

EXISTENCE RESULT FOR THE CR-YAMABE EQUATION UN RISULTATO DI ESISTENZA PER L'EQUAZIONE DI YAMABE CR

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ABSTRACT. In this note we will prove that the CR-Yamabe equation has infinitely many changing-sign solutions. The problem is variational but the associated functional does not satisfy the Palais-Smale compactness condition; by mean of a suitable group action we will define a subspace on which we can apply the minimax argument of Ambrosetti-Rabinowitz. The result solves a question left open from the classification results of positive solutions by Jerison-Lee in the '80s.

SUNTO. In questo seminario proveremo l'esistenza di (infinite) soluzioni a segno non costante per l'equazione di Yamabe CR. Il problema è variazionale, ma il funzionale associato non soddisfa le condizioni di compattezza di Palais-Smale; mediante una opportuna azione di gruppo si costruirà un sottospazio sul quale sarà comunque possibile applicare un argomento di minimax di tipo Ambrosetti-Rabinowitz. Il risultato risolve una questione rimasta aperta dopo la classificazione delle soluzioni positive fatta da Jerison-Lee negli anni '80.

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1. INTRODUCTION

The results in this note are parts of a joint paper written with Ali Maalaoui [12].

Here we will prove that the Yamabe equation on the standard CR-sphere $(\mathbb{S}^{2n+1}, \theta_0)$ has infinitely many changing-sign solutions. However, for the sake of simplicity we will start our presentation on the Heisenberg group \mathbb{H}^n , then we will move to the sphere by using the Cayley transform.

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Hence let us consider the following CR-Yamabe equation on the whole Heisenberg group

$$(1) \quad -\Delta_{\mathbb{H}} u = |u|^{\frac{4}{q-2}} u, \quad u \in S_0^1(\mathbb{H}^n)$$

Here $\Delta_{\mathbb{H}}$ denotes the sub-Laplacian of the group, $q = 2n + 2$ is the homogenous dimension of \mathbb{H}^n , and $S_0^1(\mathbb{H}^n)$ is the Folland-Stein Sobolev type space on \mathbb{H}^n (see [9]).

We recall that for the positive solutions of (1), Jerison and Lee in [11] gave a complete classification. The problem is variational but, as in the Riemannian case, the functional associated with the equation (1) fails to satisfy the Palais-Smale compactness condition. For the classical Yamabe equation on \mathbb{R}^n , the first result in this direction was proved by Ding in [8]: following the analysis by Ambrosetti and Rabinowitz [1], he found a suitable subspace X of the space of the variations for the related functional, on which he performed the minimax argument. The same argument was then used by Saintier in [14] for the Yamabe equation on \mathbb{R}^n involving the bi-Laplacian operator.

Later on, many authors proved the existence of changing-sign solutions using other kinds of variational methods (see [2], [3] and the references therein). Finally in a couple of recent works [6], [7], M. del Pino, M. Musso, F. Pacard and A. Pistoia found changing-sign solutions, different from those of Ding, by using a superposition of positive and negative *bubbles* arranged on some special sets.

We are going to use the approach of Ding. Therefore, by mean of the Cayley transform we will set the problem on the sphere \mathbb{S}^{2n+1} and with the help of the group of the isometries generated by the Reeb vector field of the standard contact form on \mathbb{S}^{2n+1} , we will be able to exhibit a suitable closed subspace on which we can apply the minimax argument for the restriction of the functional associated with the equation (1).

2. PRELIMINARIES

Let $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R} \simeq \mathbb{R}^{2n+1}$ be the Heisenberg group. If we denote by $\xi = (z, t) = (x + iy, t) \simeq (x, y, t) \in (\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R})$ then the group law is given by

$$\xi_0 \cdot \xi = (x + x_0, y + y_0, t + t_0 + 2(x \cdot y_0 - x_0 \cdot y)), \quad \forall \xi, \xi_0 \in \mathbb{H}^n$$

where \cdot denotes the inner product in \mathbb{R}^n . The left translations are defined by

$$\tau_{\xi_0}(\xi) := \xi_0 \cdot \xi$$

Finally the dilations of the group are

$$\delta_\lambda : \mathbb{H}^n \rightarrow \mathbb{H}^n, \quad \delta_\lambda(\xi) = (\lambda x, \lambda y, \lambda^2 t)$$

for any $\lambda > 0$. Moreover we will denote by $q = 2n + 2$ the homogeneous dimension of the group. The canonical left-invariant vector fields are

