

LANZANI-STEIN INEQUALITIES IN HEISENBERG GROUPS

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ABSTRACT. Lanzani & Stein consider a class of div-curl inequalities in de Rham's complex. In this note we examine the natural counterpart of that kind of inequalities for differential forms in Heisenberg groups \mathbb{H}^1 and \mathbb{H}^2 .

SUNTO. Lanzani & Stein considerano una classe di disuguaglianze tipo div-rot nel complesso di de Rham. In questa nota si esamina la naturale controparte di questo tipo di disuguaglianze nei gruppi di Heisenberg \mathbb{H}^1 e \mathbb{H}^2 .

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1. INTRODUCTION

Suppose Z is a smooth vector field, $Z(x) = (Z_1(x), Z_2(x), Z_3(x))$, of compact support in \mathbb{R}^3 . Consider in \mathbb{R}^3 the system

$$\begin{cases} \operatorname{curl} Z = f \\ \operatorname{div} Z = 0, \end{cases}$$

it is well known that $Z = (-\Delta)^{-1} \operatorname{curl} f$ is a solution of the system. Then, by the Calderon-Zygmund theory we can say that

$$\|\nabla Z\|_{L^p(\mathbb{R}^3)} \leq C_p \|f\|_{L^p(\mathbb{R}^3)}, \quad \text{for } 1 < p < \infty.$$

Using the Sobolev inequality we have then for $1 < p < 3$,

$$\|Z\|_{L^{p^*}(\mathbb{R}^3)} \leq \|f\|_{L^p(\mathbb{R}^3)},$$

where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$. When we turn to the case $p = 1$ the first inequality fails. The second remains true. This is exactly the result proved in [BB04] by Bourgain & Brezis.

More precisely, in [BB04], Bourgain & Brezis establish new estimates for the Laplacian, the div-curl system, and more general Hodge systems in \mathbb{R}^n and they show in particular that if Z is a smooth vector field, of compact support in \mathbb{R}^n , with $n \geq 3$, and if $\operatorname{curl} Z = f$ and $\operatorname{div} Z = 0$, then there exists a constant $C > 0$ so that

$$(1) \quad \|Z\|_{L^{n/(n-1)}(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}.$$

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This result does not follow straightforwardly from Calderon-Zygmund theory and Sobolev inequality. The inequality (1) is a generalization of the classical sharp Sobolev inequality (the so-called Gagliardo-Nirenberg inequality) valid for all $n \geq 1$: let u be a scalar smooth function of compact support in \mathbb{R}^n then

$$(2) \quad \|u\|_{L^{n/(n-1)}(\mathbb{R}^n)} \leq c \|\nabla u\|_{L^1(\mathbb{R}^n)}.$$

In [LS05] Lanzani & Stein proved that the classical Gagliardo–Nirenberg inequality (2) is the first link of a chain of analogous inequalities for compactly supported smooth differential h -forms in \mathbb{R}^n

$$(3) \quad \|u\|_{L^{n/(n-1)}(\mathbb{R}^n)} \leq C (\|du\|_{L^1(\mathbb{R}^n)} + \|\delta u\|_{L^1(\mathbb{R}^n)}) \quad \text{if } h \neq 1, n-1;$$

$$(4) \quad \|u\|_{L^{n/(n-1)}(\mathbb{R}^n)} \leq C (\|du\|_{L^1(\mathbb{R}^n)} + \|\delta u\|_{\mathcal{H}^1(\mathbb{R}^n)}) \quad \text{if } h = 1;$$

$$(5) \quad \|u\|_{L^{n/(n-1)}(\mathbb{R}^n)} \leq C (\|du\|_{\mathcal{H}^1(\mathbb{R}^n)} + \|\delta u\|_{L^1(\mathbb{R}^n)}) \quad \text{if } h = n-1,$$

where d is the exterior differential, and δ (the exterior codifferential) is its formal L^2 -adjoint. Here $\mathcal{H}^1(\mathbb{R}^n)$ is the real Hardy space (see e.g. [Ste93]). In other words, the main result of [LS05] provides a priori estimates for a div-curl systems with data in $L^1(\mathbb{R}^n)$. This result contains in particular the well-known Burgain-Brezis inequality [BB03], [BB04] (see also [VS04]) for divergence-free vector fields in \mathbb{R}^n . Related results have been obtained again by Burgain-Brezis in [BB07].

We refer the reader to all previous references for an extensive discussion about the presence of the Hardy space in (4), (5).

Recently, in [CVS09], Chanillo & Van Schaftingen extended Burgain-Brezis inequality to a class of vector fields in Carnot groups. Some of the results of [CVS09] are presented in Theorems 2.3 and 2.4 below.

Among Carnot groups, the simplest but, at the same time, non-trivial instance is provided by Heisenberg groups \mathbb{H}^n , with $n \geq 1$, and, in particular, by the first Heisenberg group \mathbb{H}^1 . Precise definitions will be given later (see Section 2); let us remind that \mathbb{H}^1 is a Carnot group of step 2 with 2 generators, and that it is in some sense the “model” of all topologically 3-dimensional contact structures.

More formally, the Heisenberg group \mathbb{H}^1 can be identified with \mathbb{R}^3 , with variables (x, y, t) . Set $X := \partial_x - \frac{1}{2}y\partial_t$, $Y := \partial_y + \frac{1}{2}x\partial_t$, $T := \partial_t$. The stratification of its algebra \mathfrak{h} is given by $\mathfrak{h} = V_1 \oplus V_2$, where $V_1 = \text{span}\{X, Y\}$ and $V_2 = \text{span}\{T\}$.

In spite of the extensive study of differential equations in Carnot groups (and, more generally, in sub-Riemannian spaces) carried on the last few decades, very few results are known for pde’s involving differential forms in groups (see, e.g., [Rum94], [Rum01], [BFT08], [BFTT10], [FT12], [BF12a], [BF12b], [FOV12]).

As a contribution in this direction, in this paper we attack the study of inequalities (3), (4), (5) for differential forms in Heisenberg groups \mathbb{H}^n , $n \geq 1$.

The scalar case, i.e. the Gagliardo-Nirenberg inequality in Carnot groups, is already well known, as well as its geometric counterpart, the isoperimetric inequality: see [FGW94], [FLW95], [CDG94], [GN96], [MSC95], [Pan82].

A natural setting for div-curl type systems in Heisenberg groups is provided by the so-called Rumin's complex (E_0^*, d_c) of differential forms in \mathbb{H}^n . In fact, De Rham's complex (Ω^*, d) of differential forms, endowed with the usual exterior differential, does not fit the very structure of the group, since it is not invariant under group dilations, basically since it mixes derivatives along all the layers of the stratification. Rumin's complex is meant precisely to overcome this difficulty.

As a matter of fact, the construction of the complex (E_0^*, d_c) is rather technical and will be illustrated in Section 3. However, it is important to stress here that Rumin's differential d_c may be a differential operator of higher order in the horizontal derivatives. This property affects crucially our results, that are therefore a distinct counterpart of those of Lanzani & Stein.

This note is organized as follows: in Section 2 we fix our notations and we collect some known results about Heisenberg groups. Moreover, we present two crucial estimates proved by Chanillo & Van Schaftingen ([CVS09]) for "divergence free" horizontal vector fields in Carnot groups, as well as the classical Cartan's identity that we use in this paper to reduce ourselves precisely to the case of "divergence free" horizontal vector fields. In Section 3, we sketch the construction of Rumin's complex of differential forms in Heisenberg groups, and we remind some properties of the fundamental solution for a suitable Laplace operator on Rumin's forms ([BFT08], [BFT09]). In Section 4 we collect our main results in \mathbb{H}^1 and \mathbb{H}^2 and we discuss also their sharpness. At same time, we show how they can be improved for special choices of the data.

2. PRELIMINARY RESULTS

2.1. Notations. We denote by \mathbb{H}^n the n -dimensional Heisenberg group, identified with \mathbb{R}^{2n+1} through exponential coordinates. A point $p \in \mathbb{H}^n$ is denoted by $p = (x, y, t)$, with both $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$. If p and $p' \in \mathbb{H}^n$, the group operation is defined as

$$p \cdot p' = (x + x', y + y', t + t' - \frac{1}{2} \sum_{j=1}^n (x_j y'_j - y_j x'_j)).$$

If we denote by p^{-1} the inverse of p , we remind that $p^{-1} = (-x, -y, -t)$. Sometimes, we write also pq for $p \cdot q$.

For a general review on Heisenberg groups and their properties, we refer to [Ste93], [Gro96] and to [VSCC92]. We limit ourselves to fix some notations, following [FSSC07].

For fixed $q \in \mathbb{H}^n$ and for $r > 0$, left translations $\tau_q : \mathbb{H}^n \rightarrow \mathbb{H}^n$ and non isotropic dilations $\delta_r : \mathbb{H}^n \rightarrow \mathbb{H}^n$ are defined as

$$(6) \quad \tau_q(p) := q \cdot p \quad \text{and as} \quad \delta_r(p) := (rx, ry, r^2t).$$

The Heisenberg group \mathbb{H}^n can be endowed with the homogeneous norm (Koranyi norm)

$$(7) \quad \varrho(p) = ((x^2 + y^2)^2 + t^2)^{1/4},$$

and we define the gauge distance (see [Ste93], p. 638) as

$$(8) \quad d(p, q) := \varrho(p^{-1} \cdot q).$$

It is well known that the topological dimension of \mathbb{H}^n is $2n + 1$, since as a smooth manifold it coincides with \mathbb{R}^{2n+1} , whereas the Hausdorff dimension of (\mathbb{H}^n, d) is $Q = 2n + 2$.

