RECONSTRUCTION OF A CONVOLUTION KERNEL IN A PARABOLIC PROBLEM WITH A MEMORY TERM IN THE BOUNDARY CONDITIONS
RICOSTRUZIONE DI UN NUCLEO DI CONVOLUZIONE IN UN PROBLEMA PARABOLICO CON UN TERMINE DI MEMORIA NELLE CONDIZIONI AL CONTORNO

DAVIDE GUIDETTI

Abstract. We consider the problem of the reconstruction of the convolution kernel, together with the solution, in a semilinear integrodiffential parabolic problem in the case that in the boundary conditions, there appear quite general memory operators.

Sunto. Consideriamo il problema della ricostruzione di un nucleo di convoluzione, assieme alla soluzione, in un problema parabolico semilineare, nel caso in cui nelle condizioni al contorno sono presenti operatori con memoria di tipo piuttosto generale.

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I am going to describe the content of a paper in collaboration with C. Cavaterra (Milan), which is in press (see [1]).

We want to study the evolution of the temperature u in a material with memory, occupying a region Ω in $\mathbb{R}^n$ (typically, $n = 3$). The memory mechanism is supposed to be characterized by a time-dependent convolution kernel h. We assume also of having at our disposal a thermostat, influencing the temperature by means of an action on the boundary Γ of Ω. This action is regulated by a suitable measurement $\mathcal{M}(u)$ of the temperature. These considerations are made mathematically explicit by a parabolic integrodifferential

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Dipartimento di Matematica, Università di Bologna

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system of the form

$$
\begin{cases} D_t u(t,x) = Au(t,x) + (h * Au)(t,x) + f(t,x), \quad (t,x) \in Q_T, \\ Bu(t,x') + (h * Bu)(t,x') + q(t,x') = u_e(t,x'), \quad (t,x') \in \Sigma_T, \\ u(0,x) = u_0(x), \quad x \in \Omega. \end{cases} \tag{1}
$$

Here Ω is an open, bounded subset in $\mathbb{R}^n$ with smooth boundary Γ,

$$
Q_T := (0,T) \times \Omega, \quad \Sigma_T := (0,T) \times \Gamma, \tag{2}
$$

$$
A := \sum_{i,j=1}^{n} D_{x_i}(a_{ij}(x) D_{x_j}.), \quad B := \sum_{i=1}^{n} b_i(x') D_{x_i} + b_0(x) \tag{3}
$$

such that:

(AA1) A is a strongly elliptic second order differential operator with smooth coefficients, B is a first order differential operator such that

$$
\sum_{i=1}^{n} b_i(x')\nu_i(x') \neq 0, \forall\, x' \in \Gamma.
$$

Moreover, $e^{i\theta} D_t^2 + \sum_{i,j=1}^{n} D_{x_i}(a_{ij}(x) D_{x_j})$ *is properly elliptic in* $\mathbb{R} \times \Omega$ $\forall \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ *and is covered by* B *in* $\mathbb{R} \times \Gamma$*;*

(AA2) $*$ *is the convolution (in time) in* $(0,T)$*,* $h : (0,T) \to \mathbb{C}$.

u_e stands for the action in Γ of the thermostat. This depends on a measurement of u in the form

(AA3) $\mathcal{M}u(t) = \int_\Omega \omega_1(x) u(t,x) dx + \int_\Gamma \omega_2(x') u(t,x') d\sigma$*, with* $\omega_1 \in L^2(\Omega)$*,* $\omega_2 \in L^2(\Gamma)$.

u_e is obtained as follows: we consider a suitable memory operator

$$
\mathcal{W} : \{m \in C([0,T]) : m(0) = \mathcal{M}(u_0)\} \to C([0,T]) \tag{4}
$$

Later, we shall explain what we mean with the term "memory operator". We fix ϵ in $\mathbb{R}^+$ and the problem

$$
\begin{cases} \epsilon\phi'(t) + \phi(t) = \mathcal{W}(\mathcal{M}(u))(t) + u_C(t), \quad t \in [0,T], \\ \\ \phi(0) = \phi_0, \end{cases} \tag{5}
$$

with $u_C : [0,T] \to \mathbb{C}$ suitably regular, and set

$$u_e(t,x') := \phi(t)u_A(t,x') + u_B(t,x'), \tag{6}$$

with $u_A, u_B : \Sigma_T \to \mathbb{C}$. Clearly,

$$\phi(t) = E_1 * \mathcal{W}(\mathcal{M}(u))(t) + \phi_0(t),$$

with $E_1(t) = \epsilon^{-1}e^{-t/\epsilon}$ and ϕ_0 independent of u. This construction of u_e follows the lines of [2].

