DOUBLE BALL PROPERTY: AN OVERVIEW AND THE CASE OF STEP TWO CARNOT GROUPS

PROPRIETÀ DI PALLA DOPPIA IN GRUPPI DI CARNOT DI PASSO DUE

GIULIO TRALLI

ABSTRACT. We investigate the notion of the so-called *Double Ball Property*, which concerns the nonnegative sub-solutions of some differential operators. Thanks to the axiomatic approach developed in [6], this is an important tool in order to solve the Krylov-Safonov's Harnack inequality problem for this kind of operators. In particular, we are interested in linear second order horizontally-elliptic operators in non-divergence form and with measurable coefficients. In the setting of homogeneous Carnot groups, we would like to stress the relation between the Double Ball Property and a kind of solvability of the Dirichlet problem for the operator in the exterior of some homogeneous balls. We present a recent result obtained in [15], where the double ball property has been proved in a generic Carnot group of step two.

SUNTO. Si desidera studiare la nozione della cosiddetta *proprietà di Palla Doppia*, la quale si riferisce a sottosoluzioni non-negative di certi operatori differenziali. Grazie all'approccio assiomatico sviluppato in [6], questa proprietà diventa un tassello importante per la dimostrazione di una disuguaglianza di Harnack di tipo Krylov-Safonov per questo tipo di operatori. In particolare, si considerano operatori lineari del secondo ordine in forma di non-divergenza e coefficienti misurabili che siano orizzontalmente ellittici. Si desidera sottolineare come, nell'ambito dei gruppi di Carnot omogenei, questa proprietà sia legata alla risolubilità del problema di Dirichlet per l'operatore all'esterno di certe palle omogenee. Viene qui presentato un recente risultato ottenuto in [15], dove viene dimostrata la validità della proprietà di Palla Doppia in un generico gruppo di Carnot di passo due.

2010 MSC. Primary 35R03; Secondary 35J70.

KEYWORDS. degenerate-elliptic equations, invariant Harnack inequality, homogeneous Carnot groups.

Bruno Pini Mathematical Analysis Seminar, Vol. 1 (2012) pp. 33–47

Department of Mathematics, University of Bologna. ISSN 2240-2829.

1. Introduction

In the theory of fully nonlinear elliptic equations a crucial role is played by the Krylov-Safonov's Harnack inequality for nonnegative solutions to the linear equations in non-divergence form and measurable coefficients. To be precise, let us consider a second order linear operator in non-divergence form

$$\mathcal{L} = \sum_{i,j=1}^{n} a_{ij}(x)\partial^2_{x_i x_j}, \qquad \text{for } x \in \mathbb{R}^n. \tag{1}$$

Let us suppose that the matrix $A(x) = (a_{ij}(x))_{i,j=1}^n$ is symmetric and uniformly elliptic, i.e. there exist $0 < \lambda \le \Lambda$ such that

$$\lambda \left\|\xi\right\|^2 \le \langle A(x)\xi, \xi\rangle \le \Lambda \left\|\xi\right\|^2 \tag{2}$$

for every $\xi \in \mathbb{R}^n$. Under these hypotheses, Krylov and Safonov proved in [9] the following theorem.

Theorem 1.1. *There exists a positive constant C, depending just on the dimension n and on the ellipticity constants λ, Λ, such that, for every nonnegative solution u of $\mathcal{L}u = 0$ in Ω and for every cube Q with $3Q \subset \Omega$, we have*

$$\sup_Q u \le C \inf_Q u.$$

From the previous *invariant Harnack inequality*, one can deduce an Hölder regularity result for the solutions of $\mathcal{L}$: the Hölder constants will depend just on n, λ, Λ.
By the way, in several research areas such as Complex or CR Geometry, there are fully nonlinear equations which are characterized by an underlying sub-Riemannian structure and are not elliptic at any point, see e.g. [11],[14],[12],[13],[4],[5],[10]. The existence theory for viscosity solutions to such equations is well settled, mainly thanks to the papers [14],[12],[5]. On the contrary, the problem of the solutions regularity is still widely open. This is mainly due to the lack of pointwise estimates for solutions to linear sub-elliptic equations with rough coefficients. In this context, a long standing open problem is the invariant Harnack inequality for positive solutions to horizontally elliptic equations on Lie groups, in non-divergence form and rough coefficients.
To this aim, Di Fazio, Gutiérrez and Lanconelli in [6] developed an axiomatic procedure to

establish scale invariant Harnack inequality in very general settings like doubling Hölder quasi metric spaces. This approach allows to handle both divergence and non divergence linear equations. They proved that the *double-ball property* and the *ϵ-critical density* are sufficient conditions for the Harnack inequality to hold. What are these notions? These properties arose from the techniques developed in [2] and [3] for uniformly elliptic fully nonlinear equations and for the linearized Monge-Ampère equation (see also [7], Chapter 2, and the references therein). In [6] these notions have found a precise and abstract statement, for the purpose of being used in general settings. In [8] this approach has been in fact exploited by Gutiérrez and Tournier in the setting of the Heisenberg group $\mathbb{H}$: they proved, for second order linear operators wich are elliptic with respect to the fields generating $\mathbb{H}$, the double ball property and, under an extra-assumption on the eigenvalues of the coefficient matrix, the critical density.

