REGULARITY ISSUES FOR LOCAL MINIMIZERS OF THE MUMFORD & SHAH ENERGY IN 2D
QUESTIONI DI REGOLARITÀ PER MINIMI LOCALI DEL FUNZIONALE DI MUMFORD & SHAH IN DIMENSIONE 2

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Abstract. We review some issues about the regularity theory of local minimizers of the Mumford & Shah energy in the 2-dimensional case. In particular, we stress upon some recent results obtained in collaboration with Camillo De Lellis (Universität Zürich). On one hand, we deal with basic regularity, more precisely we survey on an elementary proof of the equivalence between the weak and strong formulation of the problem established in [16]. On the other hand, we discuss fine regularity properties by outlining an higher integrability result for the density of the volume part proved in [17]. The latter, in turn, implies an estimate on the Hausdorff dimension of the singular set of minimizers according to the results in [2] (see also [18]).

Sunto. Verranno presentati alcuni aspetti della teoria di regolarità dei minimi locali del funzionale di Mumford & Shah in dimensione 2, ottenuti recentemente in collaborazione con Camillo De Lellis (Università di Zurigo). In particolare, si discuteranno un risultato di regolarità bassa, più precisamente l'equivalenza fra la formulazione debole e quella forte del problema dimostrata in [16] e un risultato di regolarità alta, o meglio la maggiore integrabilità della densità del termine di volume dei minimi provata in [17]. Da quest'ultima segue una stima sulla dimensione di Hausdorff del relativo insieme singolare grazie ai risultati contenuti in [2] (vedi anche [18]).

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1. Introduction

The Mumford & Shah model is a prominent example of variational problem in image segmentation (see [25]). A smoothed version of a black and white picture, whose levels of gray are represented by a function $g \in L^\infty(\Omega, [0,1])$, is obtained by minimizing the functional

$$(v, K) \to \mathscr{E}(v, K) + \beta \int_{\Omega \setminus K} |v - g|^2 dx, \tag{1}$$

with

$$\mathscr{E}(v, K) := \int_{\Omega \setminus K} |\nabla v|^2 \, dx + \gamma \, \mathcal{H}^1(K),$$

where $\Omega \subset \mathbb{R}^2$ is a fixed open set, K is a closed subset in Ω with finite $\mathcal{H}^1$ measure, $v \in C^1(\Omega \setminus K)$, and β and γ nonnegative parameters. For the sake of simplicity we set $\beta = \gamma = 1$ in what follows.

The squared L^2 distance in (1) plays the role of a fidelity term, in order that the output of the minimization process is close to the original input picture g in an average sense. Instead, the set of countours of the objects in the picture is represented by K, and its length is controlled by the penalization of the $\mathcal{H}^1$ measure. Finally, the Dirichlet energy of u favours sharp contours rather than zones where a thin layer of gray is used to pass smoothly from white to black or viceversa (for more details on the model see the original paper [25], in addition see also [4] and [11]).

We stress the attention upon the fact that the set K is not assigned a priori and it is not a boundary in general. Therefore, this problem is not a free-boundary problem, and new ideas and techniques had to be developed to solve it.

In passing we note that the energy in (1) has been conveniently modified and exploited in problems in other fields, e.g. in Fracture Mechanics mainly to model quasi-static irreversible crack-growth for brittle materials according to Griffith (see [4, Section 4.6.6]).

Since the appearance of the Mumford & Shah model in the late 80's the research on the problem, and on related fields, has been very active and different approaches to analyze it were developed. In this paper we shall focus mainly on the ideas and the setting proposed by De Giorgi, limited to the 2d case of interest here. More precisely, a weak formulation of the problem, from which an existence theory for minimizers of $\mathscr{E}$ can be developed, is obtained within the space SBV of *Special functions of Bounded Variation* introduced by De Giorgi and Ambrosio: the subspace of BV functions with singular part of the distributional derivative concentrated on a 1-dimensional set (throughout the paper we will use standard notations and results concerning the spaces BV and SBV, following the book [4]). To be more precise, we recall that $v \in L^1(\Omega)$ belongs to $BV(\Omega)$ if and only if the distributional derivative Dv of v is a (vector-valued) Radon measure on Ω. Then, we can decompose Dv according to

$$Dv = \nabla v\, \mathcal{L}^2 \llcorner \Omega + (v^+ - v^-)\nu_v\ \mathcal{H}^1 \llcorner S_v + D^c v,$$

where (cp. with [4, Section 3.9])

