V}, \mathcal{U})$.

If $\mathcal{U}_1 \supset \mathcal{U}_2$ is open in $\mathcal{M}$ there is a restriction map $\mathfrak{g}(\boldsymbol{V}, \mathcal{U}_1) \longrightarrow \mathfrak{g}(\boldsymbol{V}, \mathcal{U}_2)$ thanks to **($\star$).**

Notation: $\mathfrak{g}_x(\boldsymbol{V})$ =freezing of $\mathfrak{g}(\boldsymbol{V}, \mathcal{U})$ at x, independent of $\mathcal{U}$.

Proposition 1. *Let* $\mathcal{U}_1 \supset \mathcal{U}_2$ *be open subsets of* $\mathcal{M}$. *If* $\mathcal{L}_j$ *is an integral manifold of* $\mathfrak{g}(\boldsymbol{V}, \mathcal{U}_j)$, $j = 1, 2$, *and* $\mathcal{L}_1 \cap \mathcal{L}_2 \neq \varnothing$ *then* $\mathcal{L}_1 \cap \mathcal{U}_2 \subset \mathcal{L}_2$ *and thus* $\dim \mathcal{L}_1 \leq \dim \mathcal{L}_2$.

Proposition 2. *If* $\boldsymbol{V} \cap \mathcal{U}$ *is an analytic submanifold of* $\mathcal{U}$ *then an analytic vector field* X *in* $\mathcal{U}$ *belongs to* $\mathfrak{g}(\boldsymbol{V}, \mathcal{U})$ *if and only if* X *is tangent to* $\boldsymbol{V} \cap \mathcal{U}$ *at every point of* $\boldsymbol{V} \cap \mathcal{U}$.

By Nagano's Theorem there is a foliation of $\mathcal{U}$ whose leaves are integral manifolds of $\mathfrak{g}(\boldsymbol{V}, \mathcal{U})$ in $\mathcal{U}$. If $\mathfrak{g}(\boldsymbol{V}, \mathcal{U}) = \{0\}$ then every point of $\mathcal{U}$ is an integral manifold of $\mathfrak{g}(\boldsymbol{V}, \mathcal{U})$; if $\mathfrak{g}(\boldsymbol{V}, \mathcal{U}) = \mathcal{C}^\omega(\mathcal{U}; T\mathcal{M})$ then each connected component of $\mathcal{U}$ is an integral manifold of $\mathfrak{g}(\boldsymbol{V}, \mathcal{U})$.

Proposition 3. *Every* $x \in \mathcal{M}$ *belongs to a unique* $\mathcal{C}^\omega$ *submanifold* $\mathcal{L}$ *of* $\mathcal{M}$ *s. t.*

(1) $\mathcal{L}$ *is connected and* $\dim \mathcal{L} = \dim \mathfrak{g}_x(\boldsymbol{V})$ *whatever* $x \in \mathcal{L}$;

(2) *each* $x \in \mathcal{L}$ *has a neighborhood* $\mathcal{U}_x$ *s. t.* $\mathcal{L} \cap \mathcal{U}_x$ *is an integral manifold of* $\mathfrak{g}(\boldsymbol{V}, \mathcal{U}_x)$;

(3) $\mathcal{L}$ *is maximal for these properties.*

These submanifolds will be called **maximal integral manifolds** of $\mathfrak{g}(\boldsymbol{V},\mathcal{M})$.

Proposition 4. *If a maximal integral manifold $\mathcal{L}$ of $\mathfrak{g}(\boldsymbol{V},\mathcal{M})$ intersects $\boldsymbol{V}$ then $\mathcal{L}\subset\boldsymbol{V}$.*

Trivial since every connected component of $\mathcal{M}\backslash\boldsymbol{V}$ is an integral manifold of $\mathfrak{g}(\boldsymbol{V},\mathcal{M})$.