In Section 2 we are going to show the outline of the powerful approach adopted in [6]. In Section 3 we investigate the double-ball property in the particular case of the homogeneous Carnot groups: following the argument in [15], we highlight as this property is related to the solvability of a kind of exterior Dirichlet problem for the operator. More precisely, it is a consequence of the existence of some suitable interior barrier functions of Bouligand-type. In Section 4 we give a sketch of the proof of the double-ball property for a generic step two Carnot group, which is the main result of [15].

2. An axiomatic approach

As already mentioned, in [6] Di Fazio, Gutiérrez and Lanconelli presented their axiomatic approach in the abstract setting of doubling quasi metric Hölder space. For the sake of clearness, we need some definitions.

Definition 2.1. *Let Y be a non empty set. We say that Y is a quasi metric space if there exists a function $d : Y \times Y \longrightarrow [0, +\infty)$ which is symmetric, strictly positive away from $\{(x, y) \in Y \times Y : x = y\}$ and such that, for some constant $K \geq 1$, we have*

$$d(x, y) \leq K(d(x, z) + d(z, y))$$

for all $x, y, z \in Y$. The d-ball with center $x_0 \in Y$ and radius $R > 0$ is given by

$$B_R(x_0) := \{y \in Y : d(x_0, y) < R\}.$$

Definition 2.2. *Let (Y, d) be a quasi metric space and μ be a positive measure on a σ-algebra of subsets of Y containing the d-balls. We say that μ satisfies the doubling property if there exists a positive constant C_d such that*

$$0 < \mu(B_{2R}(x_0)) \leq C_d \mu(B_R(x_0))$$

for all $x_0 \in Y$ and $R > 0$.

Definition 2.3. *Let (Y, d) be a quasi metric space. The quasi distance d is Hölder continuous if there exist positive constants β and $0 < \alpha \leq 1$ such that*

$$|d(x, y) - d(x, z)| \leq \beta d(y, z)^\alpha \left(d(x, y) + d(x, z)\right)^{1-\alpha}$$

for all $x, y, z \in Y$.

Therefore, by taking into account all the previous definitions, we can now fix a doubling quasi metric Hölder space (Y, d, μ). In such a space, let Ω be an open subset of Y. Following the notations in [6], we denote by $\mathcal{K}_\Omega$ a family of μ-measurable functions with domain contained in Ω. If $u \in \mathcal{K}_\Omega$ and its domain contains a set $A \subset \Omega$, we will write $u \in \mathcal{K}_\Omega(A)$.
We are ready to give the precise statements of the double-ball property and the ϵ-critical density.

Definition 2.4. ***(Double Ball Property)*** *We say that $\mathcal{K}_\Omega$ satisfies the double ball property if there exists a positive constant γ such that, for every $B_{3R}(x_0) \subset \Omega$ and every $u \in \mathcal{K}_\Omega(B_{3R}(x_0))$ with $\inf_{B_R(x_0)} u \geq 1$, we have*

$$\inf_{B_{2R}(x_0)} u \geq \gamma.$$

Definition 2.5. ***(ϵ-Critical Density)*** *Let $0 < \epsilon < 1$. We say that $\mathcal{K}_\Omega$ satisfies the ϵ-critical density property if there exists a positive constant c such that, for every $B_{2R}(x_0) \subset \Omega$ and for every $u \in \mathcal{K}_\Omega(B_{2R}(x_0))$ with*

$$\mu(\{x \in B_R(x_0) \, : \, u(x) \geq 1\}) \geq \epsilon \mu(B_R(x_0)),$$

we have

$$\inf_{B_{\frac{R}{2}}(x_0)} u \geq c.$$

Example 2.1. *In order to understand these abstract definitions, we go back to the case of linear uniformly elliptic second order operators. So, let us consider an operator $\mathcal{L}$ as in (1) with ellipticity constants λ and Λ. In* [7]*, we can find a proof of the double ball property (Theorem 2.1.2) and the ϵ-critical density property (Theorem 2.1.1) for this class of functions*

$$\mathcal{K}_\Omega := \{u \in C^2(V, \mathbb{R}) : V \subset \Omega, u \geq 0 \text{ and } \mathcal{L}u \leq 0 \text{ in } V\}. \tag{3}$$

In this case, the doubling quasi metric Hölder space is $\mathbb{R}^n$ with the classical euclidean distance and the Lebesgue measure. We stress that this $\mathcal{K}_\Omega$ is invariant under multiplication by positive constants. Moreover, the constants appearing in the Definitions 2.4 and 2.5 *depend just on n, λ, Λ.*

Before stating the result in [6], we need two more definitions.