We denote by \mathfrak{h} the Lie algebra of the left invariant vector fields of \mathbb{H}^n . The standard basis of \mathfrak{h} is given, for $i = 1, \dots, n$, by

$$X_i := \partial_{x_i} - \frac{1}{2}y_i\partial_t, \quad Y_i := \partial_{y_i} + \frac{1}{2}x_i\partial_t, \quad T := \partial_t.$$

The only non-trivial commutation relations are $[X_j, Y_j] = T$, for $j = 1, \dots, n$. When $n = 1$ we just write $X := X_1$ and $Y := Y_1$.

The *horizontal subspace* \mathfrak{h}_1 is the subspace of \mathfrak{h} spanned by X_1, \dots, X_n and Y_1, \dots, Y_n . Coherently, from now on, we refer to $X_1, \dots, X_n, Y_1, \dots, Y_n$ (identified with first order differential operators) as to the *horizontal derivatives*. Denoting by \mathfrak{h}_2 the linear span of T , the 2-step stratification of \mathfrak{h} is expressed by

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2.$$

The vector spaces \mathfrak{h} can be endowed with an inner product, indicated by $\langle \cdot, \cdot \rangle$, making $X_1, \dots, X_n, Y_1, \dots, Y_n$ and T orthonormal.

Throught this paper, to avoid cumbersome notations, we write also

$$(9) \quad W_i := X_i, \quad W_{i+n} := Y_i, \quad W_{2n+1} := T, \quad \text{for } i = 1, \dots, n.$$

If $f : \mathbb{H}^n \rightarrow \mathbb{R}$, we denote by $\nabla_{\mathbb{H}} f$ the horizontal vector field

$$\nabla_{\mathbb{H}} f := \sum_{i=1}^{2n} (W_i f) W_i,$$

whose coordinates are $(W_1 f, \dots, W_{2n} f)$. Moreover, if $\Phi = (\phi_1, \dots, \phi_{2n})$ is an horizontal vector field, we define $\text{div}_{\mathbb{H}} \phi$ as the real valued function

$$(10) \quad \text{div}_{\mathbb{H}}(\Phi) := \sum_{j=1}^{2n} W_j \phi_j.$$

If f is a real function defined in \mathbb{H}^n , we denote by ${}^v f$ the function defined by ${}^v f(p) := f(p^{-1})$, and, if $T \in \mathcal{D}'(\mathbb{H}^n)$, then ${}^v T$ is the distribution defined by $\langle {}^v T | \phi \rangle := \langle T | {}^v \phi \rangle$ for any test function ϕ .

Following e.g. [FS82], we can define a group convolution in \mathbb{H}^n : if, for instance, $f \in \mathcal{D}(\mathbb{H}^n)$ and $g \in L^1_{\text{loc}}(\mathbb{H}^n)$, we set

$$(11) \quad f * g(p) := \int f(q) g(q^{-1} \cdot p) dq \quad \text{for } q \in \mathbb{H}^n.$$

We remind that, if (say) g is a smooth function and L is a left invariant differential operator, then

$$L(f * g) = f * Lg.$$

We remind also that the convolution is again well defined when $f, g \in \mathcal{D}'(\mathbb{H}^n)$, provided at least one of them has compact support (as customary, we denote by $\mathcal{E}'(\mathbb{H}^n)$ the class of compactly supported distributions in \mathbb{H}^n identified with \mathbb{R}^{2n+1}). In this case the following identities hold

$$(12) \quad \langle f * g | \phi \rangle = \langle g | {}^\vee f * \phi \rangle \quad \text{and} \quad \langle f * g | \phi \rangle = \langle f | \phi * {}^\vee g \rangle$$

for any test function ϕ .

If $I = (i_1, \dots, i_{2n+1})$ is a multi-index, we set $W^I = W_1^{i_1} \dots W_{2n}^{i_{2n}} T^{i_{2n+1}}$. Furthermore, we set $|I| := i_1 + \dots + i_{2n} + i_{2n+1}$ the order of the differential operator W^I , and $d(I) := i_1 + \dots + i_{2n} + 2i_{2n+1}$ its degree of homogeneity with respect to group dilations.

Suppose now $f \in \mathcal{E}'(\mathbb{H}^n)$ and $g \in \mathcal{D}'(\mathbb{H}^n)$. Then, if $\psi \in \mathcal{D}(\mathbb{H}^n)$, we have

$$(13) \quad \begin{aligned} \langle (W^I f) * g | \psi \rangle &= \langle W^I f | \psi * {}^\vee g \rangle = (-1)^{|I|} \langle f | \psi * (W^I {}^\vee g) \rangle \\ &= (-1)^{|I|} \langle f * {}^\vee W^I {}^\vee g | \psi \rangle. \end{aligned}$$

Following [Fol75], we remind now the notion of *kernel of order α* , as well as some basic properties.

Definition 2.1. A kernel of order α is a homogeneous distribution of degree $\alpha - Q$ (with respect to group dilations δ_r as in (6)), that is smooth outside of the origin.

Proposition 2.2. Let $K \in \mathcal{D}'(\Omega)$ be a kernel of order α .

- i) ${}^\vee K$ is again a kernel of order α ;
- ii) $W_\ell K$ is a kernel of order $\alpha - 1$ for any horizontal derivative $W_\ell K$, $\ell = 1, \dots, 2n$;
- iii) If $\alpha > 0$, then $K \in L_{\text{loc}}^1(\mathbb{H}^n)$.

2.2. Multilinear algebra. The dual space of \mathfrak{h} is denoted by $\bigwedge^1 \mathfrak{h}$. The basis of $\bigwedge^1 \mathfrak{h}$, dual to the basis $\{X_1, \dots, Y_n, T\}$ is the family of covectors $\{dx_1, \dots, dx_n, dy_1, \dots, dy_n, \theta\}$ where $\theta := dt - \frac{1}{2} \sum_{j=1}^n (x_j dy_j - y_j dx_j)$ is called the *contact form* in \mathbb{H}^n .

We indicate as $\langle \cdot, \cdot \rangle$ also the inner product in $\bigwedge^1 \mathfrak{h}$ that makes $dx_1, \dots, dy_n, \theta$ be an orthonormal basis.

Coherently with the previous notation (9), we set

$$\theta_i := dx_i, \quad \theta_{i+n} := dy_i, \quad \theta_{2n+1} := \theta, \quad \text{for } i = 1, \dots, n.$$

We put $\bigwedge_0 \mathfrak{h} := \bigwedge^0 \mathfrak{h} = \mathbb{R}$ and, for $1 \leq k \leq 2n + 1$,

$$\bigwedge_k \mathfrak{h} := \text{span}\{W_{i_1} \wedge \dots \wedge W_{i_k} : 1 \leq i_1 < \dots < i_k \leq 2n + 1\} =: \text{span } \Theta_k,$$

$$\bigwedge^k \mathfrak{h} := \text{span}\{\theta_{i_1} \wedge \dots \wedge \theta_{i_k} : 1 \leq i_1 < \dots < i_k \leq 2n + 1\} =: \text{span } \Theta^k.$$

The volume $(2n + 1)$ -form $\theta_1 \wedge \dots \wedge \theta_{2n+1}$ will be also written as dV .

The action of a k -covector φ on a k -vector v is denoted by $\langle \varphi | v \rangle$.

The inner product $\langle \cdot, \cdot \rangle$ extends canonically to $\bigwedge_k \mathfrak{h}$ and to $\bigwedge^k \mathfrak{h}$ making both bases Θ_k and Θ^k orthonormal. We denote by θ_i^k the i -element of the orthonormal basis Θ^k , $1 \leq i \leq \binom{2n+1}{k}$.

The same construction can be performed starting from the vector subspace $\mathfrak{h}_1 \subset \mathfrak{h}$, obtaining the *horizontal k -vectors* and *horizontal k -covectors*

$$\begin{aligned}\bigwedge_k \mathfrak{h}_1 &:= \text{span}\{W_{i_1} \wedge \cdots \wedge W_{i_k} : 1 \leq i_1 < \cdots < i_k \leq 2n\} \\ \bigwedge^k \mathfrak{h}_1 &:= \text{span}\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_k} : 1 \leq i_1 < \cdots < i_k \leq 2n\}.\end{aligned}$$

The *symplectic 2-form* $d\theta \in \bigwedge^2 \mathfrak{h}_1$ is $d\theta = -\sum_{i=1}^n dx_i \wedge dy_i$.

If $1 \leq k \leq 2n+1$, the Hodge isomorphism

$$* : \bigwedge_k \mathfrak{h} \longleftrightarrow \bigwedge_{2n+1-k} \mathfrak{h} \quad \text{and} \quad * : \bigwedge^k \mathfrak{h} \longleftrightarrow \bigwedge^{2n+1-k} \mathfrak{h},$$

is defined by

$$v \wedge *w = \langle v, w \rangle W_1 \wedge \cdots \wedge W_{2n+1},$$

$$\varphi \wedge *\psi = \langle \varphi, \psi \rangle \theta_1 \wedge \cdots \wedge \theta_{2n+1}.$$

If $v \in \bigwedge_k \mathfrak{h}$ we define $v^\sharp \in \bigwedge^k \mathfrak{h}$ by the identity $\langle v^\sharp | w \rangle := \langle v, w \rangle$, and analogously we define $\varphi^\sharp \in \bigwedge^k \mathfrak{h}$ for $\varphi \in \bigwedge^k \mathfrak{h}$.

As pointed out in the Introduction, the Lie algebra \mathfrak{h} can be identified with the tangent space at the origin $e = 0$ of \mathbb{H}^n , and hence the horizontal layer \mathfrak{h}_1 can be identified with a subspace of $T\mathbb{H}_e^n$ that we can still denote by $\bigwedge_1 \mathfrak{h}_1$. By left translation, $\bigwedge_1 \mathfrak{h}_1$ generates a subbundle of the tangent bundle, called the horizontal bundle, that, with a slight abuse of notations, we still denote by $\bigwedge_1 \mathfrak{h}_1$. A section of $\bigwedge_1 \mathfrak{h}_1$ is called a horizontal vector field.