Definition 4. *A maximal integral manifold of $\mathfrak{g}(\boldsymbol{V},\mathcal{M})$, $\mathcal{L}\subset\boldsymbol{V}$, will be called a **Nagano leaf** of $\boldsymbol{V}$.*

Proposition 5. *Every connected component of $\mathfrak{R}(\boldsymbol{V})$ is contained in a Nagano leaf of $\boldsymbol{V}$.*

Main result:

Theorem 2. *The Nagano foliation of $\boldsymbol{V}$ is a stratification* (Definition 1).

2. Local Analysis and Nagano Stratification

2.1. **The classical local partition (Osgood, Lojasiewicz).** Now Ω is an open subset of $\mathbb{R}^n$, $0\in\boldsymbol{V}\cap\Omega$; $\exists U\subset\Omega$ open, $0\in U$, and $f_j\in\mathcal{C}^\omega(U;\mathbb{R})$, $j=1,...,r$, s. t.

$$\boldsymbol{V}\cap U=$$

$$\{x\in U;\ f_j(x)=0,\ j=1...,r\}.$$

But then $\boldsymbol{V}\cap U=\{x\in U;\ F(x)=0\}$, $F=f_1^2+\cdots+f_r^2$.

Weierstrass Preparation Thm: $F=EP$, $E\in\mathcal{C}^\omega(U)$, $E(x)\neq 0$ whatever $x\in U$, and for suitable coordinates x_i,

$$P(x)=P(x';x_n)=$$

$$x_n^m+a_1(x')\,x_n^{m-1}+\cdots+a_m(x');$$

$x'=(x_1,...,x_{n-1})$, $a_j\in\mathcal{C}^\omega(U;\mathbb{R})$, $a_j(0)=0$, $j=1,...,m$; P is a **Weierstrass polynomial**.

Unique factorization of Weierstrass polynomials:

$$P=P_1^{q_1}\cdots P_\nu^{q_\nu},\ 1\leq q_j\in\mathbb{Z}.$$

The P_j are irreducible and distinct (hence *coprime*) but not necessarily real. If P_j is not real then $\exists k$ such that $P_k = \overline{P_j}$ and $q_k = q_j$.

We can assume $P = P_1 \cdots P_\nu$: it has no effect on the null-set $\boldsymbol{V} \cap U$. Then P and $\frac{\partial P}{\partial x_n}$ are coprime$\Longleftrightarrow D(x') \not\equiv 0$, $D(x')$: discriminant of P.

We take $U = U' \times (-r_n, r_n)$. If $\boldsymbol{Z}^{(0)} = \{x' \in U';\ D(x') = 0\}$ the *real* roots of $P(x'; z_n) = 0$ are *true* $\mathcal{C}^\omega$ *functions* of x' in $U' \backslash \boldsymbol{Z}^{(0)}$ (but there might be none!); the set of pts $(x', x_n) \in \boldsymbol{V}$, $z' \in U' \backslash \boldsymbol{Z}^{(0)}$, is a union (possibly $\varnothing$) of $\mathcal{C}^\omega$ graphs $\Lambda_{k,\alpha}$ $(k = 1, ..., d_\alpha^{(0)} \leq \deg P)$,

$$x_n = \rho_{k,\alpha}(x'),\ x' \in \Gamma_\alpha^{(0)},$$

where the $\Gamma_\alpha^{(0)}$ are connected components of $U' \backslash \boldsymbol{Z}^{(0)}$. For fixed α,

$$-r_n < \rho_{1,\alpha} < \cdots < \rho_{d_a^{(0)}} < r_n.$$

If $\boldsymbol{Z}^{(0)} = \varnothing$ the procedure stops: $\boldsymbol{V}$ is a $\mathcal{C}^\omega$ submanifold in a neighborhood of 0.