Definition 2.6. *We say that (Y, d, μ) has the reverse doubling condition in Ω if there exists $0 < \delta < 1$ such that*

$$\mu(B_R(x_0)) \leq \delta \mu(B_{2R}(x_0))$$

for every $B_{2R}(x_0) \subset \Omega$.

Definition 2.7. *We say that (Y, d, μ) satisfies a log-ring condition if there exists a non-negative function $\omega(\epsilon)$, with $\omega(\epsilon) = o((log(\frac{1}{\epsilon}))^{-2})$ as $\epsilon \to 0^+$, such that*

$$\mu\left(B_R(x_0) \smallsetminus B_{(1-\epsilon)R}(x_0)\right) \leq \omega(\epsilon)\mu(B_R(x_0))$$

for every ball $B_R(x_0)$ and all ϵ sufficiently small.

Among the results proved by Di Fazio, Gutiérrez and Lanconelli in [6], the one we are interested in is the following theorem.

Theorem 2.1. *Let (Y, d, μ) be a doubling quasi metric Hölder space and let $\Omega \subset Y$ be an open set such that (Y, d, μ) has the reverse doubling condition in Ω. Let us suppose* *that* (Y, d, μ) *also satisfies a log-ring condition. Let us assume in addition that the set* $\mathcal{K}_\Omega$ *is closed under multiplications by positive constants and if* $u \in \mathcal{K}_\Omega(B_R(x_0))$ *satisfies* $u \le m$ *in* $B_R(x_0)$ *then* $m - u \in \mathcal{K}_\Omega(B_R(x_0))$*. Finally, we suppose that* $\mathcal{K}_\Omega$ *satisfies the double-ball property and the* ϵ*-critical density property for some* $0 < \epsilon < 1$*. Then, there exist* $C, \eta > 1$ *independent of* u, R *and* x_0 *such that, if* $u \in \mathcal{K}_\Omega(B_{2\eta R}(x_0))$ *is nonnegative and locally bounded, we have*

$$\sup_{B_R(x_0)} u \le C \inf_{B_R(x_0)} u.$$

Example 2.2. *Keeping in mind Example 2.1, we now put*

$$\mathcal{K}_\Omega := \{u \in C^2(V, \mathbb{R}) : V \subset \Omega, u \ge 0 \text{ and } \mathcal{L}u = 0 \text{ in } V\}.$$

This class is a subset of the one defined in (3), since we have to verify the condition

$$u \in \mathcal{K}_\Omega(B_R(x_0)), \quad u \le m \text{ in } B_R(x_0) \quad \Rightarrow \quad m - u \in \mathcal{K}_\Omega(B_R(x_0)).$$

Anyway, with this choice all the assumptions of the previous theorem are satisfied. Therefore, in the case of linear uniformly elliptic second order operators, Theorem 2.1 gives back the result by Krylov and Safonov of Theorem 1.1.

3. Homogeneous Carnot groups and interior barriers

Where is it possible to apply this axiomatic approach? Besides the double-ball property and the ϵ-critical density property, in the previous section we have seen five more definitions involved in Theorem 2.1 concerning just the structure of the setting we are working in. It has been stressed in [6] that a remarkable example where these stuctural assumptions are satisfied is the setting of homogeneous Lie groups.

As a matter of fact, let $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ be an homogeneous Lie group, where

$$\delta_\lambda(x_1, \ldots, x_N) = (\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_N} x_N)$$

with $1 \le \sigma_1 \le \ldots \le \sigma_N$. In $\mathbb{G}$ there exists an homogeneous symmetric norm, i.e. a continuous function $d : \mathbb{R}^N \longrightarrow \mathbb{R}$, δ_λ-homogeneous of degree one, smooth and strictly positive outside the origin and such that $d(x^{-1}) = d(x)$ (see [1], Example 5.1.1). If we denote $d(x,y) := d(y^{-1} \circ x)$ for $x, y \in \mathbb{R}^N$, we have, for some $K \geq 1$,

$$d(x,y) \leq K(d(x,z) + d(z,y))$$

for all $x, y, z \in \mathbb{R}^N$ (see [1], Proposition 5.1.6). Thus, $(\mathbb{R}^N, d)$ is a quasi-metric space. In [6] (Remark 2.5) it is also proved that d is Hölder continuous with respect to Definition 2.3 with $\alpha = 1$: in fact, there exist β such that