We recall now the following two results due to Chanillo & Van Schaftingen that are keystones in our proofs.

Theorem 2.3 ([CVS09], Theorem 1). *Let $\Phi \in D(\mathbb{H}^n, \bigwedge_1 \mathfrak{h}_1)$ be a smooth compactly supported horizontal vector field. If $F \in L^1_{\text{loc}}(\mathbb{H}^n, \bigwedge_1 \mathfrak{h}_1)$ is \mathbb{H} -divergence free, then*

$$|\langle F, \Phi \rangle_{L^2(\mathbb{H}^n, \bigwedge_1 \mathfrak{h}_1)}| \leq C \|F\|_{L^1(\mathbb{H}^n, \bigwedge_1 \mathfrak{h}_1)} \|\nabla_{\mathbb{H}} \Phi\|_{L^2(\mathbb{H}^n, \bigwedge_1 \mathfrak{h}_1)}.$$

Let $k \geq 1$ be fixed, and let $F \in L^1(\mathbb{H}^n, \otimes^k \bigwedge_1 \mathfrak{h}_1)$ belong to the space of the horizontal k -tensors. We can write

$$F = \sum_{i_1, \dots, i_k} F_{i_1, \dots, i_k} W_{i_1} \otimes \cdots \otimes W_{i_k}.$$

We remind that F can be identified with the differential operator

$$u \rightarrow Fu := \sum_{i_1, \dots, i_k} F_{i_1, \dots, i_k} W_{i_1} \cdots W_{i_k} u.$$

Moreover, we denote by $\mathcal{D}(\mathbb{H}^n, \text{Sym}(\otimes^k \bigwedge_1 \mathfrak{h}_1))$ the subspace of compactly supported smooth symmetric horizontal k -tensors.

Then Theorem 2.3 is a special instance of the following more general result.

Theorem 2.4 ([CVS09], Theorem 5). *Let $k \geq 1$, $F \in L^1(\mathbb{H}^n, \otimes^k \wedge_1 \mathfrak{h}_1)$, $\Phi \in \mathcal{D}(\mathbb{H}^n, \text{Sym}(\otimes^k \wedge_1 \mathfrak{h}_1))$.*

Suppose

$$\int_{\mathbb{H}^n} F\psi \, dV = 0 \quad \text{for all } \psi \in \mathcal{D}(\mathbb{H}^n),$$

i.e. suppose that

$$\sum_{i_1, \dots, i_k} W_{i_k} \cdots W_{i_1} F_{i_1, \dots, i_k} = 0 \quad \text{in } \mathcal{D}'(\mathbb{H}^n).$$

Then

$$\left| \int_{\mathbb{H}^n} \langle \Phi, F \rangle \, dV \right| \leq C_k \|F\|_{L^1(\mathbb{H}^n, \otimes^k \wedge_1 \mathfrak{h}_1)} \|\nabla_{\mathbb{H}} \Phi\|_{L^Q(\mathbb{H}^n, \otimes^k \wedge_1 \mathfrak{h}_1)}.$$

We close this Section by recalling the following classical Cartan's formula in \mathbb{H}^n (see, e.g., [Hel01], identity (9) p. 21, though with a different normalization of the wedge product).

Theorem 2.5. *Let ω be a smooth h -form of (Ω^*, d) (the usual de Rham's complex), and let Z_0, Z_1, \dots, Z_h be smooth vector fields in \mathbb{H}^n . Then*

$$(14) \quad \begin{aligned} \langle d\omega | Z_0 \wedge \cdots \wedge Z_h \rangle &= \sum_{i=0}^h (-1)^i Z_i \langle \omega | Z_0 \wedge \cdots \wedge \hat{Z}_i \cdots \wedge Z_h \rangle \\ &+ \sum_{0 \leq i < j \leq h} (-1)^{i+j} \langle \omega | [Z_i, Z_j] \wedge \cdots \wedge \hat{Z}_i \wedge \cdots \wedge \hat{Z}_j \cdots \rangle. \end{aligned}$$

3. INTRINSIC COMPLEX AND FUNDAMENTAL SOLUTION

We summarize now very shortly Rumin's construction of the intrinsic complex. Though this theory can be naturally formulated in any Carnot group, we restrict ourselves to Heisenberg groups. For a general approach, we refer, for instance, to [Rum01] and [BFTT10].

Definition 3.1. If $\alpha \neq 0$, $\alpha \in \wedge^1 \mathfrak{h}_1$, we say that α has *weight* 1, and we write $w(\alpha) = 1$. If $\alpha = \theta$, we say $w(\theta) = 2$. More generally, if $\alpha \in \wedge^h \mathfrak{h}$, we say that α has pure weight k if α is a linear combination of covectors $\theta_{i_1} \wedge \cdots \wedge \theta_{i_h}$ with $w(\theta_{i_1}) + \cdots + w(\theta_{i_h}) = k$.

Notice that, if $\alpha, \beta \in \wedge^h \mathfrak{h}$ and $w(\alpha) \neq w(\beta)$, then $\langle \alpha, \beta \rangle = 0$.

We have ([BFTT10], formula (16))

$$\wedge^h \mathfrak{h} = \wedge^{h,h} \mathfrak{h} \oplus \wedge^{h,h+1} \mathfrak{h},$$

where $\wedge^{h,p} \mathfrak{h}$ denotes the linear span of $\Theta^{h,p} := \{\alpha \in \Theta^h, w(\alpha) = p\}$.

Similarly, if we denote by $\Omega^{h,p}$ the vector space of all smooth h -forms in \mathbb{H}^n of weight p , i.e. the space of all smooth sections of $\wedge^{h,p} \mathfrak{h}$, we have

$$(15) \quad \Omega^h = \Omega^{h,h} \oplus \Omega^{h,h+1}.$$

The following crucial property of the weight follows from Cartan identity: see [Rum01], Section 2.1:

Definition 3.2. Let now $\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \alpha_i \theta_i^h \in \Omega^{h,p}$ be a (say) smooth form of pure weight p . Then we can write

$$d\alpha = d_0\alpha + d_1\alpha + d_2\alpha,$$

where $d_0\alpha$ has still weight p , $d_1\alpha$ has weight $p+1$, and $d_2\alpha$ has weight $p+2$.

By Cartan's formula (14), $w(d\theta_i^h) = w(\theta_i^h)$ (because of their left-invariance), and then we can write explicitly

$$d_0\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \alpha_i d\theta_i^h$$

that does not increase the weight,

$$d_1\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \sum_{j=1}^{2n} (W_j \alpha_i) \theta_j \wedge \theta_i^h$$

that increases the weight by 1, and

$$d_2\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} (T\alpha_i) \theta \wedge \theta_i^h,$$

that increases the weight by 2.

Theorem 3.3 (see [Rum94], [Rum00]). *We have:*

- $E_0^1 = \bigwedge^1 \mathfrak{h}_1$;
- if $2 \leq h \leq n$ then $E_0^h = \bigwedge^h \mathfrak{h}_1 \cap (\bigwedge^{h-2} \mathfrak{h}_1 \wedge d\theta)^\perp$;
- if $n < h \leq 2n+1$ then $E_0^h = \{\alpha = \beta \wedge \theta, \beta \in \bigwedge^{h-1} \mathfrak{h}_1, \alpha \wedge d\theta = 0\}$.

We refer to the elements of E_0^h as to *intrinsic h -forms on \mathbb{H}^n* . Notice E_0^h inherits from $\bigwedge^h \mathfrak{h}$ the scalar product on the fibers. Observe also that all forms in E_0^h have weight h if $1 \leq h \leq n$ and weight $h+1$ if $n < h \leq 2n+1$.

We denote by $\Xi_0^h = \{\xi_j^h\}$ an orthonormal basis of E_0^h . We can take $\xi_j^1 = \theta_j$ for $j = 1, \dots, 2n$.

Remark 3.4. From now on, we shall refer to the components of a form $\alpha \in E_0^*$ with respect to the basis Ξ_0^* tout court as to *the components* of α without further specifications.

Definition 3.5. If $0 \leq h \leq n$, $1 \leq p \leq \infty$, we denote by $L^p(\mathbb{H}^n, E_0^h)$ the space of all sections of E_0^h such that their components with respect to the basis Ξ_0^h belong to $L^p(\mathbb{H}^n)$, endowed with its natural norm. Clearly, this definition is independent of the choice of the basis itself. If $h = 0$, we write $L^p(\mathbb{H}^n)$ for $L^p(\mathbb{H}^n, E_0^0)$.

The notation $\mathcal{D}(\mathbb{H}^n, E_0^h)$ has the same meaning.

We define now a (pseudo) inverse of d_0 as follows (see [BFTT10], Lemma 2.11):

Lemma 3.6. *If $\beta \in \bigwedge^{h+1} \mathfrak{h}$, then there exists a unique $\alpha \in \bigwedge^h \mathfrak{h} \cap (\ker d_0)^\perp$ such that*

$$d_0\alpha - \beta \in \mathcal{R}(d_0)^\perp.$$

We set $\alpha := d_0^{-1}\beta$. We notice that d_0^{-1} preserves the weights.

The following theorem summarizes the construction of the intrinsic differential d_c (for details, see [Rum01] and [BFTT10], Section 2).

