LIPSCHITZ ESTIMATES FOR CONVEX FUNCTIONS WITH RESPECT TO VECTOR FIELDS

VALENTINO MAGNANI

Abstract. We present Lipschitz continuity estimates for a class of convex functions with respect to Hörmander vector fields. These results have been recently obtained in collaboration with M. Scienza, [22].

Sunto. Presentiamo alcuni recenti risultati ottenuti in collaborazione con M. Scienza, [22], riguardanti la determinazione di stime quantitative sulla continuità lipschitziana di funzioni convesse rispetto a campi vettoriali di Hörmander.

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1. Introduction

This work refers to the "Bruno Pini Mathematical Analysis Seminar", that I delivered at the Mathematics Department of Bologna University on June 7, 2012. We precisely report on a recent work [22], joint with M. Scienza, about a quantitative version of the Lipschitz continuity of convex functions with respect to Hörmander vector fields.

Let $m \le n$ and let $\mathcal{X}$ be a set of smooth vector fields $X_1, \ldots, X_m$ on $\mathbb{R}^n$. We say that $\mathcal{X}$ satisfies the *Hörmander condition* if for every $x \in \mathbb{R}^n$ there exists a positive integer r' such that the following pointwise generating condition holds

$$\operatorname{span}\{X_{[S]}(x) : |S| \le r'\} = \mathbb{R}^n, \tag{1}$$

where $S = (s_1, \ldots, s_p) \in \{1, 2, \ldots, m\}^p$, $|S| = p$ and $X_{[S]} = \left[X_{s_1}, \left[\ldots, \left[X_{s_{p-1}}, X_{s_p}\right] \ldots\right]\right]$.

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Department of Mathematics, University of Bologna.

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The first important fact is that the everywhere validity of (1) implies the connectivity by *horizontal curves*, namely rectifiable curves whose velocity vectors are almost everywhere linear combinations of elements in $\mathcal{X}$. This introduces the so-called *Carnot-Carathéodory distance* associated to these vector fields, as we explain below.

The same set of vector fields $\mathcal{X}$ also yields the associated notion of convexity. Precisely, let Ω be an open set of $\mathbb{R}^n$ and let $u : \Omega \to \mathbb{R}$. We say that u is *$\mathcal{X}$-convex*, if the restriction $u \circ \gamma$ is a convex function of one variable, whenever $\gamma : I \to \Omega$ satisfies $\dot{\gamma} = \sum_{i=1}^m \alpha_i \, X_i \circ \gamma$ on the open interval I and α_i are arbitrary real numbers.

Let us observe that in the case $m = n$ and $X_i = e_i$ for all $i = 1, \ldots, n$, where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{R}^n$, then $\mathcal{X}$-convexity coincides with classical convexity for linear spaces and the metric structure associated to these vector fields is the Euclidean one, according to the notion of Carnot-Carathéodory distance.

Let us point out that $\mathcal{X}$-convexity is only partially related to the metric structure induced by the vector fields. Since $\mathcal{X}$ generates a sub-Riemannian structure, one may wonder whether $\mathcal{X}$-convexity only depends on the associated tangent distribution and on the metric on the distribution, as it happens for the sub-Riemannian distance. However, it is not difficult to realize that there is no such dependence, not even in the Riemannian setting. In fact, one can easily find a set $\mathcal{X}_0 = \{Y_1, \ldots, Y_n\}$ of vector fields that are everywhere orthonormal in $\mathbb{R}^n$ with respect to the Euclidean scalar product and observe that the associated family of $\mathcal{X}_0$-convex functions is different from the family of classically convex functions, although both $\mathcal{X}_0$ and $\{e_1, \ldots, e_n\}$ yield the same metric structure. This suggests that $\mathcal{X}$-convexity should not be considered indeed as a typical sub-Riemannian notion, but it should be interpreted as a notion assigned by a given family of curves.