If $\boldsymbol{Z}^{(0)} \neq \varnothing$ we repeat for $D(x')$ what was done for $F(x)$: $D = E^{(1)} P^{(1)}$, $E^{(1)}(x') \neq 0$ $\forall x' \in U'$, and we select the coordinates so that

$$P^{(1)}(x) = P^{(1)}(x''; x_n) =$$
$$x_{n-1}^m + a_1^{(1)}(x'')\, x_{n-1}^{m-1} + \cdots + a_m^{(1)}(x'');$$

$x'' = (x_1, ..., x_{n-2})$, $a_j^{(1)} \in \mathcal{C}^\omega(U; \mathbb{R})$, $a_j(0) = 0$. Unique factorization:

$$P^{(1)} = \left(P_1^{(1)}\right)^{q_1'} \cdots \left(P_{\nu'}^{(1)}\right)^{q'_{\nu'}},\ 1 \leq q_j' \in \mathbb{Z}$$

the $P_j^{(1)}$ irreducible and distinct. We take $P^{(1)} = P_1^{(1)} \cdots P_{\nu'}^{(1)}$ with no effect on the null-set $\boldsymbol{Z}^{(0)}$; $P^{(1)}$ and $\frac{\partial P^{(1)}}{\partial x_{n-1}}$ are coprime$\Longleftrightarrow D^{(1)}(x'') \not\equiv 0$, $D^{(1)}(x'')$: discriminant of $P^{(1)}$.

We replace $P(z'; z_n)$ by

$$P^{(0)}(z''; z_n) = \prod_{k=1}^{\deg P^{(1)}} P\left(z'', \rho_k^{(1)}(z''); z_n\right),$$

$\rho_k^{(1)}(z'')$: roots of $P^{(1)}(z''; z_{n-1}) = 0$. The coefficients of the Weierstrass polynomial $P^{(0)}(z''; z_n)$ are symmetric polynomials wrto the roots $\rho_k^{(1)}(z'')$, therefore holomorphic functions of z'' in a complex neighborhood of U''.

Unique factorization eliminates redundant factors of $P^{(0)}$: we can assume $D^{(0)}(z'') \not\equiv 0$, $D^{(0)}$: discriminant of $P^{(0)}$.

We take $U' = U'' \times (-r_{n-1}, r_{n-1})$ and set $\boldsymbol{Z}^{(1)} =$

$$\left\{x'' \in U'';\ D^{(0)}(x'')\, D^{(1)}(x'') = 0\right\}.$$

The points $x = (x'', x_{n-1}, x_n) \in \boldsymbol{V}$ s. t. $x' \in \boldsymbol{Z}^{(0)}$ are determined by the two equations

$$P^{(0)}(x''; z_n) = 0, \tag{2.1}$$

$$P^{(1)}(x''; z_{n-1}) = 0, \tag{2.2}$$

For $x'' \in U'' \backslash \boldsymbol{Z}^{(1)}$ the roots $\rho_k^{(j)}(x'')$, $k = 1, ..., \deg P^{(j)}$, $j = 0, 1$, are distinct and analytic functions of x''. If Γ'' is a connected component of $U'' \backslash \boldsymbol{Z}^{(1)}$ the points

$$\left(x'', \rho_\ell^{(1)}(x''), \rho_k^{(0)}(x'')\right) \in \boldsymbol{V}$$

describe disjoint analytic graphs over $\Gamma'' \ni x''$.

To study $\boldsymbol{V}$ over $\boldsymbol{Z}^{(1)}$ we repeat for $D^{(0)}(x'')\, D^{(1)}(x'')$ the procedure used for $F(x)$ and $D(x')$; etc. At the end we get either $\dim \boldsymbol{Z}^{(N)} = 0$ or $\boldsymbol{Z}^{(N)} = \varnothing$. We end up with a *partition*

$$\boldsymbol{V} \cap \mathfrak{Q}_r^{(n)} = \bigcup_{q=0}^{n-1} \bigcup_{\iota \in \boldsymbol{I}_q} \Lambda_\iota^{(q)}. \tag{2.3}$$

