PARTIAL RECONSTRUCTION OF THE SOURCE TERM IN A LINEAR PARABOLIC PROBLEM
RICOSTRUZIONE PARZIALE DEL TERMINE DI SORGENTE IN UN PROBLEMA PARABOLICO LINEARE

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Abstract. We consider, in some different situations, the problem of the reconstruction of the source term in a parabolic problem in a space-time domain $[0,T] \times I \times \overline{\Omega}$: this source term is assumed of the form $g(t,x)f(t,x,y)$ ($t \in [0,T]$, $x \in I$, $y \in \overline{\Omega}$), with f given and g to be determined. The novelty, with respect to the existing literature, lies in the fact that g depends on time and on some of the space variables. The supplementary information, allowing to determine g together with the solution of the problem u, is given by the knowledge, for every (t,x), of an integral of the form $\int_{\overline{\Omega}} u(t,x,y)d\mu(y)$, with μ complex Borel measure.

Sunto. Consideriamo, in varie situazioni, il problema delle ricostruzione del termine di sorgente in un'equazione parabolica in un dominio spazio-temporale $[0,T] \times I \times \overline{\Omega}$: questo termine di sorgente viene supposto della forma $g(t,x)f(t,x,y)$ ($t \in [0,T]$, $x \in I$, $y \in \overline{\Omega}$), con f dato e g da determinare. La novità, rispetto alla letteratura esistente, sta nel fatto che g dipende dal tempo e da una parte delle variabili spaziali. L'informazione supplementare, che permette di determinare g assieme alla soluzione del problema u, è data dalla conoscenza, per ogni (t,x), di un integrale della forma $\int_{\overline{\Omega}} u(t,x,y)d\mu(y)$, con μ misura di Borel complessa.

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The main aim of this seminar is the illustration of some results, concerning the following inverse problem: we consider a parabolic system which, in its most general case, has the

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form

(1)
$$\begin{cases} D_t u(t,x,y) = A(t,x,D_x)u(t,x,y) + B(t,x,y,D_y)u(t,x,y)u(t,x,y) + g(t,x)f(t,x,y), \\ \\ (t,x,y) \in [0,T] \times I \times \Omega, \\ \\ u(0,x,y) = u_0(x,y), \quad (x,y) \in I \times \Omega, \\ \\ + \text{ (possibly) boundary conditions in } \partial(I \times \Omega). \end{cases}$$

Here the space variables are divided into two families, which we have denoted with x and y, belonging, respectively, to the domains I and Ω. We consider the situation that the factor $f(t,x,y)$ in the source term is completely known, while the factor $g(t,x)$ is unknown, together with the solution u. As (roughly speaking), for large classes of g, system (1) has a unique solution, we assume that we have some further knowledge of u, in the following form: for every $(t,x) \in [0,T] \times I$, we know that

$$\int_{\overline{\Omega}} u(t,x,y)d\mu(y) = \phi(t,x), \tag{2}$$

where μ is a complex Borel function in the closure $\overline{\Omega}$ of Ω. So we want to study a system of the form (1)-(2).

A problem of this kind can be motivated by the partial knowledge, which may occur in practise, of what is usually assumed to be a datum of the system. Here we are considering the incomplete knowledge of the source term. Alternatively, we may also think of g as a sort of "regulator" that we have at our disposal, in order to obtain a certain "output" ϕ, from the solution u. For example, taking $\mu = \delta_{y^0}$, with y^0 fixed point in $\overline{\Omega}$, we are prescribing the value of $u(t,x,y^0)$, for every (t,x) in $[0,T] \times I$. We remark also that, if we want to be able to treat problems with quite general measures, which are not necessarily absolutely continuous with respect to the Lebesque measure, we are more or less obliged to work in spaces of continuous functions.

Observe that, apart the solution u, the function g we want to reconstruct depends on time and some of the spaces variables. Concerning parabolic problems of reconstruction of the source term, we quote some previous literature.