Theorem 3.7. *The de Rham complex (Ω^*, d) splits in the direct sum of two sub-complexes (E^*, d) and (F^*, d) , with*

$$E := \ker d_0^{-1} \cap \ker(d_0^{-1}d) \quad \text{and} \quad F := \mathcal{R}(d_0^{-1}) + \mathcal{R}(dd_0^{-1}).$$

We have

- i) Let Π_E be the projection on E along F (that is not an orthogonal projection). If $\alpha \in E_0^h$, then
 - $\Pi_E \alpha = \alpha - d_0^{-1}d_1\alpha$ if $1 \leq h \leq n$;
 - $\Pi_E \alpha = \alpha$ if $h > n$.
- ii) Π_E is a chain map, i.e.

$$d\Pi_E = \Pi_E d.$$

- iii) Let Π_{E_0} be the orthogonal projection from $\bigwedge^* \mathfrak{h}$ on E_0^* , then

$$(16) \quad \Pi_{E_0} = Id - d_0^{-1}d_0 - d_0d_0^{-1}, \quad \Pi_{E_0^\perp} = d_0^{-1}d_0 + d_0d_0^{-1}.$$

- iv) $\Pi_{E_0}\Pi_E\Pi_{E_0} = \Pi_{E_0}$ and $\Pi_E\Pi_{E_0}\Pi_E = \Pi_E$.

Set now

$$d_c = \Pi_{E_0} d \Pi_E : E_0^h \rightarrow E_0^{h+1}, \quad h = 0, \dots, 2n.$$

We have:

- v) $d_c^2 = 0$;
- vi) the complex $E_0 := (E_0^*, d_c)$ is exact;
- vii) $d_c : E_0^h \rightarrow E_0^{h+1}$ is an homogeneous differential operator in the horizontal derivatives of order 1 if $h \neq n$, whereas $d_c : E_0^n \rightarrow E_0^{n+1}$ is an homogeneous differential operator in the horizontal derivatives of order 2.

Remark 3.8. if $f \in E_0^0$ (i.e. f is a smooth function), then

$$d_c f = \sum_{j=1}^n X_j f dx_j + \sum_{j=1}^n Y_j f dy_j.$$

Proposition 3.9. *Denote by $\delta_c = d_c^*$ the formal adjoint of d_c in $L^2(\mathbb{G}, E_0^*)$. Then $\delta_c = (-1)^h * d_c^*$ on E_0^h .*

Example 3.10. Let $\mathbb{H}^1 \cong \mathbb{R}^3$ be the first Heisenberg group, with variables (x, y, t) . We have:

$$\begin{aligned} E_0^1 &= \text{span} \{dx, dy\}; \\ E_0^2 &= \text{span} \{dx \wedge \theta, dy \wedge \theta\}; \\ E_0^3 &= \text{span} \{dx \wedge dy \wedge \theta\}. \end{aligned}$$

Thus, if $\alpha = \alpha_1 dx + \alpha_2 dy \in E_0^1$, then

$$(a) \quad d_c \alpha = (X^2 \alpha_2 - 2XY \alpha_1 + YX \alpha_1) dx \wedge \theta + (2YX \alpha_2 - Y^2 \alpha_1 - XY \alpha_2) dy \wedge \theta$$

$$(b) \quad \delta_c \alpha = X \alpha_1 + Y \alpha_2.$$

On the other hand, if $\alpha = \alpha_{13} dx \wedge \theta + \alpha_{23} dy \wedge \theta \in E_0^2$, then

$$(c) d_c \alpha = (X\alpha_{23} - Y\alpha_{13}) dx \wedge dy \wedge \theta;$$

$$(d) \delta_c \alpha = (XY\alpha_{13} - 2YX\alpha_{13} - Y^2\alpha_{23})dx + (X^2\alpha_{13} + 2XY\alpha_{23} - YX\alpha_{23})dy.$$

Example 3.11. Choose now $\mathbb{H}^2 \equiv \mathbb{R}^5$, with variables (x_1, x_2, y_1, y_2, t) . In this case (see e.g. [BFTT10], Appendix B)

$$E_0^1 = \text{span} \{dx_1, dx_2, dy_1, dy_2\};$$

$$E_0^2 = \text{span} \{dx_1 \wedge dx_2, dx_1 \wedge dy_2, dx_2 \wedge dy_1, dy_1 \wedge dy_2,$$

$$\frac{1}{\sqrt{2}}(dx_1 \wedge dy_1 - dx_2 \wedge dy_2)\}.$$

The classes E_0^3 and E_0^4 are easily written by Hodge duality:

$$E_0^3 = \text{span} \{dy_1 \wedge dy_2 \wedge \theta, dx_2 \wedge dy_1 \wedge \theta, dx_1 \wedge dy_2 \wedge \theta, dx_1 \wedge dx_2 \wedge \theta,$$

$$\frac{1}{\sqrt{2}}(dx_1 \wedge dy_1 - dx_2 \wedge dy_2) \wedge \theta\};$$

$$E_0^4 = \text{span} \{dx_2 \wedge dy_1 \wedge dy_2 \wedge \theta, dx_1 \wedge dy_1 \wedge dy_2 \wedge \theta, dx_1 \wedge dx_2 \wedge dy_2 \wedge \theta,$$

$$dx_1 \wedge dx_2 \wedge dy_1 \wedge \theta\}$$

$$E_0^5 = \text{span} \{dx_1 \wedge dx_2 \wedge dy_1 \wedge dy_2 \wedge \theta = dV\}.$$

Thus, if $\alpha = \alpha_1 dx_1 + \alpha_2 dx_2 + \alpha_3 dy_1 + \alpha_4 dy_2 \in E_0^1$, we have

$$(a) d_c \alpha = (X_1\alpha_2 - X_2\alpha_1)dx_1 \wedge dx_2 + (Y_1\alpha_4 - Y_2\alpha_3)dy_1 \wedge dy_2$$

$$+ (X_1\alpha_4 - Y_2\alpha_1)dx_1 \wedge dy_2 + (X_2\alpha_3 - Y_1\alpha_2)dx_2 \wedge dy_1$$

$$+ \frac{X_1\alpha_3 - Y_1\alpha_1 - X_2\alpha_4 + Y_2\alpha_2}{\sqrt{2}} \frac{1}{\sqrt{2}}(dx_1 \wedge dy_1 - dx_2 \wedge dy_2).$$

$$(b) \delta_c \alpha = X_1\alpha_1 + X_2\alpha_2 + Y_1\alpha_3 + Y_2\alpha_4.$$

Definition 3.12. In \mathbb{H}^n , following [Rum94], we define the operator $\Delta_{\mathbb{H},h}$ on E_0^h by setting

$$\Delta_{\mathbb{H},h} = \begin{cases} d_c \delta_c + \delta_c d_c & \text{if } h \neq n, n+1; \\ (d_c \delta_c)^2 + \delta_c d_c & \text{if } h = n; \\ d_c \delta_c + (\delta_c d_c)^2 & \text{if } h = n+1. \end{cases}$$

Notice that $-\Delta_{\mathbb{H},0} = \sum_{j=1}^n (X_j^2 + Y_j^2)$ is the usual sub-Laplacian of \mathbb{H}^n .

Set $N_h := \dim E_0^h$. For sake of simplicity, once the basis Ξ_0^h of E_0^h is fixed, the operator $\Delta_{\mathbb{H},h}$ can be identified with a matrix-valued map, still denoted by $\Delta_{\mathbb{H},h}$

$$(17) \quad \Delta_{\mathbb{H},h} = (\Delta_{\mathbb{H},h}^{ij})_{i,j=1,\dots,N_h} : \mathcal{D}'(\mathbb{H}^n, \mathbb{R}^{N_h}) \rightarrow \mathcal{D}'(\mathbb{H}^n, \mathbb{R}^{N_h}).$$

This identification makes possible to avoid the notion of currents: we refer to [BFTT10] for this more elegant presentation.

Combining [Rum94] and [BFT09], we obtain the following result.

Theorem 3.13. *If $0 \leq h \leq 2n+1$, then the differential operator $\Delta_{\mathbb{H},h}$ is hypoelliptic of order $a=2$ if $h \neq n, n+1$ and of order $a=4$ if $h = n, n+1$ with respect to group dilations.*

Moreover for $j = 1, \dots, N_h$ there exists

$$(18) \quad K_j = (K_{1j}, \dots, K_{N_h j}), \quad j = 1, \dots, N_h$$

with $K_{ij} \in \mathcal{D}'(\mathbb{H}^n) \cap \mathcal{E}(\mathbb{H}^n \setminus \{0\})$, $i, j = 1, \dots, N$ such that

i) we have

$$\sum_i \Delta_{\mathbb{H}, h}^{i\ell} K_{ij} = \begin{cases} \delta & \text{if } \ell = j \\ 0 & \text{if } \ell \neq j; \end{cases}$$

ii) if $a < Q$, then the K_{ij} 's are kernels of type a in the sense of [Fol75], for $i, j = 1, \dots, N_h$ (i.e. they are smooth functions outside of the origin, homogeneous of degree $a - Q$, and hence belonging to $L_{\text{loc}}^1(\mathbb{H}^n)$, by Proposition 2.2). If $a = Q$, then the K_{ij} 's satisfy the logarithmic estimate $|K_{ij}(p)| \leq C(1 + |\ln \rho(p)|)$ and hence belong to $L_{\text{loc}}^1(\mathbb{H}^n)$. Moreover, their horizontal derivatives $W_\ell K_{ij}$, $\ell = 1, \dots, 2n$, are kernels of type $Q - 1$;

iii) when $\alpha \in \mathcal{D}(\mathbb{H}^n, \mathbb{R}^{N_h})$, if we set

$$(19) \quad \mathcal{K}\alpha := \left(\sum_j \alpha_j * K_{1j}, \dots, \sum_j \alpha_j * K_{N_h j} \right),$$

then $\Delta_{\mathbb{H}, h} \mathcal{K}\alpha = \alpha$. Moreover, if $a < Q$, also $\mathcal{K}\Delta_{\mathbb{H}, h} \alpha = \alpha$.

iv) if $a = Q$, then for any $\alpha \in \mathcal{D}(\mathbb{H}^n, \mathbb{R}^{N_h})$ there exists $\beta_\alpha := (\beta_1, \dots, \beta_{N_h}) \in \mathbb{R}^{N_h}$, such that

$$\mathcal{K}\Delta_{\mathbb{H}, h} \alpha - \alpha = \beta_\alpha.$$

Remark 3.14. Coherently with formula (17), the operator \mathcal{K} can be identified with an operator (still denoted by \mathcal{K}) acting on smooth compactly supported differential forms in $\mathcal{D}(\mathbb{H}^n, E_0^h)$.