So we consider a semilinear parabolic system with nonhomogeneous boundary conditions in the form

$$\begin{cases} D_t u(t,x) = Au(t,x) + (h * Au)(t,x) + f(t,x), \quad (t,x) \in Q_T, \\ Bu(t,x') + (h * Bu)(t,x') + q(t,x') = [E_1 * \mathcal{W}(\mathcal{M}u)]r(t,x'), \quad (t,x') \in \Sigma_T, \\ u(0,x) = u_0(x), \quad x \in \Omega, \end{cases} \tag{7}$$

We precise what we mean with "memory operator":

(AA4) $\mathcal{W}$ is a "memory operator" in the sense of [5]*:*

$$\mathcal{W} : D(\mathcal{W}) := \{f \in C([0,T]) : f(0) = \mathcal{M}(u_0)\} \to C([0,T])$$

(usually nonlinear), is such that, $\forall \tau \in [0,T]$, $\forall f_1, f_2 \in D(\mathcal{W})$ with $f_{1|[0,\tau]} = f_{2|[0,\tau]}$, $\mathcal{W}f_{1|[0,\tau]} = \mathcal{W}f_{2|[0,\tau]}$.

We shall need also the following further conditions:

(AA5) $\mathcal{W}$ satisfies the following regularity and continuity assumptions: it maps $D(\mathcal{W}) \cap BV([0,T])$ into $C([0,T]) \cap BV([0,T])$; moreover, there exists $L \in \mathbb{R}^+$ such that, $\forall g_1, g_2 \in D(\mathcal{W})$,

$$\|\mathcal{W}(g_1) - \mathcal{W}(g_2)\|_{C([0,T])} \le L\|g_1 - g_2\|_{C([0,T])}.$$

We observe the following: if $\tau \in (0,T]$, we can consider the operator

$$\begin{cases} \mathcal{W}_\tau : D(\mathcal{W}_\tau) := \{f \in C([0,\tau]) : f(0) = \mathcal{M}(u_0)\} \to C([0,\tau]), \\ \\ \mathcal{W}_\tau(g) := \mathcal{W}(\tilde{g})_{|[0,\tau]}, \end{cases} \tag{8}$$

with $\tilde{g} \in C([0,T])$, such that $\tilde{g}_{|[0,\tau]} = g$. It is easy to check that $\mathcal{W}_\tau$ fulfills (AA4) and (AA5), replacing T with τ.

As we shall see, assumption (AA4)-(AA5) are quite general and cover some important cases in applications, like Preisach operators and "Generalized plays". We stress the fact that, if we want to cover these cases, we are not allowed to impose Lipschitz continuity in subspaces of $C([0,T])$ whose norms envolve differences.

Now we suppose that even the convolution kernel h in (7) is unknown, together with u. As we expect that for reasonable data, for every reasonable h (7) has a solution, it is clear that, if we want to determine h together with u, we need to know something more. The further information which we prescribe is the following:

$$\Phi(u(t,\cdot)) := \int_\Omega \omega(x)u(t,x)dx = g(t), \quad t \in (0,T). \tag{9}$$

The main result of our work is the following

Theorem 1. *Consider system (7)-(9), under the conditions (AA1)-(AA5). Assume, moreover that:*

(I) f, q, u_0 are suitably regular (precisely, $f \in H^{1,0}(Q_T)$, $q, r \in H^{5/4}((0,T); L^2(\Gamma)) \cap H^1((0,T); H^{1/2}(\Gamma))$, $u_0 \in H^2(\Omega)$, $v_0 := Au_0 + f(0,\cdot) \in H^1(\Omega)$);

(II) $\omega \in H^2_0(\Omega)$;

(III) $\Phi(Au_0) \neq 0$, $g \in H^2((0,T))$;

(IV) the following compatibility conditions hold: $Bu_0 + q(0,\cdot) = 0$, $\Phi(u_0) = g(0)$, $\Phi(v_0) = g'(0)$.

Then (7)-(9) has a unique solution (u,h), with $u \in H^2((0,T); L^2(\Omega)) \cap H^1((0,T); H^2(\Omega))$.