$$d(x,y) \leq d(x,z) + \beta d(y,z) \qquad \forall\, x, y, z \in \mathbb{R}^N.$$

Moreover, if we denote by $|\cdot|$ the Lebesgue measure in $\mathbb{R}^N$, we get by the δ_λ-homogeneity of the d-balls and Proposition 1.3.21 in [1] that

$$|B_R(x_0)| = R^Q\, |B_1(0)| =: c_Q R^Q,$$

where $Q = \sum_{i=1}^N \sigma_i$. Therefore, the doubling property and the reverse doubling hold true for $(\mathbb{R}^N, d, |\cdot|)$: those inequalities are actually equalities with constant respectively $C_d = 2^Q$ and $\delta = \frac{1}{2^Q}$. Finally, also the log-ring condition is satisfied since we have

$$\left|B_R(x_0) \smallsetminus B_{(1-\epsilon)R}(x_0)\right| = c_Q R^Q (1 - (1-\epsilon)^Q) \leq Q\epsilon\, |B_R(x_0)|\,.$$

Summing up, in this setting it is worthy of taking care of double ball property and ϵ-critical density.

Suppose in addition $\mathbb{G}$ is stratified, i.e. $\mathbb{G}$ is an homogeneous Carnot group. Let us say $\mathbb{G}$ has m generators and take m left-invariant vector fields $X_1, \ldots, X_m$, δ_λ-homogeneous of degree one, generating the Lie algebra. We want to consider the linear second order operator in non-divergence form

$$\mathcal{L}_A = \sum_{i,j=1}^m a_{ij}(x) X_i X_j \qquad \text{for } x \in \mathbb{R}^N. \tag{4}$$

The symmetric matrix $A(x) = (a_{ij}(x))_{i,j=1}^m$ is supposed to be uniformly elliptic: we recall it means that there exist $0 < \lambda \leq \Lambda$ such that, for every x, we have

$$\lambda \left\|\xi\right\|^2 \leq \langle A(x)\xi, \xi\rangle \leq \Lambda \left\|\xi\right\|^2$$

for every $\xi \in \mathbb{R}^n$. We are going to denote by $M_m(\lambda, \Lambda)$ the set of the $m \times m$ symmetric matrices satisfying these bounds. We will write simply $A \in M_m(\lambda, \Lambda)$ instead of writing

$A(x) \in M_m(\lambda, \Lambda)$ for every x. These operators are called horizontally elliptic.
Let us state again the double ball condition in this context. Keeping in mind Example 2.1, we put

$$\mathcal{K}_\Omega^A := \{u \in C^2(V, \mathbb{R}) : V \subset \Omega, u \geq 0 \text{ and } \mathcal{L}_A u \leq 0 \text{ in } V\}, \tag{5}$$

for any fixed open set $\Omega \subseteq \mathbb{R}^N$. Thus, the double ball property reads as follows.

Definition 3.1. ***(Double Ball Property)*** *We say that the double ball property holds true in $\mathbb{G}$ if, for every $0 < \lambda \leq \Lambda$, there exists a positive constant γ (depending just on λ, Λ, the vectors fields X_j's and $\mathbb{G}$) such that, for every $A \in M_m(\lambda, \Lambda)$ and for every C^2-function u in $\Omega \supseteq B_{3R}(x_0)$ with*

$$u \geq 0,\ \mathcal{L}_A u \leq 0 \quad \text{in } \Omega \qquad \text{and} \qquad u \geq 1 \quad \text{in } B_R(x_0),$$

we have

$$u \geq \gamma \qquad \text{in } B_{2R}(x_0).$$

Gutiérrez and Tournier considered in [8] the case of the Heinsenberg group $\mathbb{H} = \mathbb{H}^1$ with generators $X_1 = \partial_{x_1} - \frac{x_2}{2}\partial_{x_3}$ and $X_1 = \partial_{x_2} + \frac{x_1}{2}\partial_{x_3}$. They chose the homogeneous norm $d(x_1, x_2, x_3) = ((x_1^2 + x_2^2)^2 + \mu x_3^2)^{\frac{1}{4}}$ for some fixed constant μ. They worked in $\mathbb{R}^3$, but all the arguments and the results work in $\mathbb{R}^{2n+1}$ (i.e. in $\mathbb{H}^n$). In that context, they proved the double ball property as we have just defined. They proved also the ϵ-critical density (for the same family (5)) by assuming a bound for the ratio $\frac{\Lambda}{\lambda}$.
In [15] we have stressed how the double ball property is related to the solvability of a kind of exterior Dirichlet problem for the operator. The main tool is the existence of some interior barrier functions. The important feature of these barrier functions for $\mathcal{L}_A$ is that they are uniform for $A \in M_m(\lambda, \Lambda)$: they have to be independent of the coefficients of the matrix $A(x)$ and of their regularity. Let us give the definition.