Most of the literature, treating problems of reconstruction of the source term, is concentrated on the case that g depends, either on the time t (only), or on all the space variables (only). See, for example, the well known monograph by Prilepko-Orlovsky-Vasin (see [6]) and also [3].

In our knowledge, the case that g depends, both, on time and part of the space variables has been considered, apart the work we are going to describe, only in [1]. More precisely, the author treated the case that the order of A and B is two, their coefficients are constant, $I = \mathbb{R}^+$ and $\Omega = \mathbb{R}^n$ and $\mu = \delta_0$.

Here we are going to consider three different situations, which are studied in detail in the papers [2], [4], [5]. [2] is dedicated to problems in $\mathbb{R}^m \times \mathbb{R}^n$, with operators of arbitrary even order. On the contrary, in [4] and [5] we consider only operators of second order and $m = 1$, but we have boundary conditions, respectively of Robin and Dirichlet type.

It is convenient to list here some notations we are going to employ. Let U be an open subset of $\mathbb{R}^n$, and let X be a Banach space. We shall indicate with $B(\overline{U}; X)$ the space of bounded functions from $\overline{U}$ to X, equipped with its standard norm. We shall indicate with $C(\overline{U}; X)$ the subspace of uniformly continuous and bounded functions. If $p \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, for some $\in \mathbb{N}$, we shall indicate with $C^p(\overline{U}; X)$ the class of elements of $C(\overline{U}; X)$, which are equipped with all the derivatives of order not exceeding p and belonging to $C(\overline{U}; X)$. If $\theta \in (0, 1)$, we shall indicate with $C^{p+\theta}(\overline{U}; X)$ the class of elements of $C^p(\overline{U}; X)$ whose derivatives of order p are Hölder continuous of exponent θ. In case $X = \mathbb{C}$, we shall usually omit it.

Each of these classes will be considered as equipped with a natural norm, making it a Banach space.

So we start by illustrating the main results of [2]. The problem we considered is the following

$$(3)\quad \begin{cases} D_t u(t,x,y) = A(t,x,D_x)u(t,x,y) + B(t,y,D_y)u(t,x,y) + g(t,x)f(t,x,y), \\ (t,x,y) \in [0,T] \times \mathbb{R}^m \times \mathbb{R}^n, \\ u(0,x,y) = u_0(x,y), \quad (x,y) \in \mathbb{R}^m \times \mathbb{R}^n, \\ \int_{\mathbb{R}^n} u(t,x,y)d\mu(y) = \phi(t,x), \\ (t,x) \in [0,T] \times \mathbb{R}^m, \end{cases}$$

with u, g unknown. The following result was proved in [2]:

Theorem 0.1. *Consider system (3), with the following assumptions:*

(a) $A(t,x,D_x)$ and $B(t,y,D_y)$ are strongly elliptic of order $2p$ in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, that is, for some $\nu \in \mathbb{R}^+$, $\forall (t,x,\xi) \in [0,T] \times \mathbb{R}^m \times \mathbb{R}^m$, $\forall (t,y,\eta) \in 0,T] \times \mathbb{R}^n \times \mathbb{R}^n$,

$$(-1)^p Re(\sum_{|\alpha|=2p} a_\alpha(t,x)\xi^\alpha) \leq -\nu|\xi|^{2p}, \quad (-1)^p Re(\sum_{|\beta|=2p} b_\beta(t,y)\eta^\beta) \leq -\nu|\eta|^{2p},$$

and coefficients in $C([0,T] \times \mathbb{R}^m) \cap B([0,T];\, C^\theta(\mathbb{R}^m))$ and in $C([0,T] \times \mathbb{R}^n) \cap B([0,T]; C^\theta (\mathbb{R}^n))$ respectively, for some $\theta \in (0,1)$;

(b) μ is a complex Borel measure in $\mathbb{R}^n$;

(c) $f \in C([0,T] \times \mathbb{R}^m \times \mathbb{R}^n) \cap B([0,T]; C^\theta(\mathbb{R}^m \times \mathbb{R}^n))$,

$$|\int_{\mathbb{R}^n} f(t,x,y)d\mu(y)| \geq \nu > 0, \forall (t,x) \in [0,T] \times \mathbb{R}^m;$$

(d) $u_0 \in C^{2p+\theta}(\mathbb{R}^m \times \mathbb{R}^n)$;

(e) $\phi \in C^1([0,T]; C(\mathbb{R}^m)) \cap B([0,T]; C^{2p+\theta}\, (\mathbb{R}^m))$, $D_t\phi \in C([0,T] \times \mathbb{R}^m) \cap B([0,T]; C^\theta(\mathbb{R}^m))$.