Sketch of the proof If we think of h as the unknown in (7), it appears as the solution of an integral equation of the first kind, which is a severely badly posed problem. If we differentiate the two first equation in (7) with respect to t (the assumptions allow to do it), setting $v := D_t u$, we obtain

$$(10) \quad \begin{cases} D_t v(t,x) = Av(t,x) + h(t)Au_0 + (h * Av)(t,x) + D_t f(t,x), \quad (t,x) \in Q_T, \\ Bv(t,x') + h(t)Bu_0 + (h * Bv)(t,x') + D_t q(t,x') \\ = D_t\{[E_1 * \mathcal{W}(\mathcal{M}u_0 + 1 * \mathcal{M}v)]r(t,x')\}, (t,x') \in \Sigma_T, \\ v(0,x) = v_0(x), \quad x \in \Omega. \end{cases}$$

Applying Φ to the first equation in (10), setting $\psi_1 := \chi\overline{A^*\varpi}$ (A^* := adjoint of A), $\chi := \Phi(Au_0)^{-1}$ and imposing (9) , we obtain

$$h(t) = h_0(t) - (\psi_1, v(t,\cdot)) - [h * (\chi\psi_1, v)](t),$$

with $(f,g) := \int_\Omega f(x)g(x)dx, h_0(t) := \chi\{g'(t) - \Phi[D_t f(t,\cdot)]\}$. So we are reduced to a system of the form

$$(11) \quad \begin{cases} D_t v(t,x) = Av(t,x) + h(t)Au_0 + (h * Av)(t,x) + D_t f(t,x), \quad (t,x) \in Q_T, \\ Bv(t,x') + h(t)Bu_0 + (h * Bv)(t,x') + D_t q(t,x') \\ = D_t\{[E_1 * \mathcal{W}(\mathcal{M}u_0 + 1 * \mathcal{M}v)]r(t,x')\}, (t,x') \in \Sigma_T, \\ h(t) = h_0(t) - (\psi_1, v(t,\cdot)) - [h * (\chi\psi_1, v)](t), \quad t \in (0,T), \\ v(0,x) = v_0(x), \quad x \in \Omega. \end{cases}$$

It is not difficult to show that, if $(v,h) \in H^{1,2}(Q_T) \times L^2((0,T))$ solves (11) and we set

$$u(t,x) := u_0 + \int_0^t v(s,x)ds,$$

then (u,h) solves the system (7)-(9). In the space $H^{1,2}(Q_T)$ there is a classical parabolic theory (see [4]), which we are going to illustrate. The following result holds:

Theorem 2. *Suppose that (AA1) holds. Consider the problem*

$$(12) \quad \begin{cases} D_t v(t,x) = Av(t,x) + f(t,x), \quad (t,x) \in Q_T, \\ \\ Bv(t,x') = g(x'), \quad (t,x') \in \Sigma_T, \\ \\ v(0,x) = v_0(x), \quad x \in \Omega, \end{cases}$$

Then (12) has a unique solution in the space $H^{1,2}(Q_T)$ if and only if $f \in L^2(Q_T)$, $g \in H^{1/4,1/2}(\Sigma_T)$ and $v_0 \in H^1(\Omega)$.

A natural strategy (with some simplifications) could be the following: let $\tau \in (0,T]$. We consider the system (11), replacing T with $\tau \in (0,T]$ and $\mathcal{W}$ with $\mathcal{W}_\tau$. For each $(V,H) \in H^{1,2}(Q_\tau) \times L^2((0,\tau))$ with $V(0,\cdot) = v_0$, we consider the equation

$$
\begin{cases}
D_t v(t,x) = Av(t,x) + H(t)Au_0 + (H * AV)(t,x) + D_t f(t,x), \quad (t,x) \in Q_\tau, \\
Bv(t,x') + H(t)Bu_0 + (H * BV)(t,x') + D_t q(t,x') \\
= D_t\{[E_1 * \mathcal{W}_\tau(\mathcal{M}u_0 + 1 * \mathcal{M}V)]r(t,x')\}, (t,x') \in \Sigma_\tau, \\
h(t) = h_0(t) - (\psi_1, V(t,\cdot)) - [H * (\chi\psi_1, V)](t), \quad t \in (0,\tau), \\
v(0,x) = v_0(x), \quad x \in \Omega.
\end{cases} \tag{13}
$$