According to Proposition 3.9 of [21], $\mathcal{X}$-convexity coincides with the *weakly H-convexity* introduced in [5], when the vector fields of $\mathcal{X}$ along with their iterated commutators are left invariant with respect to a Lie group operation and span a stratified Lie algebra. Notice that weakly H-convexity is also called *h-convexity*. The study of this notion appeared in connection with the development of a fully nonlinear theory of subelliptic equations in stratified groups, see [5], [20], [14], [15], [13], [28], [17], but this list is surely not exhaustive. The notion of $\mathcal{X}$-convexity for a set of vector fields has been introduced in [2] to study comparison principles for Monge-Ampere type equations with respect to Hörmander vector fields.

Now, we introduce the metric structure induced by $\mathcal{X}$. For all $x, y \in \mathbb{R}^n$, we define

$$d(x,y) = \inf\{t > 0 : \text{ there exists } \gamma \in \Gamma_{x,y}(t)\}\,, \tag{2}$$

where $\Gamma_{x,y}(t)$ is the family of all Lipschitz curves $\gamma : [0,t] \to \mathbb{R}^n$ such that $\gamma(0) = x$, $\gamma(t) = y$ and for a.e. $s \in [0,t]$ we have

$$\dot\gamma(s) = \sum_{j=1}^{m} a_j(s) X_j(\gamma(s)) \tag{3}$$

and $\max_{1\le j\le m} |a_j(s)| \le 1$. This distance along with its properties can be found in [25]. If in this definition we replace the norm $\max_{1\le j\le m} |a_j(s)|$ with $(\sum_{1\le j\le m} a_j(s)^2)^{1/2}$, then we get the Fefferman and Phong distance, [7]. We say that d is the Carnot-Carathéodory distance with respect to $\mathcal{X}$. Metric balls are denoted as follows

$$B_{x,r} = \{z \in \mathbb{R}^n : d(z,x) < r\} \quad \text{and} \quad D_{x,r} = \{z \in \mathbb{R}^n : d(z,x) \le r\}$$

for every $r > 0$ and $x \in \mathbb{R}^n$. Next, we introduce a special distance associated to $\mathcal{X}$.

Let $\Gamma^c_{x,y}(t)$ be the family of all Lipschitz curves $\gamma : [0,t] \to \mathbb{R}^n$ with $\gamma(0) = x$, $\gamma(t) = y$, such that (3) holds for a.e. $s \in [0,t]$ and $(a_1, \dots, a_m) \in \{\pm e_1, \dots, \pm e_m\}$, where the curve $(a_1, \dots, a_m)$ is piecewise constant on $[0,t]$ and $(e_1, \dots, e_m)$ is the canonical basis of $\mathbb{R}^m$. For all $x, y \in \mathbb{R}^n$ we define $\rho(x,y)$ to be the infimum among all $t > 0$ such that $\Gamma^c_{x,y}(t) \ne \emptyset$. The distance ρ was introduced by Franchi and Lanconelli, [8], [18], [9]. As pointed out by D. Morbidelli, the distances d and ρ are bi-Lipschitz equivalent on compact sets of $\mathbb{R}^n$. This fact is a consequence of the techniques of [25] and it can be also found as a consequence of Theorem 3.1 of [23]. It plays an important role in the proof of the local Lipschitz continuity of $\mathcal{X}$-convex functions.

Convexity with respect to vector fields can be also characterized by a second order condition. When $\mathcal{X}$ generates a stratified Lie group structure, this approach has been introduced in [20] and [17]. In these works the class of *v-convex functions* is introduced as the family of upper semicontinuous functions $u : \Omega \to \mathbb{R}$, defined on an open set $\Omega \subset \mathbb{R}^n$, such that $\nabla^2_{\mathcal{X}} u \ge 0$ *in the viscosity sense*, where the entries of this *horizontal*

Hessian in the case u is smooth are exactly the symmetrized second order derivatives $(X_iX_ju + X_jX_iu)/2$ for all $i, j = 1, \dots, m$.