Then, (3) has a unique solution (u,g), such that

$$u \in C^1([0,T]; C(\mathbb{R}^{m+n})) \cap B([0,T]; C^{2p+\theta}(\mathbb{R}^{m+n})),$$

$$D_t u \in B([0,T]; C^\theta(\mathbb{R}^{m+n})),$$

$$g \in C([0,T] \times \mathbb{R}^m) \cap B([0,T]; C^\theta(\mathbb{R}^m)).$$

We pass to consider mixed Cauchy-boundary value problems for second order operators. We introduce some notations. From now on, Ω will indicate a bounded, open subset of $\mathbb{R}^n$, lying on one side of its boundary $\partial\Omega$, which is a smooth submanifold of $\mathbb{R}^n$. We set also

$$Q := [0,1] \times \overline{\Omega}. \tag{4}$$

We recall some classical notations: if $\theta \in (0,1)$, we set

$$C^{\theta/2,\theta}([0,T] \times Q) := C^{\theta/2}([0,T]; C(Q)) \cap B([0,T]; C^{\theta}(Q)). \tag{5}$$

$$C^{1+\frac{\theta}{2},2+\theta}([0,T] \times Q) := \{u : D_t^{\alpha} D_z^{\beta} u \in C^{\theta/2,\theta}([0,T] \times Q) \text{ if } 2\alpha + |\beta| \leq 2\}, \tag{6}$$

If A is a linear operator in the Banach space X, we shall indicate with $\rho(A)$ its resolvent set. In their most general formulation, the results contained in [4] and [5] concern a system of the form

$$\begin{cases} D_t u(t,x,y) = a(x) D_x^2 u(t,x,y) + A(x,y,D_y) u(t,x,y) + g(t,x) f(t,x,y), \\ \\ (t,x,y) \in [0,T] \times [0,1] \times \overline{\Omega}, \\ \\ u(0,x,y) = u_0(x,y), \quad (x,y) \in Q, \end{cases} \tag{7}$$

supplemented by boundary conditions in the form

$$\begin{cases} D_x u(t,x',y) + \alpha(x') u(t,x',y) = 0, \quad (t,x',y) \in [0,T] \times \{0,1\} \times \overline{\Omega}, \\ \\ B(y',D_y) u(t,x,y') = 0, \quad (t,x,y') \in [0,T] \times [0,1] \times \partial\Omega, \end{cases} \tag{8}$$

with B suitable first order operator, or in the form

$$u(t,x,y) = 0, \quad (t,x,y) \in [0,T] \times \partial Q. \tag{9}$$

As g is unknown together with u, we add to (7)-(8) and to (7)-(9) a condition of the form

$$\int_{\overline{\Omega}} u(t,x,y) d\mu(y) = \phi(t,x), \tag{10}$$

with μ complex Borel measure in $\overline{\Omega}$.

In the following, for simplicity, we are going to take $a(x) \equiv 1$, $A(x, y, D_y) = \Delta_y$ and, in (8), $\alpha(x') \equiv 0$, $B(y', D_y) = D_\nu$ (the outer normal derivative at $\partial\Omega$). So, we are going to consider the following simplified version of (7)-(8):

$$
\begin{cases}
D_t u(t,x,y) = D_x^2 u(t,x,y) + \Delta_y u(t,x,y) + g(t,x) f(t,x,y), \\
(t,x,y) \in [0,T] \times [0,1] \times \overline{\Omega}, \\
D_x u(t,x',y) = 0, \quad (t,x',y) \in [0,T] \times \{0,1\} \times \overline{\Omega}, \\
D_\nu u(t,x,y') = 0, \quad (t,x,y') \in [0,T] \times [0,1] \times \partial\Omega, \\
u(0,x,y) = u_0(x,y), \quad (x,y) \in Q, \\
\int_{\overline{\Omega}} u(t,x,y) d\mu(y) = \phi(t,x).
\end{cases} \tag{11}
$$