(i) ∇v is the density of the absolutely continuous part of Dv with respect to the Lebesgue measure on Ω (and the approximate gradient of v in the sense of Geometric Measure Theory as well),

(ii) S_v is the set of approximate discontinuities of v, an $\mathcal{H}^1$-rectifiable set (so that $\mathcal{L}^2(S_v) = 0$) endowed with approximate normal $\nu_v(x)$ for $\mathcal{H}^1$ a.e. x,

(iii) $v^\pm$ are the one-sided traces left by v on S_v defined $\mathcal{H}^1$ a.e.,

(iv) $D^c v$ is a measure singular w.r.to the Lebesgue measure, and not charging sets that are (σ-)finite w.r.to the $\mathcal{H}^1$ measure.

Definition 1.1 ([14], Section 4.1 [4]). *$v \in BV(\Omega)$ is a* Special function of Bounded Variation*, in short $v \in SBV(\Omega)$, if $D^c v = 0$, i.e. $Dv = \nabla v\, \mathcal{L}^2 \llcorner \Omega + (v^+ - v^-)\nu_v\ \mathcal{H}^1 \llcorner S_v$.*

So, no Cantor staircase type behaviour is allowed for this class of functions. The latter requirement is fairly natural by taking into account the structure of the energy MS, the minimization on the whole space BV leading otherwise to a trivial result.

Simple examples are collected in the ensuing list:

(i) $W^{1,1}(\Omega) \subset SBV(\Omega)$, with $S_v = \emptyset$ and $Dv = \nabla v\, \mathcal{L}^2 \llcorner \Omega$ for all $v \in W^{1,1}(\Omega)$;

(ii) if $\chi_E \in BV(\Omega)$, i.e. E is a set of *finite perimeter*, then actually $\chi_E \in SBV(\Omega)$ with $D\chi_E = \nu_{S_{\chi_E}}\ \mathcal{H}^1 \llcorner S_{\chi_E}$. The set S_{χ_E} is called the *essential boundary* of E and denoted by $\partial^* E$.

More generally, $\sum_{i=1}^{M} a_i \chi_{E_i} \in SBV(\Omega)$, if $\chi_{E_i} \in BV(\Omega)$, $a_i \in \mathbb{R}$ and $M \in \mathbb{N}$;

(iii) let $\lambda \in \mathbb{R}$, the function $v = \lambda\sqrt{\rho} \cdot \sin(\theta/2)$, $\theta \in (-\pi, \pi)$ and $\rho > 0$, belongs to $SBV(B_r)$, for all $r > 0$, with $S_v \cap B_r = (-r, 0) \times \{0\}$.

In particular, the latter example shows that the direct sum of the subspaces of absolutely continuous functions and piecewise constant ones in items (i) and (ii) above is strictly included in SBV.

Other examples can be obtained as follows (see [4, Proposition 4.4]): if $K \subset \Omega$ is a closed set such that $\mathcal{H}^1(K) < +\infty$ and $v \in W^{1,1} \cap L^\infty(\Omega \setminus K)$, then $v \in SBV(\Omega)$ and

$$\mathcal{H}^1(S_v \setminus K) = 0. \tag{2}$$

Clearly, property (2) above is not valid for a generic member of SBV, but it does for a significant class of functions: local minimizers of the energy under consideration (see below for the definition).

Keeping in mind this example, the weak formulation of the problem is obtained naively by taking $K = S_v$. Loosely speaking, in this approach the set of contours K is identified by the (Borel) set S_v of (approximate) discontinuities of the function v that is not fixed apriori. This is the reason for the terminology *free-discontinuity* problem introduced by De Giorgi. The Mumford & Shah energy of a function v in $SBV(\Omega)$ on an open subset $A \subseteq \Omega$ then reads as

$$\mathrm{MS}(v, A) + \int_A |v - g|^2 dx,$$

where

$$\mathrm{MS}(v, A) := \int_A |\nabla v|^2 dx + \mathcal{H}^1(S_v \cap A). \tag{3}$$

For the sake of simplicity, we drop the dependence on the set of integration on the term on the left hand side above in case $A = \Omega$.