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, \dots, n$$

The horizontal (or intrinsic) gradient of the group is

$$D_{\mathbb{H}} = (X_1, \dots, X_n, Y_1, \dots, Y_n)$$

The Kohn Laplacian (or sub-Laplacian) on \mathbb{H}^n is the following second order operator invariant with respect to the left translations and homogeneous of degree two with respect to the dilations:

$$\Delta_{\mathbb{H}} = \sum_{j=1}^n X_j^2 + Y_j^2$$

Let now

$$q^* = \frac{2q}{q-2}$$

then the following Sobolev-type inequality holds

$$\|\varphi\|_{q^*}^2 = \left(\int_{\mathbb{H}^n} |\varphi|^{q^*} \right)^{\frac{2}{q^*}} \leq C \int_{\mathbb{H}^n} |D_{\mathbb{H}}\varphi|^2 = C \|D_{\mathbb{H}}\varphi\|_2^2, \quad \forall \varphi \in C_0^\infty(\mathbb{H}^n)$$

with C a positive constant. For every domain $\Omega \subseteq \mathbb{H}^n$, the Folland-Stein Sobolev space $S_0^1(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|\varphi\| = \|D_{\mathbb{H}}\varphi\|_2$$

The exponent q^* is called critical since the embedding

$$S_0^1(\Omega) \hookrightarrow L^{q^*}(\Omega)$$

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 is continuous but not compact for every domain $\Omega \subseteq \mathbb{H}^n$.

Let us now consider the following Yamabe type equation on the Heisenberg group \mathbb{H}^n

$$(2) \quad -\Delta_{\mathbb{H}} u = |u|^{q^*-2} u, \quad u \in S_0^1(\mathbb{H}^n)$$

The idea is to consider the problem, after the Cayley transform, on the sphere \mathbb{S}^{2n+1} and there we will be able to find a suitable closed subspace $X \subseteq S^1(\mathbb{S}^{2n+1})$, compactly embedded in $L^{q^*}(\mathbb{S}^{2n+1})$, on which we can apply the minimax argument for the functional associated with the equation.

We recall that a solution of the problem (2) on \mathbb{H}^n can be found as critical point of the following functional

$$J : S_0^1(\mathbb{H}^n) \rightarrow \mathbb{R}, \quad J(u) = \frac{1}{2} \int_{\mathbb{H}^n} |D_{\mathbb{H}} u|^2 - \frac{1}{q^*} \int_{\mathbb{H}^n} |u|^{q^*}$$

Moreover any variational solution is actually a classical solution [9].

We will prove the following

Theorem 2.1. *There exists a sequence of solutions $\{u_k\}$ of (2), with*

$$\int_{\mathbb{H}^n} |D_{\mathbb{H}} u_k|^2 \longrightarrow \infty, \quad \text{as } k \rightarrow \infty$$

Theorem (2.1) will imply then that equation (2) has infinitely many changing-sign solutions: in fact by a classification result by Jerison and Lee [11] all the positive solutions of the equation (2) are in the form

$$u = \omega_{\lambda, \xi} = \lambda^{\frac{2-q}{2}} \omega \circ \delta_{\frac{1}{\lambda}} \circ \tau_{\xi^{-1}}$$

for some $\lambda > 0$ and $\xi \in \mathbb{H}^n$, where

$$\omega(x, y, t) = \frac{c_0}{((1 + |x|^2 + |y|^2)^2 + t^2)^{\frac{q-2}{4}}}$$

with c_0 a positive constant; in particular all the solutions $\omega_{\lambda, \xi}$ have the same energy.