Then, the following result holds:

Theorem 0.2. *Consider system (11), with the following further conditions:*

(I) $f \in C^{\theta/2,\theta}([0,T] \times Q)$, $\int_{\overline{\Omega}} f(t,x,y) d\mu(y) \neq 0 \ \forall (t,x) \in [0,T] \times [0,1]$;

(II) $u_0 \in C^{2+\theta}(Q)$, $D_\nu u_0(x,y) \equiv 0$ $((x,y) \in \partial Q)$;

(III) $\phi \in C^{1+\frac{\theta}{2},2+\theta}([0,T] \times [0,1])$, $D_x\phi(t,0) = D_x\phi(t,1) = 0 \ \forall t \in [0,T]$, $\phi(0,x) = \int_{\overline{\Omega}} u_0(x,y) d\mu(y) \ \forall x \in [0,1]$.

Then (11) has a unique solution

$$(u,g) \in C^{1+\frac{\theta}{2},2+\theta}([0,T] \times Q) \times C^{\frac{\theta}{2},\theta}([0,T] \times Q).$$

Now we consider the mixed Cauchy-Dirichlet problem. The following result holds:

Theorem 0.3. *We consider the system*

$$
(12)\qquad \begin{cases}
D_t u(t,x,y) = D_x^2 u(t,x,y) + \Delta_y u(t,x,y) + g(t,x) f(t,x,y), \\[2ex]
(t,x,y) \in [0,T] \times [0,1] \times \overline{\Omega}, \\[2ex]
u(t,x',y) = 0, \quad (t,x',y) \in [0,T] \times \partial Q, \\[2ex]
u(0,x,y) = u_0(x,y), \quad (x,y) \in Q, \\[2ex]
\int_{\overline{\Omega}} u(t,x,y) d\mu(y) = \phi(t,x).
\end{cases}
$$

Assume that:

(I) $f \in C^{\theta/2,\theta}([0,T] \times Q)$, $\int_\Omega f(t,x,y) d\mu(y) \neq 0$ $\forall (t,x) \in [0,T] \times [0,1]$, $f(t,x,y) = 0$ *if* $(t,x,y) \in [0,T] \times (\{0,1\} \times \partial\Omega)$*;*

(II) $u_0 \in C^{2+\theta}(Q)$, $u_0(x,y) \equiv 0$ *(*$(x,y) \in \partial Q$*);*

(III) $\phi \in C^{1+\frac{\theta}{2},2+\theta}([0,T] \times [0,1])$, $\phi(t,0) = \phi(t,1) = 0, \quad \forall t \in [0,T]$,

$$\phi(0,x) = \int_{\overline{\Omega}} u_0(x,y) d\mu(y) \quad \forall x \in [0,1];$$

(IV) if we set $I(t,x) := \int_\Omega f(t,x,y) d\mu(y)$ *(*$\neq 0$ $\forall (t,x) \in [0,T] \times [0,1]$*)*, $\forall (x,y) \in \partial Q$,

$$D_x^2 u_0(x,y) + \Delta_y u_0(x,y) + \tfrac{f(0,x,y)}{I(0,x)} [D_t \phi(0,x) - D_x^2 \phi(0,x) - \textstyle\int_\Omega \Delta_y u_0(x,z) d\mu(z)] = 0.$$

Then, (7)-(9) has a unique solution (u,g) *in* $C^{1+\frac{\theta}{2},2+\theta}([0,T] \times Q) \times C^{\frac{\theta}{2},\theta}([0,T] \times Q)$*.*

Remark 0.1. In some cases, the integration in $\overline{\Omega}$ in the statement of Theorem 0.2 is replaced by the integration in Ω in Theorem 0.3. In this second case, as $u(t,x,y) = 0$ $\forall (t,x,y) \in [0,T] \times \partial Q$, in order that the last equation in (12) be effective, the support of μ cannot be contained in $\partial\Omega$.