Definition 3.2. *Let O be an open set of $\mathbb{R}^N$ with non-empty boundary. Fix $p \in \partial O$ and $0 < \lambda \leq \Lambda$. A function h is an interior $\mathcal{L}$-barrier function for O at p if*

- *h is a C^2 function defined on an open bounded neghborhood U of p,*
- *h and U depend just on O, p, Λ, λ (and on $\mathbb{G}, d$ and the X_j's),*

· *$\mathcal{L}_A h \le 0$ in U for any $A \in M_m(\lambda, \Lambda)$,*

· *$h(p) = 0$,*

· *$\{x \in U \ : \ h \le 0\} \setminus \{p\} \subseteq O$.*

In [15] we considered the case of step two Carnot groups and we proved that the existence, for any $\lambda \le \Lambda$, of an interior $\mathcal{L}$-barrier for $B_1(0)$ at every point of its boundary implies the double ball property. The generalization of this fact to every homogeneous Carnot group is straightforward. Here we give the details by arguing in the same way.

Lemma 3.1. *Let T be a compact subset of an open set $O \subset \mathbb{R}^n$. There exists $\nu_0 > 1$ such that*

$$\delta_\nu T \subset O$$

for all $\nu \in [1, \nu_0]$.

Proof. The sets T and $\mathbb{R}^N \setminus O$ are close and disjoint. Thus, their distance δ is a positive number. Since T is bounded, there exists $M > 0$ such that, if $x = (x_1, \dots, x_N) \in T$, we have $|x_j| \le M$. Therefore, for $x \in T$ and $\nu \ge 1$, we get

$$dist(\delta_\nu(x), T) \le \|\delta_\nu(x) - x\| \le M \sum_{j=1}^{N} (\nu^{\sigma_j} - 1).$$

It is easy to choose $\nu_0 > 1$ such that $\sup_{x \in T} dist(\delta_\nu(x), T) < \delta$ for all $\nu \in [1, \nu_0]$. ☐

We set

$$K_0^A = \{u \in C^2(B_{\frac{3}{2}}(0)) \, : \, u \ge 0 \text{ and } \mathcal{L}_A u \le 0 \text{ in } B_{\frac{3}{2}}(0), \; u \ge 1 \text{ in } B_1(0)\}.$$

The next lemma is an application of the weak maximum principle for the operator $\mathcal{L}_A$.

Lemma 3.2. *Suppose that, for every $0 < \lambda \le \Lambda$ and every $p \in \partial B_1(0)$, there exists an interior $\mathcal{L}$-barrier function for $B_1(0)$ at p. Then, there exists $\nu \in (1, \frac{3}{2})$ such that*

$$u \ge \frac{1}{2} \text{ in } B_\nu(0)$$

for any $u \in K_0^A$ and any $A \in M_m(\lambda, \Lambda)$.

Proof. Fix $\lambda \leq \Lambda$ and $A \in M_m(\lambda, \Lambda)$. Fix also $p \in \partial B_1(0)$ and consider the barrier function $h = h_p$ defined in $U = U_p$. If we set $V = \Big(U \cap B_{\frac{3}{2}}(0)\Big) \smallsetminus \overline{B_1(0)}$, we have that $h \geq 0$ and $\mathcal{L}_A h \leq 0$ in V. Let us now consider the boundary $\partial V = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \partial V \cap \partial B_1(0)$ and $\Gamma_2 = \partial V \smallsetminus \Gamma_1$. The number $m = \inf_{\Gamma_2} h$ is strictly positive because $\{x \in \partial V \,:\, h(x) = 0\} = \{p\}$. So, the function $w = 1 - \frac{1}{m}h$ is well defined. We have

$$\mathcal{L}_A w = -\frac{1}{m}\mathcal{L}_A h \geq 0 \text{ in } V, \quad w \leq 1 \text{ on } \Gamma_1 \text{ and } w \leq 0 \text{ on } \Gamma_2.$$

If $u \in K_0^A$, we get

$$\mathcal{L}_A u \leq \mathcal{L}_A w \text{ in } V, \quad u \geq w \text{ on } \partial V.$$