In this setting, the set $\mathcal{X}$ is constituted by a basis of the first layer of the stratified Lie algebra. On the other hand, the notion of v-convexity makes perfectly sense for every family of vector fields, as it was observed in [20]. If $\mathcal{X}$ yields a stratified group structure on $\mathbb{R}^n$, then $\mathcal{X}$-convex functions that are locally bounded from above coincide with v-convex functions. This result has been first proved in Heisenberg groups without assuming the boundedness from above, [1], and also in general stratified groups [17], [21], [26], [30].

The main result of [2] is that *in the class of upper semicontinuous functions, v-convexity and $\mathcal{X}$-convexity coincide*, for an arbitrary set of C^2 vector fields $\mathcal{X}$. It is worth to mention that this characterization for C^2 functions can be easily established as in the case of groups, see Proposition 5.1 of [21]. In fact, taking $X = \sum_{j=1}^m \alpha_j X_j$ and $\gamma : I \longrightarrow \Omega$ such that $\dot\gamma = \sum_{j=1}^m \alpha_j \, X_j \circ \gamma$, then $u \circ \gamma$ is convex if and only if

$$(u\circ\gamma)'' = \sum_{j,i=1}^m \alpha_i\alpha_j\,(X_iX_ju)\circ\gamma = \sum_{j,i=1}^m \alpha_i\alpha_j\,(\nabla_{\mathcal{X}}u)_{ij}\circ\gamma \tag{4}$$

that immediately implies the characterization, due to the arbitrary choice of all α_j's. Notice that this proof automatically extends to differentiable manifolds and might be a first step to extend the study of $\mathcal{X}$-convexity to this framework.

In the case of stratified groups, the quantitative estimates on the Lipschitz continuity are stated as follows:

$$\sup_{w\in B_{x,r}} |u(w)| \;\le\; C_0 \fint_{B_{x,2r}} |u(w)|\,dw \tag{5}$$

$$\operatorname*{ess\,sup}_{w\in B_{x,r}} |\nabla_H u(w)| \;\le\; \frac{C_0}{r} \fint_{B_{x,2r}} |u(w)|\,dw\,. \tag{6}$$

These estimates are proved in [5] for continuous h-convex functions and in [20] and [17] for v-convex functions. These two classes of functions indeed coincide with locally Lipschitz continuous h-convex functions. Notice that the constant C_0 is independent of $r > 0$ and $x \in \mathbb{R}^n$, since these estimates are invariant under rescaling by the intrinsic dilations of the group. The estimates (5) and (6) play a key role in the study of fine properties of h-convex functions. For instance, they are necessary to prove the a.e. second order differentiability of weakly H-convex functions, starting from the a.e. L^1 differentiability of second order, see for instance [6], [21]. This step has been used in the proof of the second order differentiability of h-convex functions in two step stratified groups, where the key problem is the monotonicity of certain integral operators, [6], [13], [14], [15], [28]. In this connection, we mention some recent results on monotonicity of new Hessian operators in Heisenberg groups, [29]. Another application of (5) and (6) is in the proof of the distributional characterization of h-convex functions, [3].

Our purpose is to present how the estimates (5) and (6) can be extended to $\mathcal{X}$-convex functions with respect to Hörmander vector fields, according to Theorem 1.1, below stated. The characterization of $\mathcal{X}$-convexity by v-convexity in stratified groups allows for different approaches to their Lipschitz continuity. In [20], see also [17], the above mentioned estimates have been established for v-convex functions by a purely PDE approach. One of the central points here is the notion of "subelliptic cone" that also requires comparison principle and Harnack inequality for ∞-harmonic functions in order to establish that v-convex functions are locally bounded. To reach (5), the authors observe that viscosity subsolutions of the sub-Laplacian are subharmonic, then show by standard arguments that bounded subharmonic functions are distributional subsolutions of the sub-Laplacian and finally apply the regularity results for nonnegative weak subsolutions, [9], [19], [16], [4]. The comparison principle for subelliptic cones leads the authors also to (6).