Remark 0.2. It is not difficult to check that the assumptions of Theorems 0.2-0.3 are necessary, in order to get the desired conclusions, with the exception of (I).

Now we shall try to give some ideas, concerning the proofs of Theorems 0.2-0.3.

We shall indicate with B_j ($j \in \{0,1\}$), the boundary condition with order j. In case $j = 0$,

$$\int_\Omega u(t,x,y)d\mu(y) = \int_{\overline{\Omega}} u(t,x,y)d\mu(y) = \phi(t,x).$$

We set

$$\Omega_0 := \Omega, \quad \Omega_1 = \overline{\Omega}. \tag{13}$$

We apply the measure to the main equation:

$$\int_{\Omega_j} D_t u(t,x,y)d\mu(y) = \int_{\Omega_j} D_x^2 u(t,x,y)d\mu(y)$$

$$+ \int_{\Omega_j} \Delta_y u(t,x,y)d\mu(y) + g(t,x)I(t,x),$$

so that

$$D_t\phi(t,x) = D_x^2\phi(t,x) + \int_{\Omega_j} \Delta_y u(t,x,y)d\mu(y)$$

$$+g(t,x)I(t,x), \quad t \in [0,T], x \in [0,1].$$

As $I(t,x) \neq 0$, we obtain

$$g(t,x) = F_1(t,x) - \frac{1}{I(t,x)} \int_{\Omega_j} \Delta_y u(t,x,y)d\mu(y), \tag{14}$$

with

$$F_1(t,x) = \frac{D_t\phi(t,x) - D_x^2\phi(t,x)}{I(t,x)}. \tag{15}$$

Replacing g with the second term in (14) we obtain the system

$$\begin{cases} D_t u(t,x,y) = D_x^2 u(t,x,y) + \Delta_y u(t,x,y) - I(t,x)^{-1} f(t,x,y) \\ \times \int_{\Omega_j} \Delta_y u(t,x,z)d\mu(z) + F_1(t,x)f(t,x,y), \quad (t,x,y) \in [0,T] \times Q, \\ B_j u(t,x,y) = 0, (t,x,y) \in [0,T] \times \partial Q, \\ u(0,x,y) = u_0(x,y), \quad (x,y) \in Q. \end{cases} \tag{16}$$

It is not difficult to verify that, if u solves (8), setting

$$g(t,x) := F_1(t,x) - \frac{1}{I(t,x)} \int_{\Omega_j} \Delta_y u(t,x,y) d\mu(y),$$

(u, g) solves (11) or (12). So we are reduced to consider a problem of the form

$$(17) \qquad \begin{cases} D_t u(t,x,y) = D_x^2 u(t,x,y) + \Delta_y u(t,x,y) \\ +c(t,x,y) \int_{\overline{\Omega}} \Delta_y u(t,x,z) d\mu_j(z) + F(t,x,y), \\ \quad (t,x,y) \in [0,T] \times Q, \\ B_j u(t,x,y) = 0, (t,x,y) \in [0,T] \times \partial Q, \\ u(0,x,y) = u_0(x,y), \quad (x,y) \in Q. \end{cases}$$

with $d\mu_0 := \chi_\Omega d\mu$, $d\mu_1 = d\mu$. The following theorem holds:

Theorem 0.4. *Consider system (9), with μ_j complex Borel measure in $\overline{\Omega}$. Assume that $|\mu_j|(\partial\Omega) = 0$ in case $j = 0$ and $c \in C^{\theta/2,\theta}([0,T] \times Q)$. Then the following conditions are necessary and sufficient, in order that (17) have a unique solution u in $C^{1+\frac{\theta}{2},2+\theta}$ $([0,T] \times Q)$:*