Ambrosio's SBV closure and compactness theorem (see [4, Theorems 4.7 and 4.8]) ensures the existence of a minimizer in SBV. It is worth noting that, by truncation, minimizing sequences have norms bounded in L^∞ by that of the datum g.

Instead, existence of minimizers for the strong formulation of the problem is obtained via a regularity property enjoyed by (the discontinuity set of) the minimizers of the weak counterpart. To this aim we need to analyze the scaling of the energy in order to understand the local behaviour of minimizers. This operation has to be done with some care since the volume and length terms in the energy MS scale differently under affine change of variables of the domain. Let $v \in SBV(B_\rho(x))$, set

$$v_\rho(y) := \rho^{-1/2} v(x + \rho\, y), \tag{4}$$

then $v_\rho \in SBV(B_1)$, with

$$\mathrm{MS}(v_\rho, B_1) = \rho^{-1}\mathrm{MS}(v, B_\rho(x))$$

and

$$\int_{B_1} |v_\rho - g_\rho|^2 dz = \rho^{-3} \int_{B_\rho(x)} |v - g|^2 dy.$$

Thus,

$$\frac{1}{\rho}\left(\mathrm{MS}(v, B_\rho(x)) + \int_{B_\rho(x)} |v - g|^2 dz\right) = \mathrm{MS}(v_\rho, B_1) + \rho^2 \int_{B_1} |v_\rho - g_\rho|^2 dy.$$

This calculation shows that at first order the leading term in the energy is that related to the MS functional, the other being a contribution of higher order that can be neglected in a preliminary analysis. Motivated by this, we introduce a notion of minimality involving only the leading part of the energy. In what follows, u will always denote a *local minimizer* of the functional MS, that is any $u \in SBV(\Omega)$ with $\mathrm{MS}(u) < +\infty$ and such that

$$\mathrm{MS}(u) \leq \mathrm{MS}(w) \quad \text{ whenever } \{w \neq u\} \subset\subset \Omega.$$

The class of all local minimizers shall be denoted by $\mathcal{M}(\Omega)$. Actually, we shall often refer to local minimizers simply as minimizers if no confusion can arise. Regularity properties for minimizers of the whole energy can be obtained by perturbing the theory for local minimizers (cp. with Corollary 2.2 below).

As established in [15] in all dimensions (and alternatively in [9] and [10] in dimension two), if $u \in \mathcal{M}(\Omega)$ then the pair $(u, \overline{S_u})$ is a minimizer of $\mathscr{E}$. The main point is the identity $\mathcal{H}^1(\overline{S_u} \setminus S_u) = 0$, which holds for every $u \in \mathcal{M}(\Omega)$. The groundbreaking paper [15] proves this identity via the following density lower bound estimate (actually established in any dimension with the obvious changes in the statement, see [4, Theorem 7.21]).

Theorem 1.1 (De Giorgi, Carriero & Leaci [15])**.** *Let $u \in \mathcal{M}(\Omega)$, then there exists a dimensional constant θ independent of u such that*

$$\frac{\mathrm{MS}(u, B_r(z))}{2r} \geq \theta \qquad \text{for all } z \in \overline{S_u}, \text{ and all } r \in (0, \mathrm{dist}(z, \partial\Omega)). \tag{5}$$

Building upon the same ideas, in [8] it is proved a slightly more precise result (see again [4, Theorem 7.21]).