3. PROOF OF THEOREM (2.1)

Let us consider the sphere $\mathbb{S}^{2n+1} \subseteq \mathbb{C}^{n+1}$ defined by

$$\mathbb{S}^{2n+1} = \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} \text{ s.t. } |z_1|^2 + \dots + |z_{n+1}|^2 = 1\}$$

As in the Riemannian case, we will use an analogous of the stereographic projection. The Cayley transform is the CR-diffeomorphism between the sphere minus a point and the Heisenberg group

$$F : \mathbb{S}^{2n+1} \setminus \{0, \dots, 0, -1\} \rightarrow \mathbb{H}^n$$

$$F(z_1, \dots, z_{n+1}) = \left(\frac{z_1}{1 + z_{n+1}}, \dots, \frac{z_n}{1 + z_{n+1}}, \operatorname{Re}\left(i \frac{1 - z_{n+1}}{1 + z_{n+1}}\right) \right)$$

Denoting by θ_0 the standard contact form on \mathbb{S}^{2n+1} and by Δ_{θ_0} the related sub-Laplacian, a direct computation show that equation (2) becomes

$$(3) \quad -\Delta_{\theta_0} v + c_n v = |v|^{\frac{4}{q-2}} v, \quad v \in S^1(\mathbb{S}^{2n+1})$$

with c_n a suitable positive dimensional constant related to the (constant) Webster curvature of the sphere (see [10] for a full detailed exposition); in particular by setting $u \circ F = v \varphi$ (where φ is the function that gives the conformal factor in the change of the contact form) we have that every solution u of (2) corresponds to a solution v of (3) and it holds

$$\int_{\mathbb{H}^n} |D_{\mathbb{H}} u|^2 = \int_{\mathbb{S}^{2n+1}} |v|^{q^*}$$

At this point we can consider the variational problem on the sphere

$$I : S^1(\mathbb{S}^{2n+1}) \rightarrow \mathbb{R}, \quad I(v) = \frac{1}{2} \int_{\mathbb{S}^{2n+1}} (|D_{\theta_0} v|^2 + c_n v^2) - \frac{1}{q^*} \int_{\mathbb{S}^{2n+1}} |v|^{q^*}$$

Now we notice that I meets the structural conditions of the Mountain-Pass Lemma, but since the embedding

$$S^1(\mathbb{S}^{2n+1}) \hookrightarrow L^{q^*}(\mathbb{S}^{2n+1})$$

is not compact, the functional I does not satisfy the Palais-Smale condition. The following lemma by Ambrosetti and Rabinowitz gives a condition on some particular subspaces of the space of variations on which it is allowed to perform the minimax argument; we will omit the proof (see Theorems 3.13 and 3.14 in [1])

Lemma 3.1. *Let X be a closed subspace of $S^1(\mathbb{S}^{2n+1})$. Suppose that the embedding $X \hookrightarrow L^{q^*}(\mathbb{S}^{2n+1})$ is compact. Then $I|_X$, the restriction of I on X , satisfies the Palais-Smale condition. Furthermore, if X is infinite-dimensional, then $I|_X$ has a sequence of critical points $\{v_k\}$ in X , such that*

$$\int_{\mathbb{S}^{2n+1}} |v_k|^{q^*} \longrightarrow \infty, \quad \text{as } k \rightarrow \infty$$

Now let us suppose that we can find a closed and compact group G such that the functional I is invariant under the action of G , namely:

$$I(v) = I(v \circ g), \quad \forall g \in G$$

Let us set

$$X_G = \{v \in S^1(\mathbb{S}^{2n+1}) \text{ s.t. } v = v \circ g, \forall g \in G\}$$

Then if X_G satisfies the condition of Lemma (3.1), by the Principle of Symmetric Criticality [13], any critical point of the restriction $I|_{X_G}$ is also a critical point of I on the whole space of variations. We are going to prove that such a X_G exists. First we observe the following fact:

Lemma 3.2. *The Reeb vector field T related to the standard contact form θ_0 on \mathbb{S}^{2n+1} is a Killing vector field.*

Proof. The proof is straightforward. The vector field T is Killing if

$$(4) \quad g_0(\nabla_V T, W) + g_0(V, \nabla_W T) = 0, \quad \forall V, W \in T(\mathbb{S}^{2n+1})$$

where g_0 is the metric induced by \mathbb{C}^{n+1} and ∇ is the Levi-Civita connection (we will call g_0 and ∇ also the standard metric and the related Levi-Civita connection in \mathbb{C}^{n+1}). Since we are on the sphere we can consider an orthonormal basis on $T(\mathbb{S}^{2n+1})$ of eigenvectors of the Weingarten operator A (we recall that for every $V \in T(\mathbb{S}^{2n+1})$, then $A(V) := -\nabla_V N$) with eigenvalues all equal to 1:

$$E = \{X_1, \dots, X_n, Y_1, \dots, Y_n, T\}, \quad Y_j = JX_j, \quad j = 1, \dots, n$$

where J is the standard complex structure on \mathbb{C}^{n+1} . Moreover $T = JN$, with N the inward unit normal to \mathbb{S}^{2n+1} . We have that

$$g_0(\cdot, \cdot) = g_0(J\cdot, J\cdot), \quad J\nabla_{(\cdot)}\cdot = \nabla_{(\cdot)}J\cdot$$