By the Weak Maximum Principle for $\mathcal{L}_A$, $u \geq w$ in V. Since $w(p) = 1$, there exists an open neighborhood W_p of p contained in $U \cap B_{\frac{3}{2}}(0)$ where $w \geq \frac{1}{2}$. The compact set $\partial B_1(0)$ is contained in the open set $O = \cup_{p \in \partial B_1(0)} W_p$. By the previous lemma, there exists $\nu > 1$ such that $(B_\nu(0) \smallsetminus B_1(0)) \subset O$: the constant ν depends on A just through the ellipticity contants λ, Λ (since it depends on the barriers). Therefore, we deduce

$$u \geq \frac{1}{2} \text{ on } B_\nu(0)$$

for all $u \in K_0^A$. □

Proposition 3.1. *Suppose there exists, for any $0 < \lambda \leq \Lambda$, an interior $\mathcal{L}$-barrier function for $B_1(0)$ at every point of $\partial B_1(0)$. Then, the double ball property holds true in $\mathbb{G}$.*

Proof. Fix $\lambda \leq \Lambda$ and $A \in M_m(\lambda, \Lambda)$. We are going to prove first the condition in Definition 3.1 by assuming $x_0 = 0$ and $R = 1$. If $u \in \mathcal{K}_\Omega^A(B_3(0))$ (in the sense of (5)) with $u \geq 1$ in $B_1(0)$, in particular $u \in K_0^A$. By the last lemma, we have $u \geq \frac{1}{2}$ in $B_\nu(0)$ for a fixed $1 < \nu < \frac{3}{2}$. Let us consider the function

$$v = 2u \circ \delta_\nu.$$

It is a non-negative function of class C^2 defined at least in $B_{\frac{3}{\nu}}(0) \supseteq B_{\frac{3}{2}}(0)$ (since $\nu < 2$). We have that $v \geq 1$ in $B_1(0)$. By denoting $\tilde{A}(x) = A(\delta_\nu(x))$, we get

$$\mathcal{L}_{\tilde{A}} v(x) = 2\nu^2 (\mathcal{L}_A u)(\delta_\nu(x)) \leq 0$$

because of the homogeneity of the vector fields. This means that $v \in K_0^{\tilde{A}}$. The matrix $\tilde{A}$ has the same ellipticity constants of A and ν depends just on these. Thus, $v \geq \frac{1}{2}$ in $B_\nu(0)$, that is $u \geq \frac{1}{4}$ in $B_{\nu^2}(0)$. If $\nu^2 \geq 2$, we have just proved the statement. If it is not, the argument can be reapplied. Since $\nu > 1$, there exists an integer n_0 such that $\nu^{n_0} \geq 2$. Therefore, we get

$$u \geq \frac{1}{2^{n_0}} =: \gamma \quad in \;\; B_2(0).$$

If x_0 and R are arbitrary, we can argue in a similar way. As a matter of fact, we consider the function

$$\tilde{u}(x) = u(x_0 \circ \delta_R(x))$$

for $u \in C^2(B_{3R}(x_0))$. The homogeneity and the left-invariance properties of horizontally elliptic operators imply that

$$\sum_{i,j} A_{i,j}(x_0 \circ \delta_R(x)) X_i X_j \tilde{u}(x) = R^2 (\mathcal{L}_A u)(x_0 \circ \delta_R(x)).$$

Therefore, if $u \in \mathcal{K}_\Omega^A(B_{3R}(x_0))$ with $u \geq 1$ in $B_R(x_0)$, we have $\tilde{u} \in \mathcal{K}_\Omega^{\tilde{A}}$ and $\tilde{u} \geq 1$ in $B_1(0)$, where $\tilde{A}(x) = A(x_0 \circ \delta_R(x))$. By what we have just proved, we get $\tilde{u} \geq \gamma$ in $B_2(0)$, i.e. $u \geq \gamma$ in $B_{2R}(x_0)$. □

Example 3.1. *Proposition 3.1 allows us to prove the double ball property for the euclidean case in a different way from the one in* [7] *(Theorem 2.1.2). As a matter of fact, if $\mathcal{L}$ is as in (1), we can build up interior $\mathcal{L}$-barrier functions for the euclidean ball $B_1^e(0) = B$ at the boundary points which are similar to the ones in the Hopf's lemma. Fix a point $p \in \partial B$ and take an euclidean ball $B_\delta^e(\xi_0)$ which is tangent to ∂B at p and strictly contained in B, that is $\overline{B_\delta^e}(\xi_0) \setminus \{p\} \subset B$ (let's say $\xi_0 = \frac{1}{2}p$ and $\delta = \frac{1}{2}$). For any $\lambda \leq \Lambda$, we can choose α big enough such that the function*

$$h(x) = e^{-\alpha\delta^2} - e^{-\alpha\|x-\xi_0\|^2}$$

is an $\mathcal{L}$-barrier for $B_1^e(0)$ at p.