The assumptions (AA1)-(AA5), together with Theorem 2, allow to say that (13) has a unique solution $(v,h) = S(V,H)$ in $H^{1,2}(Q_\tau) \times L^2((0,\tau))$ for each (V,H). We would like to determine an appropriate closed subset $\mathcal{U}_\tau$ of $\{(V,H) \in H^{1,2}(Q_\tau) \times L^2((0,\tau)) : V(0,\cdot) = v_0\}$ which is mapped by S into itself and such that S is a contraction in it. The contraction mapping theorem would guarantee the existence of a unique solution to (11) in $\mathcal{U}_\tau$. In this order of ideas, we should need the fact that the nonlinear operator

$$
v \to D_t[E_1 * \mathcal{W}_\tau(\mathcal{M}u_0 + 1 * \mathcal{M}v)]r
$$

were Lipschitz continuous from $H^{1,2}(Q_\tau)$ to $H^{1/4,1/2}(\Sigma_\tau)$. Now our assumptions allow to say that this operator really carries $H^{1,2}(Q_\tau)$ into $H^{1/4,1/2}(\Sigma_\tau)$, but is not Lipschitz continuous, as the norm in $H^{1/4,1/2}(\Sigma_\tau)$ involves differences in the time variable. So this approach does not seem to work. However, we might look for a less regular solution. The main tool could be the following result, again from from ([4]):

Theorem 3. *Assume that (AA1) holds. Consider the linear operator $S : L^2(Q_T) \times H^{1/4,1/2}(\Sigma_T) \times H^1(\Omega)$ to $H^{1,2}(Q_T)$ such that $S(f,g,v_0)$ is the solution to (10). Then S admits a unique continuous extension from $H^{1/4,1/2}(Q_T)' \times L^2(\Sigma_T) \times H^{1/2}(\Omega)$ into $H^{3/4,3/2}(Q_T)$.*

The key fact for our purposes is that $v \to D_t[E_1 * \mathcal{W}(\mathcal{M}u_0 + 1 * \mathcal{M}v)]r$ is Lipschitz continuous from $H^{3/4,3/2}(Q_T)$ to $L^2(\Sigma_T)$. It is conceivable that Theorem 3 is not so amenable to apply, as it requires (among other things) to study terms of the form $h * F$, with $F \in H^{1/4,1/2}(Q_T)'$ which is a space of distribution. Nevertheless, this can be done and the scheme we outlined before is applicable, modifying the spaces. So we get a unique solution in $X_\tau \times L^2((0,\tau))$, for τ sufficiently small, with

$$X_\tau = H^{3/4,3/2}(Q_\tau).$$

The method can be iterated and this allows to construct a unique solution in (v,h) in $X_T \times L^2((0,T))$. Finally, one can show that the solution we have found belongs, in fact, to $H^{1,2}(Q_T) \times L^2((0,T))$, which gives a unique solution (u,h) to (7)-(9) in $[H^2((0,T);L^2(\Omega)) \cap H^1((0,T);H^2(\Omega))] \times L^2((0,T))$.

□

Now we are going to describe some types of memory operators to which our previous results are applicable. We start by considering the so called "generalized plays", following the presentation in [3]. We consider the following general situation:

(A1) $\lambda_l, \lambda_r \in C(\mathbb{R};\mathbb{R})$, they are nondecreasing and $\lambda_r(u) \le \lambda_l(u)$, $\forall u \in \mathbb{R}$. Let $m \in C([a,b])$ be such that $\lambda_r[m(a)] \le x_0 \le \lambda_l[m(a)]$ and assume that m is piecewise monotonic, in the sense that there exist $t_0, ..., t_N$, with $a = t_0 < ... < t_N = b$, such that in each interval $[t_{j-1}, t_j]$ m is monotonic. We start by defining $\mathcal{W}(m)(t)$ if $a \le t \le t_1$: we set

$$\mathcal{W}(m)(t) = \begin{cases} max\{x_0, \lambda_r(m(t))\} & \text{if } m_{|[a,t_1]} \text{is nondecreasing ,} \\ min\{x_0, \lambda_l(m(t))\} & \text{if } m_{|[a,t_1]} \text{is nonincreasing .} \end{cases}$$

We have

$$\lambda_r(m(t)) \le \mathcal{W}(m)(t) \le \lambda_l(m(t)), \quad \forall t \in [a, t_1].$$