We also recall that T is a geodesic vector field, namely

$$g_0(\nabla_T T, V) = 0, \quad \forall V \in T(\mathbb{S}^{2n+1})$$

Since the formula (4) is linear in V and W , we can check it on the basis E . Let us first fix $W = T$. Then $g_0(\nabla_V T, T) = 0$ since T is unitary, and $g_0(V, \nabla_T T) = 0$ since T is geodesic. Now suppose $(V, W) = (X_j, X_k)$ or $(V, W) = (Y_j, Y_k)$. Then

$$g_0(\nabla_{X_j} T, X_k) = -g_0(\nabla_{X_j} N, Y_k) = 0$$

$$g_0(\nabla_{Y_j} T, Y_k) = g_0(\nabla_{Y_j} N, X_k) = 0$$

as X_j and Y_j are eigenvectors for A . The same holds for $(V, W) = (X_j, Y_k)$ with $j \neq k$. Finally let us consider $(V, W) = (X_j, Y_j)$. Then we have

$$g_0(\nabla_{X_j} T, Y_j) + g_0(X_j, \nabla_{Y_j} T) = g_0(\nabla_{X_j} N, X_j) - g_0(Y_j, \nabla_{Y_j} N) = 0$$

□

Then T generates a one-parameter family of diffeomorphisms, namely a closed group G , and since T is Killing these diffeomorphisms are isometries. Moreover on \mathbb{S}^{2n+1} we have the following crucial property of commutation

$$(5) \quad T \Delta_{\theta_0} = \Delta_{\theta_0} T$$

We notice that the previous formula (5) is true for every K -contact manifold (see Lemma 4.3. in [15]) and the sphere $(\mathbb{S}^{2n+1}, \theta_0)$ is a particular case of K -contact manifold according the definition in [4].

Hence we can conclude that the the functional I is invariant under the action of G . Moreover G is a closed subgroup of the compact Lie group $O(2n+2)$ of the isometries on \mathbb{S}^{2n+1} , in particular G is compact. Now we will call basic a function that belongs to the following set:

$$X_G := \{v \in S^1(\mathbb{S}^{2n+1}) \text{ s.t. } Tv = 0\}$$

In other words, a basic function is a function on \mathbb{S}^{2n+1} invariant for T . There is an explicit way to characterize the basic functions on \mathbb{S}^{2n+1} : since the orbits of T are circles, we can consider the following Hopf fibration:

$$\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n$$

where the fibers are exactly the orbits of T . Therefore a function v is basic if and only if can be written as

$$v = w \circ \pi$$

for some function $w : \mathbb{C}P^n \rightarrow \mathbb{R}$. Let us consider now the subspace of functions $X_G \subseteq S^1(\mathbb{S}^{2n+1})$. We have

Lemma 3.3. *The embedding*

$$X_G \hookrightarrow L^{q^*}(\mathbb{S}^{2n+1})$$

is compact.

Proof. The proof is based on the following simple observation. First note that the *Riemannian* critical exponent

$$p^* = \frac{2m}{m-2}, \quad m = 2n + 1$$

is always greater than the *sub-Riemannian* critical exponent q^* . In addition for every basic function, the horizontal gradient (and the sub-Laplacian as well) coincides with the usual Riemannian gradient (the Laplacian, respectively). Therefore, denoting by $H^1(\mathbb{S}^{2n+1})$ the usual Sobolev space on \mathbb{S}^{2n+1} , we have the following chain of embeddings

$$X_G \hookrightarrow H^1(\mathbb{S}^{2n+1}) \hookrightarrow L^{q^*}(\mathbb{S}^{2n+1})$$

and the second one is compact since q^* is subcritical for the (Riemannian) Sobolev embedding. \square

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