(I) $F \in C^{\theta/2,\theta}([0,T] \times Q)$; if $j = 0$, $F(t,x,y) = 0$ if $(x,y) \in \{0,1\} \times \partial\Omega$;

(II) $u_0 \in C^{2+\theta}(Q)$, $B_j u_0 \equiv 0$ in ∂Q;

(III) if $j = 0$ and $(x,y) \in \partial Q$,

$$D_x^2 u_0(x,y) + \Delta_y u_0(x,y) + c(0,x,y) \int_{\overline{\Omega}} \Delta_y u_0(x,z) d\mu_0(z) + F(0,x,y) = 0.$$

Applying Theorem 0.4, it is easy to obtain Theorems 0.2-0.3.

Concerning Theorem 0.4, which is the core of the proof, some remarks are in order.

(a) In case $j = 0$, the condition $F(t,x,y) = 0$ if $(x,y) \in \{0,1\} \times \partial\Omega$ is a consequence of the fact that $\partial Q = \partial(]0,1[\times \Omega)$ is not regular.

(b) Condition (III) is a consequence of the fact that, as $u(t,x,y) \equiv 0$ in $[0,T] \times \partial Q$, $D_t u(t,x,y) \equiv 0$ in $[0,T] \times \partial Q$. In particular, $D_t u(0,x,y) = 0$ if $(x,y) \in \partial Q$.

(c) We discuss the assumption $|\mu_0|(\partial\Omega) = 0$. This is motivated by the fact that, given $c \in C(\overline{\Omega})$, the operator

$$v \to \int_{\overline{\Omega}} \Delta u(y) d\mu_j(y) c(\cdot)$$

should be a perturbation of Δ with homogeneous boundary conditions, in the sense that it should not change the parabolic character of th system. This is true in the case of first order boundary conditions, whatever the measure μ_1 is, in a sense we are going to make precise. We start by recalling the following theorem, due to H. B. Stewart (see [7]):

Theorem 0.5. *Let $j \in \{0,1\}$. Consider the following operator A in the Banach space $C(\overline{\Omega})$, where Ω is an open bounded subset of $\mathbb{R}^n$, with $\partial\Omega$ of class C^2:*

$$\begin{cases} D(A) = \{v \in \cap_{1 \le p < \infty} W^{2,p}(\Omega) : B_j v_{|\partial\Omega} = 0\}, \\ \\ Av = \Delta v. \end{cases}$$

Then $\mathbb{C} \setminus \{\lambda \in \mathbb{R} : \lambda \le 0\} \subseteq \rho(A)$, and, if $0 \le \alpha < \pi$, and $|Arg(\lambda)| \le \alpha$,

$$\|(\lambda - A)^{-1} f\|_{C(\overline{\Omega})} \le C(\alpha)|\lambda|^{-1}\|f\|_{C(\overline{\Omega})}.$$

As it often happens for parabolic systems, the first step of the proof of Theorem 0.4 is the study of the following problem:

$$\lambda v - Av - \Phi(Av)c = f, \tag{18}$$

with $\lambda \in \mathbb{C} \setminus \{0\}$, $|Arg(\lambda)| \le \alpha < \pi$, $c, f \in C(\overline{\Omega})$, $\Phi(v) := \int_{\overline{\Omega}} v(y) d\mu_j(y)$, equivalent to

$$v = (\lambda - A)^{-1} f + \Phi(Av)(\lambda - A)^{-1} c, \tag{19}$$

It should be

$$Av = A(\lambda - A)^{-1} f + \Phi(Av) A(\lambda - A)^{-1} c,$$

$$\Phi(Av) = \Phi[A(\lambda - A)^{-1} f] + \Phi(Av)\Phi[A(\lambda - A)^{-1} c].$$

If $|\Phi[A(\lambda - A)^{-1} c]|$ were less than one, we could deduce

$$\Phi(Av) = \frac{\Phi[A(\lambda - A)^{-1} f]}{1 - \Phi[A(\lambda - A)^{-1} c]},$$

and solve (18). Stewart's theorem implies that, $\forall f \in C(\overline{\Omega})$,

$$\|A(\lambda - A)^{-1} f\|_{C(\overline{\Omega})} \leq C_1(\alpha)\|f\|_{C(\overline{\Omega})}.$$