Theorem 1.2 (Carriero & Leaci [8])**.** *Let $u \in \mathcal{M}(\Omega)$, then for some dimensional constant θ_0 independent of u it holds*

$$\frac{\mathcal{H}^1(S_u \cap B_r(z))}{2r} \geq \theta_0 \qquad \text{for all } z \in \overline{S_u}, \text{ and all } r \in (0, \mathrm{dist}(z, \partial\Omega)). \tag{6}$$

The argument for (5) used by De Giorgi, Carriero & Leaci in [15], and similarly in [8] for (6), is indirect: it relies on Ambrosio's SBV compactness theorem, an SBV Poincaré-Wirtinger type inequality established in [15] (see also [4, Theorem 4.14]) and the asymptotic analysis of blow-up limits of minimizers with vanishing line energy, that is limits of sequences $(u_{\rho_k})_{k\in\mathbb{N}}$ with $\mathcal{H}^1(S_{u_{\rho_k}}) \downarrow 0$ for any $\rho_k \downarrow 0$ as $k \uparrow \infty$, with u_ρ defined as in (4) and $u \in \mathcal{M}(\Omega)$ (see [4, Theorem 7.21]). In the paper [16] a simpler proof in 2 dimensions is given, that does not require any Poincaré-Wirtinger inequality, nor any compactness argument. Indeed, the proof in [16] is based on an observation of geometric nature and on a direct variational comparison argument, it differs from those exploited in [9] and [10] to derive (6) in the two dimensional case as well (see Section 2 for more details).

Theorem 1.3 (De Lellis & Focardi [16])**.** *Let $u \in \mathcal{M}(\Omega)$. Then*

$$\frac{\mathrm{MS}(u, B_z(r))}{r} \geq 1 \qquad \text{for all } z \in \overline{S_u} \text{ and all } r \in (0, \mathrm{dist}(z, \partial\Omega)). \tag{7}$$

More precisely, the set $\Omega_u := \{z \in \Omega : (7) \text{ fails}\}$ is open and $\Omega_u = \Omega \setminus \overline{S_u}$.

Furthermore, Corollary 2.1 provides a similar conclusion involving only the $\mathcal{H}^1$ measure of the discontinuity set in analogy with Theorem 1.2 above.

Having established the existence of (local) minimizers $(u, \overline{S_u})$ for $\mathscr{E}$, we discuss next some among their regularity properties, or better of the (closure of the) discontinuity set $\overline{S_u}$ being easy to check that u is harmonic on $\Omega \setminus \overline{S_u}$. The interest of the researchers in this problem is motivated by the ensuing conjecture due to Mumford & Shah.

Conjecture 1.4 (Mumford & Shah [25])**.** *If $u \in \mathcal{M}(\Omega)$, then $\overline{S_u}$ is the union of (at most) countably many injective C^1 arcs $\gamma_i : [a_i, b_i] \to \Omega$ with the following properties:*

(c1) *Any compact $K \subset \Omega$ intersects at most finitely many arcs;*

(c2) *Two arcs can have at most an endpoint p in common, and if this is the case, then p is in fact the endpoint of three arcs, forming equal angles of $2\pi/3$.*

According to this conjecture only two possible singular configurations occur: either three arcs meet in an end with angles of $2\pi/3$, or an arc has a free-end. In what follows, we shall call *triple junction* the first configuration and *crack-tip* the second.

It was shown by Alberti, Bouchitté & Dal Maso [1] that the prototype of triple junctions, i.e. three segments meeting in a common point with equal angles, is indeed a local minimizer by developing a suitable theory of calibrations for free-discontinuity problems (see [23] and [24] for related results). Instead, Bonnet & David [6] have shown that the prototype of crack-tip functions, i.e. $u(r,\theta) = \sqrt{\frac{2}{\pi} r} \cdot \sin(\theta/2)$, is a *global* minimizer of the Mumford & Shah energy, a slightly different notion including a topological condition (see [5]). We do not know yet whether they are local minimizers as well or not.

Let us now survey the state of the art about Conjecture 1.4. We begin with a regularity result stated in the 2-dimensional setting of interest here.

Theorem 1.5 (Ambrosio, Fusco & Pallara [3])**.** *Let $u \in \mathcal{M}(\Omega)$, then there exists $\Sigma_u \subset \overline{S_u}$ relatively closed in Ω with $\mathcal{H}^1(\Sigma_u) = 0$, and such that $\overline{S_u} \setminus \Sigma_u$ is locally a $C^{1,1}$ arc.*