In [5], the approach to the proof of (5) and (6) is more geometric. The estimate (6) is obtained directly from (5), using the convexity inequality on the first order expansion. The proof of (5) uses Jensen's inequality and suitable averages along horizontal planes. In particular, the integration of the first order expansion inequality, that involves the horizontal gradient, has a key cancellation of the linear part, since the domain of integration is symmetric. In the case of groups, the important aspect is that the composition of different iterated horizontal flows yields an open set of the space, due to the Lie bracket generating condition. This allows for iterating the previously mentioned averaging procedure, then reaching the integral estimates on a bounded open set.

The local Lipschitz continuity of upper semicontinuous and locally bounded $\mathcal{X}$-convex functions has been proved in Theorem 6.1 of [2], where the set $\mathcal{X}$ is only assumed to generate a finite Carnot-Carathéodory distance. On the other hand, it is not clear yet whether this generality still allows for having integral estimates similar to (5) and (6).

It is possible to show that (5) and (6) suitably extend to $\mathcal{X}$-convex functions, where $\mathcal{X}$ is a set of Hörmander vector fields. This is precisely stated in the next theorem.

Theorem 1.1 ([22]). *Let $\Omega \subset \mathbb{R}^n$ be open, let $K \subset \Omega$ be compact and let $\mathcal{X}$ be a set of Hörmander vector fields. Then there exist $C > 0$ and $R > 0$, depending on K, such that each $\mathcal{X}$-convex function $u : \Omega \to \mathbb{R}$, that is locally bounded from above, for every $x \in K$ satisfies the following estimates*

$$\sup_{B_{x,r}} |u| \leq C \fint_{B_{x,2r}} |u(w)|\, dw\,, \tag{7}$$

$$|u(y) - u(z)| \leq C\, \frac{d(y,z)}{r} \fint_{B_{x,2r}} |u(w)|\, dw\,, \tag{8}$$

for every $0 < r < R$ and every $y, z \in B_{x,r}$.

Contrary to the case of stratified groups, the constant $C > 0$ in the previous theorem cannot be chosen independently of K, since general Carnot-Carathéodory spaces need not have either a group operation or dilations and their Hausdorff dimension can change in different parts of the space.

The approach of [22] to prove (7) and (8) differs from both the geometric approach of [5] and the PDE approach of [20] and [17]. It partly relies on a PDE approach for local upper estimates and partly on a geometric approach to turn these estimates into local Lipschitz estimates. The next section is devoted to the presentation of this method, along with some other new results from [22].

The estimates (7) and (8) are also related to some results for k-convex functions with respect to Hörmander vector fields, [28]. A *smooth k-convex function* has the property that all j-th elementary symmetric functions of the horizontal Hessian $\nabla^2_{\mathcal{X}} u$ are nonnegative for all $j = 1, \ldots, k$ and $k \leq m$. Nonsmooth k-convex functions with respect to $\mathcal{X}$ have been introduced by N. S. Trudinger in [28], taking L^1_{loc}-limits of smooth k-convex functions. In the sub-Riemannian setting, this approach was first considered by C. Gutierrez and A. Montanari, taking locally uniform limits of smooth k-convex functions, [14], [15].

In [28], among other results, it is also proved in particular that all locally summable k-convex functions with respect to Hörmander vector fields are α-Hölder continuous when $k < m$ and $\mathrm{div} X_j = 0$ for all $X_j \in \mathcal{X}$. The last condition can be rephrased as follows $X_j^* = -X_j$. We also have an interesting formula for the Hölder exponent

$$\alpha = \frac{k(Q+m-2)-m(Q-1)}{k(m-1)} < 1\,, \tag{9}$$

where Q is the doubling dimension. The α-Hölder continuity of [28] has $\alpha < 1$ also in the case $k = m$, although formula (9) yields $\alpha = 1$ for $k = m$. Since the second order characterization of smooth $\mathcal{X}$-convex functions says that $\mathcal{X}$-convex functions coincide with smooth m-convex functions, as a consequence of Theorem 1.1, all nonsmooth m-convex functions with respect to Hörmander vector fields are indeed locally Lipschitz continuous. This shows that formula (9) also holds for $k = m$.