$\mathfrak{Q}_r^{(n)} = \{x \in \Omega;\ |x_i| < r_i,\ i = 1, ..., n\}$; $\Lambda_\iota^{(q)}$: $\mathcal{C}^\omega$ submanifolds, $\dim \Lambda_\iota^{(q)} = q$. Some index sets $\boldsymbol{I}_q$ might be empty; (2.3) is coordinate dependent:

Example 1. *$\boldsymbol{V} = \{x \in \mathbb{R}^2;\ x_2 = x_1^2\}$ has 1 stratum $\Lambda_\iota^{(1)}$; $\boldsymbol{V} = \{x \in \mathbb{R}^2;\ x_2^2 = x_1\}$ has 3 strata: two $\Lambda_\iota^{(1)}$ and $\Lambda^{(0)} = \{0\}$.*

Theorem 3. *The partition (2.3) is a stratification* (Definition 1).

For further details about the partition (2.3).

2.2. **Nagano stratification.** In order to complete the proof of Thm 2 we need two results:

(1) $\Lambda_\iota^{(q)} \cap \overline{\Lambda_{\iota'}^{(q')}} \neq \varnothing \Longrightarrow \Lambda_\iota^{(q)} \subset \partial\Lambda_{\iota'}^{(q')}$;

(2) $\forall\,(q,\iota)$, $\exists$ Nagano leaf (per force unique) $\mathcal{L}$ of $\boldsymbol{V}$ s. t. $\Lambda_\iota^{(q)} \subset \mathcal{L}$.

#1 is an easy consequence of the construction of (2.3); for details see the original Lojasiewicz article.

To prove #2 one starts with the $\Lambda_\iota^{(q)}$ of highest dimension, d: $\Lambda_\iota^{(q)} \subset \mathfrak{R}_d(\boldsymbol{V})$, *regular part* of $\boldsymbol{V}$ of dimension d. We use this general fact:

Proposition 6. *Let $\boldsymbol{W}$ be an analytic subvariety of a $\mathcal{C}^\omega$ manifold $\mathcal{M}$. If* $\dim \boldsymbol{W} = d$ *then $\boldsymbol{W}\backslash\mathfrak{R}_d(\boldsymbol{W})$ is an analytic variety. If a Nagano leaf $\mathcal{L}$ of $\boldsymbol{W}$ intersects $\boldsymbol{W}\backslash\mathfrak{R}_d(\boldsymbol{W})$ then* $\dim \mathcal{L} < d$ *and $\mathcal{L} \subset \boldsymbol{W}\backslash\mathfrak{R}_d(\boldsymbol{W})$.*

To show that each connected component of $\mathfrak{R}_d(\boldsymbol{V}) \cap \mathfrak{Q}_r^{(n)}$ is contained in a Nagano leaf $\mathcal{L}$ of $\boldsymbol{V}$ we "wiggle" the coordinate frames, as can be shown on Example 1.

To summarize:

$$\boldsymbol{V} = \bigcup_{Nag\ leaves} \mathcal{L}.$$

If $\mathcal{L}$ and $\mathcal{L}'$ are Nagano leaves of $\boldsymbol{V}$, $\mathcal{L} \subset \overline{\mathcal{L}'} \Longleftrightarrow \mathcal{L} \subset \partial\mathcal{L}'$, a consequence of the analogous property of the $\Lambda_\iota^{(q)}$. Also a "Whitney property":

Proposition 7. *Let $\mathcal{L}$ and $\mathcal{L}'$ be Nagano leaves of $\boldsymbol{V}$ such that $\mathcal{L} \subset \partial\mathcal{L}'$. Every vector tangent to $\mathcal{L}$ at a point $\wp \in \mathcal{L}$ is the limit of vectors tangent to $\mathcal{L}'$ at points that converge to $\wp$.*

Detailed proofs will be provided in a forthcoming article.

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