4. MAIN RESULT

Theorem 4.1. Denote by (E_0^*, d_c) the complex of intrinsic forms in \mathbb{H}^1 . Then there exists $C > 0$ such that for any h -form $u \in \mathcal{D}(\mathbb{H}^1, E_0^h)$, $0 \leq h \leq 3$, satisfying

$$\begin{cases} d_c u = f \\ \delta_c u = g, \end{cases}$$

we have

$$\begin{aligned} \|u\|_{L^{Q/(Q-1)}(\mathbb{H}^1)} &\leq C \|f\|_{L^1(\mathbb{H}^1, E_0^1)} \quad \text{if } h = 0; \\ \|u\|_{L^{Q/(Q-2)}(\mathbb{H}^1, E_0^1)} &\leq C (\|f\|_{L^1(\mathbb{H}^1, E_0^2)} + \|d_c g\|_{\mathcal{H}^1(\mathbb{H}^1)}) \quad \text{if } h = 1; \\ \|u\|_{L^{Q/(Q-2)}(\mathbb{H}^1, E_0^2)} &\leq C (\|d_c f\|_{\mathcal{H}^1(\mathbb{H}^1, E_0^3)} + \|g\|_{L^1(\mathbb{H}^1, E_0^1)}) \quad \text{if } h = 2; \\ \|u\|_{L^{Q/(Q-1)}(\mathbb{H}^1, E_0^3)} &\leq C \|g\|_{L^1(\mathbb{H}^1, E_0^2)} \quad \text{if } h = 3. \end{aligned}$$

Proof. First of all, we notice that, since the complex (E_0^*, d_c) is invariant under Hodge-star duality, we may restrict ourselves to forms in E_0^h , with $h = 0, 1$. The case $h = 0$ is well known ([FGW94], [CDG94], [MSC95]). On the other hand, keeping in mind Theorem 3.13, if $u, \phi \in \mathcal{D}(\mathbb{H}^1, E_0^1)$, we can write

$$(20) \quad \begin{aligned} \langle u, \phi \rangle_{L^2(\mathbb{H}^1, E_0^1)} &= \langle u, \Delta_{\mathbb{H}, 1} \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^1, E_0^1)} \\ &= \langle u, (\delta_c d_c + (d_c \delta_c)^2) \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^1, E_0^1)}. \end{aligned}$$

Consider now the term

$$\langle u, \delta_c d_c \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^1, E_0^1)} = \langle d_c u, d_c \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^1, E_0^2)}.$$

If we write $f := d_c u$, then f is a 2-form in E_0^2 and therefore it can be written as $f = f_1 dx \wedge \theta + f_2 dy \wedge \theta$. Analogously, we can write $d_c \mathcal{K}\phi = (d_c \mathcal{K}\phi)_1 dx \wedge \theta + (d_c \mathcal{K}\phi)_2 dy \wedge \theta$. Thus

$$\langle u, \delta_c d_c \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^1, E_0^1)} = \langle f_1, (d_c \mathcal{K}\phi)_1 \rangle_{L^2(\mathbb{H}^1)} + \langle f_2, (d_c \mathcal{K}\phi)_2 \rangle_{L^2(\mathbb{H}^1)}.$$

Let us estimate, for instance, the first term of the sum. We remind that, since f is closed, we have $Xf_2 - Yf_1 = 0$ (by Example 3.10, (c)), i.e. if

$$F := (f_2, -f_1) \quad \text{then} \quad \operatorname{div}_{\mathbb{H}} F = 0.$$

Thus, if we choose $\Phi := (0, (d_c \mathcal{K}\phi)_1)$, we can apply Theorem 1 in [CVS09] to obtain

$$|\langle F, \Phi \rangle| = |\langle f_1, (d_c \mathcal{K}\phi)_1 \rangle_{L^2(\mathbb{H}^1)}| \leq C \|d_c u\|_{L^1(\mathbb{H}^1, E_0^2)} \|\nabla_{\mathbb{H}} d_c \mathcal{K}\phi\|_{L^Q(\mathbb{H}^1, E_0^2)}.$$

The term $\langle f_2, (d_c \mathcal{K}\phi)_2 \rangle_{L^2(\mathbb{H}^1)}$ can be handled in the same way, choosing $\Phi := ((d_c \mathcal{K}\phi)_2, 0)$. After all, we obtain

$$|\langle u, \delta_c d_c \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^1, E_0^1)}| \leq C \|d_c u\|_{L^1(\mathbb{H}^1, E_0^2)} \|\nabla_{\mathbb{H}} d_c \mathcal{K}\phi\|_{L^Q(\mathbb{H}^1, E_0^2)}.$$

Furthermore, $\nabla_{\mathbb{H}} d_c \mathcal{K}\phi$ can be expressed as a sum of terms with components of the form

$$\alpha_j * W^I \tilde{K}_{ij}, \quad \text{with } d(I) = 3.$$

By Theorem 3.13, iv) and Proposition 2.2, ii) $W^I \tilde{K}_{ij}$ are kernels of order 1, so that, by [Fol75], Proposition 1.11 we have

$$(21) \quad |\langle u, \delta_c d_c \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^1, E_0^1)}| \leq C \|f\|_{L^1(\mathbb{H}^1, E_0^2)} \|\phi\|_{L^{Q/2}(\mathbb{H}^1, E_0^1)}.$$

Consider now the second term in (20)

$$\langle u, (d_c \delta_c)^2 \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^1, E_0^1)} = \langle d_c \delta_c u, d_c \delta_c \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^1, E_0^1)}.$$

By Theorem 3.13, formula (19), keeping in mind that δ_c is an operator of order 1 in the horizontal derivatives when acting on 1-forms, as well as d_c when acting on 0-forms, the quantity $d_c \delta_c \mathcal{K}\phi$ can be written as a sum of terms with components of the form

$$\phi_j * W^I \tilde{K}_{ij}, \quad \text{with } d(I) = 2 \text{ and } \phi_j \in \mathcal{D}(\mathbb{H}^1).$$

On the other hand, if $d_c \delta_c u = d_c g = (d_c g)_1 dx + (d_c g)_2 dy$, then we are reduced to estimate

$$\langle (d_c g)_i, \phi_j * W^I \tilde{K}_{ij} \rangle_{L^2(\mathbb{H}^1)} = \langle (d_c g)_i * {}^v(W^I \tilde{K}_{ij}), \phi_j \rangle_{L^2(\mathbb{H}^1)},$$

for $i = 1, 2$. Moreover,

$$|\langle (d_c g)_i, \phi_j * W^I \tilde{K}_{ij} \rangle_{L^2(\mathbb{H}^1)}| \leq \|(d_c g)_i * {}^v(W^I \tilde{K}_{ij})\|_{L^{Q/(Q-2)}(\mathbb{H}^1)} \|\phi_j\|_{L^{Q/2}(\mathbb{H}^1)}.$$

Notice the $W^I \tilde{K}_{ij}$'s and hence the ${}^v(W^I \tilde{K}_{ij})$'s are kernels of type 2 since $d(I) = 2$. Thus, by Theorem 6.10 in [FS82],

$$|\langle (d_c g)_i, \phi_j * W^I \tilde{K}_{ij} \rangle_{L^2(\mathbb{H}^2)}| \leq C \|d_c g\|_{\mathcal{H}^1(\mathbb{H}^1, E_0^1)} \|\phi\|_{L^Q(\mathbb{H}^1, E_0^1)}.$$

Thus

$$(22) \quad |\langle u, (d_c \delta_c)^2 \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^1, E_0^1)}| \leq C \|d_c g\|_{\mathcal{H}^1(\mathbb{H}^1, E_0^1)} \|\phi\|_{L^{Q/2}(\mathbb{H}^1, E_0^1)}.$$

Thus, combining (21) and (22), by duality we obtain eventually

$$\|u\|_{L^{Q/(Q-2)}(\mathbb{H}^1, E_0^1)} \leq C(\|f\|_{L^1(\mathbb{H}^1, E_0^2)} + \|d_c g\|_{\mathcal{H}^1(\mathbb{H}^1, E_0^1)}),$$

which concludes the proof. \square

Theorem 4.2. *Denote by (E_0^*, d_c) the complex of intrinsic forms in \mathbb{H}^2 . Then there exists $C > 0$ such that for any h -form $u \in \mathcal{D}(\mathbb{H}^2, E_0^h)$, $0 \leq h \leq 5$, such that*

$$\begin{cases} d_c u = f \\ \delta_c u = g \end{cases}$$

we have

$$\begin{aligned} \|u\|_{L^{Q/(Q-1)}(\mathbb{H}^2)} &\leq C\|f\|_{L^1(\mathbb{H}^2, E_0^1)} && \text{if } h = 0; \\ \|u\|_{L^{Q/(Q-1)}(\mathbb{H}^2, E_0^5)} &\leq C\|g\|_{L^1(\mathbb{H}^2, E_0^4)} && \text{if } h = 5; \\ \|u\|_{L^{Q/(Q-1)}(\mathbb{H}^2, E_0^1)} &\leq C(\|f\|_{L^1(\mathbb{H}^2, E_0^2)} + \|g\|_{\mathcal{H}^1(\mathbb{H}^2)}) && \text{if } h = 1; \\ \|u\|_{L^{Q/(Q-1)}(\mathbb{H}^2, E_0^4)} &\leq C(\|f\|_{\mathcal{H}^1(\mathbb{H}^2, E_0^5)} + \|g\|_{L^1(\mathbb{H}^2, E_0^3)}) && \text{if } h = 4; \\ \|u\|_{L^{Q/(Q-2)}(\mathbb{H}^2, E_0^2)} &\leq C(\|f\|_{L^1(\mathbb{H}^2, E_0^3)} + \|d_c g\|_{L^1(\mathbb{H}^2, E_0^2)}) && \text{if } h = 2; \\ \|u\|_{L^{Q/(Q-2)}(\mathbb{H}^2, E_0^3)} &\leq C(\|d_c f\|_{L^1(\mathbb{H}^2, E_0^5)} + \|g\|_{L^1(\mathbb{H}^2, E_0^2)}) && \text{if } h = 3. \end{aligned}$$

Proof. As in Theorem 4.1, the cases $h = 0$ and $h = 5$ are well known, and we may restrict ourselves to forms in E_0^h , with $h = 1, 2$, since the complex (E_0^*, d_c) is invariant under Hodge-star duality.