The extension of $\mathcal{W}(m)$ to $[0,t_2]$ can be obtain analogously, replacing x_0 with $\mathcal{W}(m)(t_1)$. In general, assuming of having defined $\mathcal{W}(m)(t)$ in each interval $[t_0,t_1], \dots, [t_{j-1},t_j]$ with

$$\lambda_r(m(t)) \le \mathcal{W}(m)(t) \le \lambda_l(m(t)), \quad \forall t \in [a, t_j],$$

we extend $\mathcal{W}(m)$ to $[t_j, t_{j+1}]$ setting $x_j := \mathcal{W}(m)(t_j)$ and

$$\mathcal{W}(m)(t) = \begin{cases} max\{x_j, \lambda_r(m(t))\} & \text{if } m_{|[t_j,t_{j+1}]} \text{is nondecreasing ,} \\ min\{x_j, \lambda_l(m(t))\} & \text{if } m_{|[t_j,t_{j+1}]} \text{is nonincreasing .} \end{cases}$$

Obviously, with such procedure we obtain a function $\mathcal{W}(m)$ in $C([a,b];\mathbb{R})$. Let $m \in C([a,b];\mathbb{R})$ be such that, again, $\lambda_r(m(a)) \le x_0 \le \lambda_l(m(a))$. We construct a sequence

(m_k) of piecewise monotonic functions $(m_k)_{k\in\mathbb{N}}$, converging to m uniformly. Then it is possible to show that the sequence $\mathcal{W}(m_k)$ converges uniformly to an element $\mathcal{W}(m)$ which does not depend on (m_k). In this way, we are able to define $\mathcal{W}(m)$ for every m in $C([a,b];\mathbb{R})$ such that $\lambda_r(m(a)) \leq x_0 \leq \lambda_l(m(a))$. It is easy to see that this is a memory operator. Assume that λ_r and λ_l are globally Lipschitz continuous. Then it is possible to show that, if $m \in C([a,b];\mathbb{R}) \cap BV([a,b])$ and $\lambda_r(m(a)) \leq x_0 \leq \lambda_l(m(a))$, $\mathcal{W}(m) \in C([a,b];\mathbb{R}) \cap BV([a,b])$. Moreover, the map $m \to \mathcal{W}(m)$ is Lipschitz continuous with respect to the norm in $C([a,b])$.

Now we describe Preisach operators. We begin by introducing the Preisach plane

$$\mathcal{P} := \{\rho = (\rho_1, \rho_2) \in \mathbb{R}^2 : \rho_1 < \rho_2\}.$$

We fix ρ in $\mathcal{P}$ and ξ_ρ in $\{-1, 1\}$. Let $m \in C([a,b];\mathbb{R})$. We start by defining

$$h_\rho(m)(a) = \begin{cases} 1 & \text{if } m(a) \geq \rho_2 \text{ or } m(a) > \rho_1 \text{ and } \xi_\rho = 1, \\ -1 & \text{if } m(a) \leq \rho_1 \text{ or } m(a) < \rho_2 \text{ and } \xi_\rho = -1. \end{cases}$$

Now we define $h_\rho(m)(t)$ for $t \in (a,b]$. Assume that $h_\rho(m)(a) = 1$. Then $h_\rho(m)(t) = 1$ up to some t_1 in $(a,b]$ such that $m(t_1) = \rho_1$. We set $h_\rho(m)(t_1) = -1$: Then, $h_\rho(m)(t) = -1$ up to some t_2 in $(t_1, b]$ such that $m(t_2) = \rho_2$. We set $h_\rho(t_2) = 1$ and we continue extending $h_\rho(m)(t)$ up to $t = b$. Next, we fix a positive bounded Borel measure μ in $\mathcal{P}$ and a Borel function $\rho \to \xi_\rho$, with values in $\{-1, 1\}$. Finally, we set, for every m in $C([a,b];\mathbb{R})$,

$$\mathcal{W}(m)(t) := \int_{\mathcal{P}} h_\rho(m)(t) d\mu(\rho).$$

The following theorem holds:

Theorem 4. *Suppose that $d\mu = f_1(\rho_1) f_2(\rho_2) d\rho_1 d\rho_2$, with f_1, f_2 nonnegative, bounded and summable in $\mathbb{R}$, with respect to the Lebesgue measure. Then, $\mathcal{W}$ maps $C([a,b];\mathbb{R}) \cap BV([a,b])$ into itself and is Lipschitz continuous from $C([a,b];\mathbb{R})$ into itself.*

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Dipartimento di matematica, Piazza di Porta S. Donato 5, 40126 Bologna, Italy
E-mail address: davide.guidetti@unibo.it