More precisely, there exists $\varepsilon_0 > 0$ such that

$$\Sigma_u = \{x \in \overline{S_u} : \liminf_{\rho \downarrow 0} (\mathscr{D}(x,\rho) + \mathscr{A}(x,\rho)) \geq \varepsilon_0\} \tag{8}$$

where

$$\mathscr{D}(x,\rho) := \rho^{-1} \int_{B_\rho(x)} |\nabla u|^2 dy, \quad \textit{(scaled Dirichlet energy)}$$

$$\mathscr{A}(x,\rho) := \rho^{-3} \min_{T\ line} \int_{S_u \cap B_\rho(x)} \mathrm{dist}^2(y,T) d\mathcal{H}^1(y), \quad \textit{(scaled mean flatness).}$$

Note that $\mathscr{D}(x,\cdot)$ and $\mathscr{A}(x,\cdot)$ are equal, by a change of variables, to the Dirichlet energy and the mean flatness of the blow-up maps u_ρ on B_1 defined in (4), respectively.

Theorem 1.5, or better the characterization of the *singular set* Σ_u in (8), can be employed to subdivide Σ_u according to the Mumford & Shah conjecture as follows: $\Sigma_u = \Sigma_u^1 \cup \Sigma_u^2 \cup \Sigma_u^3$, where

$$\Sigma_u^1 := \{x \in \Sigma_u : \lim_{\rho \downarrow 0} \mathscr{D}(x,\rho) = 0\}, \quad \text{the subset of triple junctions}$$
$$\Sigma_u^2 := \{x \in \Sigma_u : \lim_{\rho \downarrow 0} \mathscr{A}(x,\rho) = 0\}, \quad \text{the subset of crack-tips}$$
$$\Sigma_u^3 := \{x \in \Sigma_u : \liminf_{\rho \downarrow 0} \mathscr{D}(x,\rho) > 0, \liminf_{\rho \downarrow 0} \mathscr{A}(x,\rho) > 0\}.$$

According to the Mumford & Shah conjecture we should expect $\Sigma_u^3 = \emptyset$.

In the paper [2], Ambrosio, Fusco & Hutchinson have investigated the connection between the higher integrability of ∇u and the Mumford & Shah conjecture. If Conjecture 1.4 does hold, then $\nabla u \in L^p_{loc}$ for all $p < 4$ (cp. with [2, Proposition 6.3] under $C^{1,1}$ regularity assumptions on $\overline{S_u}$, see also Proposition 1.10 below). Viceversa, the higher integrability can be translated into an estimate for the size of the singular set Σ_u of $\overline{S_u}$ (see [2, Corollary 5.7]): in particular this set has Hausdorff dimension $2-p/2$ under the apriori assumption that $\nabla u \in L^p_{loc}$ for some $p > 2$. In fact [2] proves also an higher-dimensional analog of this second result.

Theorem 1.6 (Ambrosio, Fusco & Hutchinson [2]). *If* $u \in \mathcal{M}(\Omega)$ *and* $|\nabla u| \in L^p_{loc}(\Omega)$ *for some* $2 < p < 4$*, then*

$$\dim_{\mathcal{H}} \Sigma_u \le 2 - \frac{p}{2} \in (0,1).$$

Few remarks are in order:

(i) the limitation $p < 4$ is motivated not only because we need the rhs in the estimate above to be positive, but also because explicit examples show that it is the best exponent one can hope for (see the crack-tip example below);

(ii) if we were able to prove the higher integrability property for every $p < 4$ then we would infer that $\dim_{\mathcal{H}} \Sigma_u = 0$. Clearly, a big step towards the solution in positive of the Mumford & Shah conjecture. For further progress in this direction see Proposition 1.10 below.

Theorem 1.6 is a straightforward corollary of a much deeper and technically demanding result, that we report in the 2-dimensional case of interest here though it holds true with a similar statement in any dimension as well.

Theorem 1.7 (Ambrosio, Fusco & Hutchinson, [2]). *The subset of triple junctions* Σ_u^1 *has Hausdorff dimension zero.*

Given Theorem 1.7 for granted, Theorem 1.6 is a simple consequence of soft measure theoretic arguments. We shall comment further on Theorem 1.7 in Section 3. Instead, here we outline the proof of Theorem 1.6 to show the role of higher integrability.