2. Method and other results

This section is mainly concerned with the main ideas to establish Theorem 1.1. Here an important step is the fact that local boundedness from above of $\mathcal{X}$-convex functions implies local Lipschitz continuity.

Theorem 2.1 ([22])**.** *Let $\mathbb{R}^n$ be equipped with Hörmander vector fields $\mathcal{X}$ and let $\Omega \subset \mathbb{R}^n$ be an open set. It follows that for every compact set $K \subset \Omega$ there exists $C > 0$ such that for every $\mathcal{X}$-convex function $u : \Omega \to \mathbb{R}$ that is locally bounded from above and $0 < r < \mathrm{dist}_d(K, \Omega^c)$ for all $x, y \in K$ we have*

$$|u(x) - u(y)| \leq \frac{C}{r}\, d(x,y) \sup_{K_r} |u|\,, \tag{10}$$

where $K_r = \{z \in \mathbb{R}^n : \mathrm{dist}_d(K, z) \leq r\} \subset \Omega$

This result precisely extends Theorem 3.18 of [21] to the general setting of Hörmander vector fields. It amounts to a regularity theorem, since we start from a function that is only assumed to be locally bounded from above and convex along a given family of curves and then establish its Lipschitz continuity, without any measurability assumption. Let us mention that in the case $\mathcal{X}$ generates a stratified Lie group strucure on $\mathbb{R}^n$, then measurable h-convex functions are locally Lipschitz continuous, [27]. This result is actually missing in the setting of Hörmander vector fields. On the other hand, even in higher step stratified groups it is not known yet whether h-convex functions are locally Lipschitz continuous, when no additional condition is assumed.

The proof of Theorem 2.1 follows the scheme used in [21], where the generating property of Proposition 3.12 of [21] is replaced by the properties of the so-called *approximate exponential* introduced by D. Morbidelli in [23]. By this tool, it is still possible to cover the Carnot-Carathéodory ball by suitable iterated compositions of flows of the vector fields of $\mathcal{X}$ and this can be performed in a quantitative way, as for stratified groups. This allows us to extend the Lipschitz estimates along horizontal curves, arising from the one dimensional convexity, to Lipschitz estimates on some bounded open set.

The local Lipschitz continuity of u implies in particular that it belongs to the anisotropic Sobolev space $W^{1,2}_{\mathcal{X},loc}(\Omega)$, since it belongs indeed to $W^{1,\infty}_{\mathcal{X},loc}(\Omega)$, see [10], [12]. We have defined $W^{1,p}_{\mathcal{X}}(\Omega)$, with $1 \leq p \leq \infty$, as follows

$$W^{1,p}_{\mathcal{X}}(\Omega) = \{f \in L^p(\Omega),\ X_j f \in L^p(\Omega),\ j = 1, \dots, m\},$$

where $C^\infty_c(\Omega)$ denotes the class of smooth functions with compact support and $X_j u$ is the *distributional derivative* of $u \in L^1_{loc}(\Omega)$, namely

$$\langle X_i u, \phi \rangle = \int_\Omega u\ X_i^* \phi\ dx, \quad \phi \in C^\infty_0(\Omega),$$

and X_i^* is the formal adjoint of X_i, namely, $X_i^* = -X_i - \mathrm{div} X_i$. The next step is to show that our $\mathcal{X}$-convex function, which is also $W^{1,2}_{\mathcal{X},loc}$, is locally a *weak subsolution* of a suitable sub-Laplacian. A function $u \in W^{1,2}_{\mathcal{X}}(\Omega)$ is an $\mathcal{L}$-*weak subsolution* of