Case $h = 1$. If $u, \phi \in \mathcal{D}(\mathbb{H}^2, E_0^1)$, we can write

$$(23) \quad \begin{aligned} \langle u, \phi \rangle_{L^2(\mathbb{H}^2, E_0^1)} &= \langle u, \Delta_{\mathbb{H}, 1} \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^2, E_0^1)} \\ &= \langle u, (\delta_c d_c + d_c \delta_c) \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^2, E_0^1)}. \end{aligned}$$

Consider now the first term in the previous sum,

$$\langle u, \delta_c d_c \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^2, E_0^1)} = \langle d_c u, d_c \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^2, E_0^2)}.$$

If we write $f := d_c u$, then $0 = d_c f := \Pi_{E_0} d \Pi_E f = \Pi_{E_0} \Pi_E df$, by Theorem 3.7, ii). If we apply Π_E to this equation, we get

$$(24) \quad 0 = \Pi_E \Pi_{E_0} \Pi_E df = \Pi_E df = d \Pi_E f,$$

by Theorem 3.7, iv), i.e., $\Pi_E f$ is closed in the usual sense. Moreover, since $f, d_c \mathcal{K}\phi \in E_0^2$, we can write $f = \sum_{\ell=1}^5 f_\ell \xi_\ell^2$, $d_c \mathcal{K}\phi = \sum_{\ell=1}^5 (d_c \mathcal{K}\phi)_\ell \xi_\ell^2$, and hence we can reduce ourselves to estimate

$$(25) \quad \langle f_\ell, (d_c \mathcal{K}\phi)_\ell \rangle_{L^2(\mathbb{H}^2)} \quad \text{for } \ell = 1, \dots, 5.$$

Consider now the horizontal 2-tensors $F, G \in \mathcal{D}(\mathbb{H}^2, \otimes^2 \wedge_1 \mathfrak{h}_1)$ defined as

- $F := -\frac{1}{2}f_4(X_1 \otimes X_2 + X_2 \otimes X_1) - \frac{1}{2}f_3(X_1 \otimes Y_2 + Y_2 \otimes X_1) + \frac{1}{2}f_2(X_2 \otimes Y_1 + Y_1 \otimes X_2) - \frac{1}{2}f_1(Y_1 \otimes Y_2 + Y_2 \otimes Y_1)$,
- $G := \frac{\sqrt{2}}{2}f_5(X_1 \otimes X_2 + X_2 \otimes X_1) - f_3 X_1 \otimes X_1 + f_2 X_2 \otimes X_2 - f_1(Y_1 \otimes X_1 + X_2 \otimes Y_2)$,

that are identified with the following differential operator

- $F := -f_4 X_2 X_1 - f_3 Y_2 X_1 + f_2 X_2 Y_1 - f_1 Y_2 Y_1$,
- $G := \sqrt{2}f_5 X_2 X_1 - f_3 X_1^2 + f_2 X_2^2 - f_1(Y_1 X_1 + X_2 Y_2)$,

since the only nontrivial commutation rules are $[X_1, Y_1] = [X_2, Y_2] = T$. We claim that

$$(26) \quad \int_{\mathbb{H}^n} F\psi \, dV = \int_{\mathbb{H}^n} G\psi \, dV = 0 \quad \text{for all } \psi \in \mathcal{D}(\mathbb{H}^n),$$

Suppose for a while (26) holds, and let us achieve the estimate of (25). Suppose for instance $f_\ell = f_1$. We consider now the symmetric horizontal 2-tensor Φ :

$$\Phi := (d_c \mathcal{K}\phi)_1(Y_1 \otimes Y_2 + Y_2 \otimes Y_1),$$

so that

$$\langle f_1, (d_c \mathcal{K}\phi)_1 \rangle_{L^2(\mathbb{H}^2)} = \langle F, \Phi \rangle_{L^2(\mathbb{H}^2, \otimes^2 \Lambda_1 \mathfrak{h}_1)}.$$

By Theorem 2.4

$$(27) \quad \begin{aligned} |\langle f_1, (d_c \mathcal{K}\phi)_1 \rangle_{L^2(\mathbb{H}^2)}| &\leq \|F\|_{L^1(\mathbb{H}^2, \otimes^2 \Lambda_1 \mathfrak{h}_1)} \|\nabla_{\mathbb{H}}(d_c \mathcal{K}\phi)_1\|_{L^Q(\mathbb{H}^2)} \\ &\leq \|f\|_{L^1(\mathbb{H}^2, E_0^2)} \|\nabla_{\mathbb{H}} d_c \mathcal{K}\phi\|_{L^Q(\mathbb{H}^2, E_0^2)}. \end{aligned}$$

On the other hand, $\nabla_{\mathbb{H}} d_c \mathcal{K}\phi$ can be expressed as a sum of terms with components of the form

$$\phi_j * W^I \tilde{K}_{ij} \quad \text{with } d(I) = 2.$$

By Theorem 3.13, iv) and Proposition 2.2, ii) $W^I \tilde{K}_{ij}$ are kernels of type 0, so that, by [Fol75], Proposition 1.9 we have

$$(28) \quad |\langle f_1, (d_c \mathcal{K}\phi)_1 \rangle_{L^2(\mathbb{H}^2)}| \leq C \|f\|_{L^1(\mathbb{H}^2, E_0^2)} \|\phi\|_{L^Q(\mathbb{H}^2, E_0^1)}.$$

The same arguments applies to f_2, f_3, f_4 . As for f_5 , we can use the same argument, replacing F by G and considering $\langle G, \Phi \rangle$ with the symmetric horizontal 2-tensor Φ :

$$\Phi := \sqrt{2} (d_c \mathcal{K}\phi)_5(X_1 \otimes X_2 + X_2 \otimes X_1).$$

We obtain eventually

$$(29) \quad |\langle f, d_c \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^2, E_0^2)}| \leq C \|f\|_{L^1(\mathbb{H}^2, E_0^2)} \|\phi\|_{L^Q(\mathbb{H}^2, E_0^1)}.$$

Consider now the second term in (23)

$$\langle u, d_c \delta_c \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^2, E_0^1)} = \langle \delta_c u, \delta_c \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^2)}.$$

By Theorem 3.13, formula (19), keeping in mind that δ_c is an operator of order 1 in the horizontal derivatives when acting on E_0^1 the quantity $\delta_c \mathcal{K}\phi$ can be written as a sum of terms such as

$$\phi_j * W_\ell \tilde{K}_{ij}, \quad \text{with } \ell = 1, 2, 3, 4.$$

On the other hand,

$$\langle \delta_c u, \phi_j * W_\ell \tilde{K}_{ij} \rangle_{L^2(\mathbb{H}^2)} = \langle g, \phi_j * W_\ell \tilde{K}_{ij} \rangle_{L^2(\mathbb{H}^2)} = \langle g * {}^\vee(W_\ell \tilde{K}_{ij}), \phi_j \rangle_{L^2(\mathbb{H}^2)}$$

Notice the $W_\ell \tilde{K}_{ij}$'s and hence the ${}^\vee(W_\ell \tilde{K}_{ij})$'s are kernels of type 1. Thus, by Theorem 6.10 in [FS82],

$$|\langle \delta_c u, \phi_j * W_\ell \tilde{K}_{ij} \rangle_{L^2(\mathbb{H}^2)}| \leq C \|g\|_{\mathcal{H}^1(\mathbb{H}^2)} \|\phi\|_{L^Q(\mathbb{H}^2, E_0^1)}.$$

Combining this estimate with the one in (29), we get eventually

$$|\langle u, \phi \rangle_{L^2(\mathbb{H}^2, E_0^1)}| \leq C (\|f\|_{L^1(\mathbb{H}^2, E_0^2)} + \|g\|_{\mathcal{H}^1(\mathbb{H}^2)}) \|\phi\|_{L^Q(\mathbb{H}^2, E_0^1)},$$

and hence

$$\|u\|_{L^{Q/(Q-1)}(\mathbb{H}^2, E_0^1)} \leq C(\|f\|_{L^1(\mathbb{H}^2, E_0^2)} + \|g\|_{\mathcal{H}^1(\mathbb{H}^2)}).$$

Thus, to achieve the proof in the case $h = 1$ we are left to prove the claim (26).