4. Carnot groups of step two

We want to study the case of an N-dimensional Carnot group of step two with m generators. Up to a canonical isomorphism (see [1], Theorem 3.2.2), we can thus consider an homogeneous Carnot group $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ such that the composition law $\circ$ is defined by

$$(x,t) \circ (x_1,t_1) = \left(x + x_1, t + t_1 + \frac{1}{2} \langle Bx, x_1 \rangle \right) \tag{6}$$

for $(x,t),(x_1,t_1) \in \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^N$. Here we have denoted by $\langle Bx, x_1 \rangle$ the vector of $\mathbb{R}^n$ whose components are $\left\langle B^k x, x_1 \right\rangle$ (for $k = 1, \ldots, n$) and $B^1, \ldots, B^n$ are $m \times m$ linearly independent skew-symmetric matrices. The group of dilations is defined as

$$\delta_\lambda((x,t)) = (\lambda x, \lambda^2 t)$$

and the inverse of (x,t) is $(-x,-t)$. We can choose as homogeneous symmetric norm the function $d : \mathbb{R}^N \longrightarrow \mathbb{R}$ such that

$$d\left((x,t)\right) = \left(\|x\|^4 + \|t\|^2 \right)^{\frac{1}{4}};$$

from here on we denote by $\|\cdot\|$ both the euclidean norms in $\mathbb{R}^m$ and in $\mathbb{R}^n$. Hence, we have $B_R(x_0) = x_0 \circ B_R(0)$ where

$$B_R(0) = \{(x,t) \in \mathbb{R}^N \, : \, \|x\|^4 + \|t\|^2 < R^4\}.$$

Let us fix m vector fields generating the Lie algebra of $\mathbb{G}$, for example the ones of the Jacobian basis (this choice does not affect our problem, by changing it we would change at most λ and Λ): they are given by

$$X_i(x,t) = \partial_{x_i} + \frac{1}{2} \sum_{k=1}^{n} (B^k x)_i \partial_{t_k} \qquad \text{for } i = 1, \ldots, m.$$

For $A \in M_m(\lambda, \Lambda)$, we build up the operators as in (4), i.e.

$$\mathcal{L}_A = \sum_{i,j=1}^{m} a_{ij}(x,t) X_i X_j \qquad \text{for } (x,t) \in \mathbb{R}^N.$$

Remark 4.1. *The hypothesis of being Carnot is crucial. In fact, if we consider an homogeneous Lie group with a composition law as in* (6)*, it is a Carnot group if and only if the matrices B^k's are linear independent (see* [1]*, Section 3.2). Thus, if the group has not the Carnot property, up to a change of variable we can say that B^n is the null matrix. This would imply that every function u depending just on the variable t_n is a solution of $\mathcal{L}u = 0$. If this is the case, it is not difficult to falsify the double ball property.*

In order to apply Proposition 3.1, we have to prove that there exists an interior $\mathcal{L}$-barrier function for $B_1(0)$ at every point of $\partial B_1(0)$. The characteristic points of $\partial B_1(0)$ are the points where the horizontal gradient $\nabla_X = (X_1, \ldots, X_m)$ of the defining function of $B_1(0)$ vanishes. In [15] we showed that, at the non-characteristic points, the functions like the ones used in Example 3.1 for the euclidean case work as interior $\mathcal{L}$-barriers.

Remark 4.2. *If we denote with F the defining function of $B_1(0)$, i.e.*

$$F(x,t) = d^4((x,t)) = \|x\|^4 + \|t\|^2 - 1,$$

we have

$$\nabla_X F(x,t) = (X_1 F, \ldots, X_m F)(x,t) = 4\|x\|^2 x + \sum_{k=1}^{n} t_k B^k x.$$

Since the matrices B^k's are skewsymmetric, the vectors x and $B^k x$ are orthogonal for every $k = 1, \ldots, n$. So, we can state that

$$\nabla_X F(x,t) = 0 \quad \Leftrightarrow \quad x = 0.$$

Thus, the problem is to find, for any $\lambda \leq \Lambda$, an interior $\mathcal{L}$-barrier function at the points $(0, t_0)$ with $\|t_0\| = 1$.

Remark 4.3. *Suppose for a moment that $\mathbb{G}$ is an H-group in the sense of Métivier. This means that every non-vanishing linear combination of the matrices B^k's is non singular (see* [1]*, Proposition 3.7.4). Denoting $t' = t - \langle t, t_0\rangle t_0$, for β big enough and δ small enough the function*

$$h_M(x,t) = e^{-\beta} - e^{-\beta(\|x\|^4 + \|t'\|^2 + \langle t, t_0\rangle)},$$

defined in $\{(x,t) \in \mathbb{R}^n : \langle t, t_0\rangle > 0, \|t'\| < \delta\}$, is an interior $\mathcal{L}$-barrier for $B_1(0)$ at $(0, t_0)$. The Heisenberg group $\mathbb{H}^n$ belongs to the class of the H-groups in the sense of Métivier:

thus, by exploiting the functions h_M, we have a different proof for the result by Gutiérrez and Tournier ([8]*, Theorem 4.1) in* $\mathbb{H}^n$.