In case $f \in D(A)$, we have

$$\|A(\lambda - A)^{-1} f\|_{C(\overline{\Omega})} = \|(\lambda - A)^{-1} Af\|_{C(\overline{\Omega})} = O(|\lambda|^{-1}) \quad (|\lambda| \to \infty).$$

This implies that, if $c \in \overline{D(A)}$,

$$\|A(\lambda - A)^{-1} c\|_{C(\overline{\Omega})} = o(1) \quad (|\lambda| \to \infty), \tag{20}$$

so that $|\Phi[A(\lambda - A)^{-1} c]| = o(1)$ $(|\lambda| \to \infty)$. Therefore, (18) is solvable if $|\lambda|$ is sufficiently large, and we can also obtain the estimate

$$\|v\|_{C(\overline{\Omega})} \leq C(\alpha)|\lambda|^{-1}\|f\|_{C(\overline{\Omega})}.$$

If $j = 1$, $D(A)$ is dense in $C(\overline{\Omega})$. In case $j = 0$, $\overline{D(A)} = \{g \in C(\overline{\Omega}) : g_{|\partial\Omega} = 0\}$. If $c_{|\partial\Omega} \neq 0$, (20) does not hold. However, one can prove that, $\forall c \in C(\overline{\Omega})$, $[A(\lambda - A)^{-1} c](x) \to 0$ as $|\lambda| \to \infty$, uniformly in compact subsets of Ω. This implies that, if $|\mu_0|(\partial\Omega) = 0$,

$$\lim_{|\lambda| \to \infty, |Arg(\lambda)| \leq \alpha} \Phi(A(\lambda - A)^{-1} c) = 0.$$

We conclude with an example. We consider the system

$$\begin{cases} D_t u(t,x,y) = D_x^2 u(t,x,y) + \Delta_y u(t,x,y) + g(t,x) f(t,x,y), \quad (t,x,y) \in [0,T] \times Q, \\ u(t,x,y) = 0, \quad (t,x,y) \in [0,T] \times \partial Q, \\ u(0,x,y) = u_0(x,y), \quad (x,y) \in Q, \\ u(t,x,y_0) = 0, \quad t \in [0,T], x \in [0,1], \end{cases} \tag{21}$$

with $y_0 \in \Omega$ and u and g unknown. We assume that

(I) $f \in C^{\theta/2,\theta}([0,T] \times Q)$, $f(t,x,y_0) \neq 0$ $\forall (t,x) \in [0,T] \times [0,1]$, $f(t,x,y) = 0$ if $(t,x,y) \in [0,T] \times (\{0,1\} \times \partial\Omega)$;

(II) $u_0 \in C^{2+\theta}(Q)$, $u_0(x,y) = 0$ if $(x,y) \in \partial Q$, $u_0(x,y_0) = 0 \; \forall x \in [0,1]$;

(III)

$$\forall (x,y) \in \partial Q, \quad D_x^2 u_0(x,y) + \Delta_y u_0(x,y) = \tfrac{f(0,x,y)}{f(0,x,y_0)} \Delta_y u_0(x,y_0). \tag{22}$$

Then, (12) has a unique solution (u,g), such that

$$(u,g) \in C^{1+\frac{\theta}{2},2+\theta}([0,T] \times Q) \times C^{\frac{\theta}{2},\theta}([0,T] \times Q).$$

Taking into account the condition $u_0 \equiv 0$ in ∂Q, (22) is equivalent to the two conditions

$$\begin{cases} D_x^2 u_0(x,y) = 0, x \in \{0,1\}, y \in \overline{\Omega}, \\ \\ \Delta_y u_0(x,y) = \frac{f(0,x,y)}{f(0,x,y_0)} \Delta_y u_0(x,y_0), \\ x \in [0,1], y \in \partial\Omega. \end{cases}$$

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