We prove first that

$$\int_{\mathbb{H}^n} F\psi dV = 0 \quad \text{for all } \psi \in \mathcal{D}(\mathbb{H}^n).$$

To this end, we apply Cartan's formula (14) with $\omega = \Pi_E f$ and $Z_0 = T$, $Z_1 = X_1, Z_2 = Y_1$. Since $\Pi_E f = f - d_0^{-1}d_1f$, and keeping in mind that $d\Pi_E f = 0$ (by (24)), we can write

$$\begin{aligned} (30) \\ 0 &= Z_0\langle f|Z_1 \wedge Z_2 \rangle - Z_1\langle f|Z_0 \wedge Z_2 \rangle + Z_2\langle f|Z_0 \wedge Z_1 \rangle \\ &- \left(Z_0\langle d_0^{-1}d_1f|Z_1 \wedge Z_2 \rangle - Z_1\langle d_0^{-1}d_1f|Z_0 \wedge Z_2 \rangle + Z_2\langle d_0^{-1}d_1f|Z_0 \wedge Z_1 \rangle \right) \\ &- \langle f|[Z_0, Z_1] \wedge Z_2 \rangle + \langle f|[Z_0, Z_2] \wedge Z_1 \rangle - \langle f|[Z_1, Z_2] \wedge Z_0 \rangle \\ &- \left(-\langle d_0^{-1}d_1f|[Z_0, Z_1] \wedge Z_2 \rangle + \langle d_0^{-1}d_1f|[Z_0, Z_2] \wedge Z_1 \rangle - \langle d_0^{-1}d_1f|[Z_1, Z_2] \wedge Z_0 \rangle \right) \\ &:= A_1 + A_2 + A_3 + A_4. \end{aligned}$$

By our choice of Z_i , trivially, $A_3 = A_4 = 0$, since each term of the sum vanishes. Indeed, we have $[T, X_1] = [T, Y_1] = 0$ and $[X_1, Y_1] \wedge T = T \wedge T = 0$. Moreover, in A_1 the second and the third term vanish since $T \wedge X_1$ and $T \wedge Y_1$ have weight 3 whereas f has weight 2. In A_2 the first term vanishes since $d_0^{-1}d_1f$ has weight 3 whereas $X_1 \wedge Y_1$ has weight 2. Then

$$0 = A_1 + A_2 = T\langle f|X_1 \wedge Y_1 \rangle + X_1\langle d_0^{-1}d_1f|T \wedge Y_1 \rangle - Y_1\langle d_0^{-1}d_1f|T \wedge X_1 \rangle.$$

Keeping in mind that

$$\begin{aligned} (31) \\ \xi_1^2 &= dx_1 \wedge dx_2, \quad \xi_2^2 = dx_1 \wedge dy_2, \quad \xi_3^2 = dx_2 \wedge dy_1, \quad \xi_4^2 = dy_1 \wedge dy_2, \\ \xi_5^2 &= \frac{1}{\sqrt{2}}(dx_1 \wedge dy_1 - dx_2 \wedge dy_2), \end{aligned}$$

(see Example 3.11) a straightforward computation gives:

$$A_1 = \frac{Tf_5}{\sqrt{2}}.$$

On the other hand, let us compute explicitly d_1f . We have:

$$\begin{aligned} d_1f &= (X_1f_3 - \frac{X_2f_5}{\sqrt{2}} + Y_1f_1)dx_1 \wedge dx_2 \wedge dy_1 \\ &- \left(\frac{X_1f_5}{\sqrt{2}} + X_2f_2 - Y_2f_1 \right)dx_1 \wedge dx_2 \wedge dy_2 \\ &+ (X_1f_4 - Y_1f_2 + \frac{Y_2f_5}{\sqrt{2}})dx_1 \wedge dy_1 \wedge dy_2 \\ &+ (X_2f_4 + Y_2f_3 + \frac{Y_1f_5}{\sqrt{2}})dx_2 \wedge dy_1 \wedge dy_2 \end{aligned}$$

Moreover,

$$\begin{aligned} d_0(dx_1 \wedge \theta) &= -dx_1 \wedge dx_2 \wedge dy_2, & d_0(dx_2 \wedge \theta) &= dx_1 \wedge dx_2 \wedge dy_1, \\ d_0(dy_1 \wedge \theta) &= dx_2 \wedge dy_1 \wedge dy_2, & d_0(dy_2 \wedge \theta) &= -dx_1 \wedge dy_1 \wedge dy_2, \end{aligned}$$

so that

$$\begin{aligned} dx_1 \wedge \theta &= -d_0^{-1}(dx_1 \wedge dx_2 \wedge dy_2), & dx_2 \wedge \theta &= d_0^{-1}(dx_1 \wedge dx_2 \wedge dy_1), \\ dy_1 \wedge \theta &= d_0^{-1}(dx_2 \wedge dy_1 \wedge dy_2), & dy_2 \wedge \theta &= -d_0^{-1}(dx_1 \wedge dy_1 \wedge dy_2). \end{aligned}$$

Hence

$$\begin{aligned} A_2 &= X_1 \langle d_0^{-1} d_1 f | T \wedge Y_1 \rangle - Y_1 \langle d_0^{-1} d_1 f | T \wedge X_1 \rangle \\ &= X_1 \langle (X_2 f_4 + Y_2 f_3 + \frac{Y_1 f_5}{\sqrt{2}}) dy_1 \wedge \theta | T \wedge Y_1 \rangle \\ &\quad - Y_1 \langle (\frac{X_1 f_5}{\sqrt{2}} + X_2 f_2 - Y_2 f_1) dx_1 \wedge \theta | T \wedge X_1 \rangle \\ &= -X_1 X_2 f_4 - X_1 Y_2 f_3 - \frac{X_1 Y_1 f_5}{\sqrt{2}} + \frac{Y_1 X_1 f_5}{\sqrt{2}} + Y_1 X_2 f_2 - Y_1 Y_2 f_1 \\ &= -X_1 X_2 f_4 - X_1 Y_2 f_3 - \frac{T f_5}{\sqrt{2}} + Y_1 X_2 f_2 - Y_1 Y_2 f_1 \end{aligned}$$

Therefore,

$$0 = A_1 + A_2 = -X_1 X_2 f_4 - X_1 Y_2 f_3 + Y_1 X_2 f_2 - Y_1 Y_2 f_1.$$

Hence the first identity in (26) is proved. The second identity in (26) can be proved analogously by choosing, in the Cartan's formula (14), $\omega = \Pi_E f$ and $Z_0 = T$, $Z_1 = X_1$, $Z_2 = X_2$.

Case $\mathbf{h} = 2$. If $u, \phi \in E_0^2$ are smooth compactly supported forms, then we can write

$$\begin{aligned} \langle u, \phi \rangle_{L^2(\mathbb{H}^2, E_0^2)} &= \langle u, \Delta_{\mathbb{H}, 2} \mathcal{K} \phi \rangle_{L^2(\mathbb{H}^2, E_0^2)} \\ (32) \quad &= \langle u, (\delta_c d_c + (d_c \delta_c)^2) \mathcal{K} \phi \rangle_{L^2(\mathbb{H}^2, E_0^2)}. \end{aligned}$$

Consider now the term

$$\langle u, \delta_c d_c \mathcal{K} \phi \rangle_{L^2(\mathbb{H}^2, E_0^2)} = \langle d_c u, d_c \mathcal{K} \phi \rangle_{L^2(\mathbb{H}^2, E_0^3)}.$$

Let us write $f := d_c u$. We can write $f = \sum_{\ell} f_{\ell} \xi_{\ell}^3$, where

$$\begin{aligned} \xi_1^3 &= dx_1 \wedge dx_2 \wedge \theta, & \xi_2^2 &= dx_1 \wedge dy_2 \wedge \theta, \\ (33) \quad \xi_3^2 &= dx_2 \wedge dy_1 \wedge \theta, & \xi_4^2 &= dy_1 \wedge dy_2 \wedge \theta, \\ \xi_5^2 &= \frac{1}{\sqrt{2}}(dx_1 \wedge dy_1 - dx_2 \wedge dy_2) \wedge \theta, \end{aligned}$$

As above, $0 = d_c f = \Pi_{E_0} d \Pi_E f$, and hence $d \Pi_E f = 0$. But, on 3-forms $\Pi_E f = f$, since f has weight 4, that is already the maximum weight among all (even not intrinsic) 3-forms in \mathbb{H}^2 , so that eventually $df = 0$. Again

$$\langle d_c u, d_c \mathcal{K} \phi \rangle_{L^2(\mathbb{H}^2, E_0^3)} = \sum_{\ell} \langle f_{\ell}, (d_c \mathcal{K} \phi)_{\ell} \rangle_{L^2(\mathbb{H}^2)}.$$

As above, we prove that for any $\ell = 1, 2, 3, 4, 5$, f_{ℓ} is one of the components of an horizontal vector field with vanishing horizontal divergence. However,

the subsequent estimates are different from the case $h = 1$ because of the different order of the operators involved.

Consider now the following horizontal fields in \mathbb{H}^2 :

- $F = (2f_3, -\sqrt{2}f_5, 2f_1, 0)$;
- $G = (-\sqrt{2}f_5, -2f_2, 0, 2f_1)$;
- $K = (0, 2f_4, \sqrt{2}f_5, 2f_3)$.

We claim that

$$(34) \quad \operatorname{div}_{\mathbb{H}} F = \operatorname{div}_{\mathbb{H}} G = \operatorname{div}_{\mathbb{H}} K = 0.$$

Notice each component of f appears at least once as a component of one of the horizontal vector fields F, G, K .