In the case of a generic step two Carnot group, the functions h_M do not always work as barriers. The difference is that, for any fixed unit vector $t_0 = (t_0^1, \dots, t_0^n)$, the matrix $\sum_{k=1}^n t_0^k B^k$ might have a non-trivial kernel. Hence, let us denote by P the orthogonal projector on $Range(\sum_{k=1}^n t_k^0 B^k) = Ker(\sum_{k=1}^n t_k^0 B^k)^\perp$ and with Q the orthogonal projector on $Ker(\sum_{k=1}^n t_k^0 B^k)$. In [15] we proved that, for any $\lambda \le \Lambda$, there are convenient choices of γ, β and δ such that the function

$$h(x,t) = e^{-\beta} - e^{-\beta\left(\|x\|^4 + (\|Qx\|^2 - \gamma\|Px\|^2)^2 + \|t'\|^2 + \langle t, t_0\rangle\right)},$$

defined in $\{(x,t) \,:\, \langle t, t_0\rangle > 0,\, \|t'\| < \delta\}$, is an interior $\mathcal{L}$-barrier function for $B_1(0)$ at $(0, t_0)$.

Summing up all these facts, the main result in [15] reads as follows.

Theorem 4.1. *The double ball property holds true in every Carnot group of step two.*

References

[1] A. Bonfiglioli, E. Lanconelli, F. Uguzzoni. Stratified Lie Groups and Potential Theory for their Sub-Laplacians, Springer Monographs in Mathematics, Springer, Berlin (2007).

[2] L.A. Caffarelli. *Interior a priori estimates for solutions of fully non-linear equations.* Ann. of Math. (2), **130** (1989) 189-213.

[3] L.A. Caffarelli, C.E. Gutiérrez. *Properties of the solutions of the linearized Monge-Ampère equation.* Amer. J. Math., **119** (1997) 423-465.

[4] G. Citti, E. Lanconelli, A. Montanari. *Smoothness of Lipschitz continuous graphs with non vanishing Levi curvature.* Acta Math., **188** (2002) 87-128.

[5] F. Da Lio, A. Montanari. *Existence and Uniqueness of Lipschitz Continuous Graphs with Prescribed Levi Curvature.* Ann. Inst. H. Poincaré Anal. Non Linéaire, **23** (2006) 1-28.

[6] G. Di Fazio, C.E. Gutiérrez, E. Lanconelli. *Covering theorems, inequalities on metric spaces and applications to pde's.* Math. Ann., **341** (2008) 255-291.

[7] C.E. Gutiérrez. The Monge-Ampère equation, Progress in Nonlinear Differential Equations and their Applications, Birkhäuser, Boston (2001).

[8] C.E. Gutiérrez, F. Tournier. *Harnack inequality for a degenerate elliptic equation.* Comm. Partial Differential Equations, **36** (2011) 2103-2116.

[9] N.V. Krylov, M.V. Safonov. *A property of the solutions of parabolic equations with measurable coefficients.* Izv. Akad. Nauk SSSR Ser. Mat., **44** (1980) 161-175.

[10] V. Martino, A. Montanari. *Integral formulas for a class of curvature PDE's and applications to isoperimetric inequalities and to symmetry problems.* Forum Math., **22** (2010) 255-267.

[11] A. Montanari, E. Lanconelli. *Pseudoconvex fully nonlinear partial differential operators. Strong comparison Theorems.* J. Differential Equations, **202** (2004) 306-331.

[12] Z. Slodkowski, G. Tomassini. *Weak solutions for the Levi equation and envelope of holomorphy.* J. Funct. Anal., **101** (1991) 392-407.

[13] Z. Slodkowski, G. Tomassini. *The Levi equation in higher dimension and relationship to the envelope of holomorphy.* Amer. J. Math., **116** (1994) 479-499.

[14] G. Tomassini. *Geometric properties of solutions of the Levi-equation.* Ann. Mat. Pura Appl. (4), **152** (1988) 331-344.

[15] G. Tralli. *Double Ball Property for non-divergence horizontally elliptic operators on step two Carnot groups.* Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl., to appear.

Dipartimento di Matematica, Università di Bologna, Piazza di Porta S.Donato 5, 40126 Bologna, Italy.

E-mail address: giulio.tralli2@unibo.it