$$\mathcal{L}u = \sum_{i=1}^m X_i^2 u = 0, \tag{11}$$

if for every nonnegative $\eta \in W^{1,2}_{\mathcal{X},0}(\Omega)$, we have

$$\sum_{i=1}^m \int_\Omega X_i u\ X_i^* \eta\ dx\ \geq 0.$$

The second derivatives along the vector fields X_i are nonnegative due to the $\mathcal{X}$-convexity assumption. However, to turn this information into its "distributional version", eventually involving the sub-Laplacian $\mathcal{L} = \sum_{j=1}^m X_j^2$, we need all the vector fields X_i to be nonvanishing. At this point we face an important obstacle, related to the choice of general Hörmander vector fields, since many of them may vanish at some fixed point. This clearly cannot occur in the case of stratified groups. We overcome this difficulty constructing new locally nonvanishing Hörmander vector fields and consider their associated sub-Laplacian. We have the following result.

Theorem 2.2 ([22]). *Let $x_0 \in \Omega$ and let $u : \Omega \to \mathbb{R}$ be a $\mathcal{X}$-convex function that is locally bounded from above. There exist $\delta_0 > 0$ and a new family of Hörmander vector fields $\mathcal{X}_1 = \{Y_1, \ldots, Y_m\}$, both depending on x_0, such that $B_{x_0,\delta_0} \subset \Omega$ and u is a weak subsolution of the equation*

$$\sum_{i=1}^m Y_i^2 v = 0 \quad on \quad B_{x_0,\delta_0}\,, \tag{12}$$

where the vector fields of $\mathcal{X}_1$ are linear combinations of elements in $\mathcal{X}$.

The previous theorem allows us to appeal to the now classical local upper integral estimates for weak subsolutions to the sub-Laplacian equation, [9], [19], [16], [4]. As a consequence, we obtain local estimates of the following form

$$\sup_{B_{y,\frac{r}{2}}} u \leq \kappa_x \fint_{B_{y,r}} |u(z)| dz \tag{13}$$

for all $0 < r < \sigma_x$ and y sufficiently close to x, where κ_x clearly depends on x. The lower estimate can be obtained using again the approximate exponential, that extends this estimate to a lower estimate on a small metric ball. This yields

$$2^{N_x}\, u(x) - (2^{N_x} - 1) \sup_{B_{x,\bar{N}\delta}} u \leq \inf_{B_{x,b\delta}} u\,, \tag{14}$$

where N_x depends on x and it satisfies the condition $1 \leq N_x \leq \bar{N}$ on some compact set. This leads us to the following theorem.

Theorem 2.3 ([22]). *Let $\Omega \subset \mathbb{R}^n$ be open, let $K \subset \Omega$ be compact and let $u : \Omega \to \mathbb{R}$ be a $\mathcal{X}$-convex function that is locally bounded from above. Then there exists $C_0 > 0$, $b_0 > 0$ and $N_0 > 1$, depending on K, such that for every $x \in K$ there holds*

$$\sup_{B_{x,r}} |u| \leq C_0 \fint_{B_{x,N_0 r}} |u(z)|\, dz$$

whenever $0 < r < b_0$ and $K_0 = \{z \in \mathbb{R}^n : \mathrm{dist}(K, z) \leq N_0\, b_0\} \subset \Omega$.

The local doubling property of the Lebesgue measure leads to the proof of (7). The estimate (8) essentially follows from Theorem 2.1 joined with (7). We can resume our approach saying that the PDE tools give the upper estimates and the geometric tools provided by the approximate exponentials yield the lower estimates, finally leading to quantitative estimates on the Lipschitz constant of the $\mathcal{X}$-convex function.

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Dipartimento di Matematica, Largo Bruno Pontecorvo 5, I-56127, Pisa

E-mail address: magnani@dm.unipi.it