Suppose now for a while (34) holds, and let us achieve the estimate of $\langle f_\ell, (d_c \mathcal{K}\phi)_\ell \rangle_{L^2(\mathbb{H}^2)}$. Suppose for instance $f_\ell = f_1$. We define a new horizontal vector field Φ as

$$\Phi := (0, 0, (d_c \mathcal{K}\phi)_1, 0),$$

so that

$$\langle f_1, (d_c \mathcal{K}\phi)_1 \rangle_{L^2(\mathbb{H}^2)} = \langle F, \Phi \rangle_{L^2(\mathbb{H}^2, H\mathbb{H}^2)}.$$

By Theorem 2.3

$$|\langle f_1, (d_c \mathcal{K}\phi)_1 \rangle_{L^2(\mathbb{H}^2)}| \leq \|f\|_{L^1(\mathbb{H}^2, E_0^2)} \|\nabla_{\mathbb{H}} d_c \mathcal{K}\phi\|_{L^Q(\mathbb{H}^2, E_0^2)}$$

On the other hand, $\nabla_{\mathbb{H}} d_c \mathcal{K}\phi$ can be expressed as a sum of terms with components of the form

$$\phi_j * W^I \tilde{K}_{ij}, \quad \text{with } d(I) = 3,$$

since $d_c : E_0^2 \rightarrow E_0^3$ is an operator of order 2 in the horizontal derivatives. By Theorem 3.13, iv) and Proposition 2.2, ii) $W^I \tilde{K}_{ij}$ are kernels of type 1, so that, by [Fol75], Proposition 1.11 we have

$$|\langle f_1, (d_c \mathcal{K}\phi)_1 \rangle_{L^2(\mathbb{H}^2)}| \leq C \|f\|_{L^1(\mathbb{H}^2, E_0^3)} \|\phi\|_{L^{Q/2}(\mathbb{H}^2, E_0^2)}.$$

The same argument can be carried out for all the components of f , yielding

$$(35) \quad |\langle f, d_c \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^2, E_0^3)}| \leq C \|f\|_{L^1(\mathbb{H}^2, E_0^3)} \|\phi\|_{L^{Q/2}(\mathbb{H}^2, E_0^2)}.$$

Consider now the second term in (32). We have

$$\begin{aligned} \langle u, (d_c \delta_c)^2 \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^2, E_0^2)} &= \langle d_c \delta_c u, d_c \delta_c \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^2, E_0^2)} \\ &= \langle d_c g, d_c \delta_c \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^2, E_0^2)}. \end{aligned}$$

We notice now that $d_c g$ is a d_c -closed form in E_0^2 , and then we can repeat the arguments leading to (27) for f in the case $h = 1$, obtaining

$$(36) \quad |\langle d_c g, d_c \delta_c \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^2, E_0^2)}| \leq \|d_c g\|_{L^1(\mathbb{H}^2, E_0^2)} \|\nabla_{\mathbb{H}} d_c \delta_c \mathcal{K}\phi\|_{L^Q(\mathbb{H}^2, E_0^2)}$$

As above, $\nabla_{\mathbb{H}} d_c \delta_c \mathcal{K}\phi$ can be expressed as a sum of terms with components of the form

$$\phi_j * W^I \tilde{K}_{ij}, \quad \text{with } d(I) = 3,$$

since $\delta_c : E_0^2 \rightarrow E_0^1$ is an operator of order 1 in the horizontal derivatives, as well as $d_c : E_0^1 \rightarrow E_0^2$. By Theorem 3.13, iv) and Proposition 2.2, ii) $W^I \tilde{K}_{ij}$ are kernels of type 1, so that, by [Fol75], Proposition 1.11 we have

$$|\langle d_c g, d_c \delta_c \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^2, E_0^2)}| \leq C \|d_c g\|_{L^1(\mathbb{H}^2, E_0^2)} \|\phi\|_{L^{Q/2}(\mathbb{H}^2, E_0^2)}.$$

Combining this estimate with the one in (35), we get eventually

$$|\langle u, \phi \rangle_{L^2(\mathbb{H}^2, E_0^2)}| \leq C(\|f\|_{L^1(\mathbb{H}^2, E_0^3)} + \|d_c g\|_{L^1(\mathbb{H}^2, E_0^2)}) \|\phi\|_{L^{Q/2}(\mathbb{H}^2, E_0^2)},$$

and hence

$$\|u\|_{L^{Q/(Q-2)}(\mathbb{H}^2, E_0^2)} \leq C(\|f\|_{L^1(\mathbb{H}^2, E_0^3)} + \|d_c g\|_{L^1(\mathbb{H}^2, E_0^2)}).$$

Thus, to achieve the proof in the case $h = 2$ we have to prove the claim (34).

To prove that $\operatorname{div}_{\mathbb{H}} F = 0$, we apply now Cartan's formula (14) with $\omega = f$ and $Z_0 = X_1, Z_1 = X_2, Z_2 = Y_1, Z_3 = T$.

Keeping in mind the commutation rules, we have

$$\begin{aligned} 0 &= Z_0 \langle f | Z_1 \wedge Z_2 \wedge Z_3 \rangle - Z_1 \langle f | Z_0 \wedge Z_2 \wedge Z_3 \rangle \\ &\quad + Z_2 \langle f | Z_0 \wedge Z_1 \wedge Z_3 \rangle - Z_3 \langle f | Z_0 \wedge Z_1 \wedge Z_2 \rangle \\ &\quad - \langle f | [Z_0, Z_1] \wedge Z_2 \wedge Z_3 \rangle + \langle f | [Z_0, Z_2] \wedge Z_1 \wedge Z_3 \rangle \\ &\quad - \langle f | [Z_0, Z_3] \wedge Z_1 \wedge Z_2 \rangle - \langle f | [Z_1, Z_2] \wedge Z_0 \wedge Z_3 \rangle \\ &\quad + \langle f | [Z_1, Z_3] \wedge Z_0 \wedge Z_2 \rangle - \langle f | [Z_2, Z_3] \wedge Z_0 \wedge Z_1 \rangle \\ (37) \quad &= Z_0 \langle f | Z_1 \wedge Z_2 \wedge Z_3 \rangle - Z_1 \langle f | Z_0 \wedge Z_2 \wedge Z_3 \rangle \\ &\quad + Z_2 \langle f | Z_0 \wedge Z_1 \wedge Z_3 \rangle - Z_3 \langle f | Z_0 \wedge Z_1 \wedge Z_2 \rangle \\ &\quad + \langle f | T \wedge X_2 \wedge T \rangle \\ &= X_1 \langle f | X_2 \wedge Y_1 \wedge T \rangle - X_2 \langle f | X_1 \wedge Y_1 \wedge T \rangle \\ &\quad + Y_1 \langle f | X_1 \wedge X_2 \wedge T \rangle - T \langle f | X_1 \wedge X_2 \wedge Y_1 \rangle \end{aligned}$$

By (33), identity (37) becomes

$$0 = X_1 f_3 - \frac{1}{\sqrt{2}} X_2 f_5 + Y_1 f_1, \quad \text{i.e.} \quad \operatorname{div}_{\mathbb{H}} F = 0.$$

This proves the first identity in (34). To prove the remaining two identities in (34), we apply again Cartan's formula as above with $Z_0 = X_1, Z_1 = X_2, Z_2 = Y_2, Z_3 = T$ and $Z_0 = Y_1, Z_1 = X_2, Z_2 = Y_2, Z_3 = T$, respectively. This achieves the proof of the theorem. \square

We conclude this section with some remarks about the sharpness of previous results.

Let us consider, for instance, the following estimates in \mathbb{H}^1 stated in Theorem 4.1: there exists a constant $C > 0$ so that

$$\|u\|_{L^{Q/(Q-2)}(\mathbb{H}^1, E_0^1)} \leq C(\|f\|_{L^1(\mathbb{H}^1, E_0^2)} + \|d_c g\|_{\mathcal{H}^1(\mathbb{H}^1)}) \quad \text{if } h = 1;$$

$$\|u\|_{L^{Q/(Q-2)}(\mathbb{H}^1, E_0^2)} \leq C(\|d_c f\|_{\mathcal{H}^1(\mathbb{H}^1, E_0^3)} + \|g\|_{L^1(\mathbb{H}^1, E_0^1)}) \quad \text{if } h = 2;$$

The presence of the terms $d_c g$ and $d_c f$ might seem somehow artificial, but is due to the fact that, on 1-forms, d_c has order 2, whereas δ_c has order 1 (dually, on 2-forms, d_c has order 1, whereas δ_c has order 2). By the way, also the norm in $L^{Q/(Q-2)}$ in the left hand side is due to the presence of a second order operator in the right hand side. We notice also that if we consider, for instance, co-closed 1-forms (i.e. we assume $g = 0$), then a straightforward homogeneity argument shows that the exponent $Q/(Q-2)$ is sharp. On the other hand, if $f = 0$ or $g = 0$, then the statement can be sharpened. More

precisely, as shown in [BF13], Theorem 5.1 and Theorem 5.2 for the case \mathbb{H}^1 and \mathbb{H}^2 respectively, the following results hold:

Theorem 4.3. *Let $u \in \mathcal{D}(\mathbb{H}^1, E_0^h)$ solve the system*

$$\begin{cases} d_c u = f \\ \delta_c u = g. \end{cases}$$

If $h = 1$ and $f = 0$, then

$$\|u\|_{L^{Q/(Q-1)}(\mathbb{H}^1, E_0^1)} \leq C \|g\|_{\mathcal{H}^1(\mathbb{H}^1)}.$$

If $h = 2$ and $g = 0$, then

$$\|u\|_{L^{Q/(Q-1)}(\mathbb{H}^1, E_0^2)} \leq C \|f\|_{\mathcal{H}^1(\mathbb{H}^1, E_0^3)}.$$

Theorem 4.4. *Let $u \in \mathcal{D}(\mathbb{H}^2, E_0^h)$ solve the system*

$$\begin{cases} d_c u = f \\ \delta_c u = g. \end{cases}$$

If $h = 2$ and $f = 0$, then

$$\|u\|_{L^{Q/(Q-1)}(\mathbb{H}^2, E_0^2)} \leq C \|g\|_{L^1(\mathbb{H}^2, E_0^1)}.$$

If $h = 3$ and $g = 0$, then

$$\|u\|_{L^{Q/(Q-1)}(\mathbb{H}^2, E_0^3)} \leq C \|f\|_{L^1(\mathbb{H}^2, E_0^4)}.$$

The results presented in this note are obtained in a joint work with Bruno Franchi, [BF13].

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