Sketch of the Proof of Theorem 1.6. Suppose that $|\nabla u| \in L^p_{loc}$, then for all $s \in (2-p/2, 1)$ the set

$$\Lambda_s := \left\{ x \in \Omega : \limsup_{\rho} \rho^{-s} \int_{B_\rho(x)} |\nabla u|^p dy > 0 \right\}$$

satisfies $\mathcal{H}^s(\Lambda_s) = 0$ by an elementary covering argument.

Hence, if we rewrite Σ_u as the disjoint union of $\Sigma_u \cap \Lambda_s$ and of $\Sigma_u \setminus \Lambda_s$, we deduce the estimate $\dim_{\mathcal{H}}(\Sigma_u \cap \Lambda_s) \le s$.

Furthermore, it is easy to prove that $\Sigma_u \setminus \Lambda_s \subseteq \Sigma_u^1$, since if $x \in \Sigma_u \setminus \Lambda_s$ by the higher integrability it follows that

$$\mathscr{D}(x, \rho) = \rho^{-1} \int_{B_\rho(x)} |\nabla u|^2 dy \le \pi^{1-\frac{2}{p}} \rho^{1+\frac{2}{p}(s-2)} \left(\rho^{-s} \int_{B_\rho(x)} |\nabla u|^p dy \right)^{\frac{2}{p}} \stackrel{\rho \downarrow 0^+}{\longrightarrow} 0.$$

By taking into account Theorem 1.7 we have that $\dim_{\mathcal{H}}(\Sigma_u \setminus \Lambda_s) = 0$.

In conclusion, we infer that for all $s \in (2 - p/2, 1)$

$$\dim_{\mathcal{H}} \Sigma_u = \dim_{\mathcal{H}}(\Sigma_u \cap \Lambda_s) \le s,$$

by letting $s \downarrow (2 - p/2)^+$ we are done. □

The estimate $\dim_{\mathcal{H}} \Sigma_u < 1$ was already present in literature (see David [10], Maddalena & Solimini [21]), though not related to the higher integrability property of the gradient.

So far, in [17, Theorem 1.1] we have been able to prove the following statement that was conjectured by De Giorgi in all space dimensions (cp. with [13, Conjecture 1]).

Theorem 1.8 (De Lellis & Focardi [17])**.** *There is $p > 2$ such that $\nabla u \in L^p_{\mathrm{loc}}(\Omega)$ for all $u \in \mathcal{M}(\Omega)$ and for all open sets $\Omega \subseteq \mathbb{R}^2$.*

For a hint of the proof and further comments see Section 3.

Let us now go back to the role of the exponent 4 in the higher integrability result. We consider crack-tip minimizers (Bonnet & David [6]), i.e. functions that up to a rigid motion can be written as

$$u(\rho, \theta) = C \pm \sqrt{\frac{2}{\pi}\rho} \cdot \sin(\theta/2)$$

for $\theta \in (-\pi, \pi)$ and $\rho > 0$, and some constant $C \in \mathbb{R}$. Instead, the prefactor $\sqrt{2/\pi}$ is the only admissible constant for crack-tip like functions to be minimizers, as one can check by using the Euler-Lagrange necessary conditions.

Simple calculations imply that crack-tip minimizers satisfy

$$|\nabla u| \in L^p_{loc}(\mathbb{R}^2) \setminus L^4_{loc}(\mathbb{R}^2) \quad \text{for all } p < 4.$$

Actually, beyond the scale of L^p spaces the following slightly more precise piece of information holds true: $|\nabla u| \in L^{4,\infty}_{loc}(\mathbb{R}^2)$. The latter is a weak-Lebesgue space, i.e. if $U \subseteq \mathbb{R}^2$ is open, then $f \in L^{4,\infty}_{loc}(U)$ if and only if for all $U' \subset\subset U$ there exists $K = K(U') > 0$ such that

$$|\{x \in U' : |f(x)| > \lambda\}| \leq K\lambda^{-4} \qquad \text{for all } \lambda > 0.$$

As a side effect of our considerations, we remark a small improvement of the result in [2] in the 2-dimensional case: a weaker form of the Mumford-Shah conjecture in 2d is equivalent to a sharp L^p estimate of the gradient of the minimizers.

Conjecture 1.9. *If $u \in \mathcal{M}(\Omega)$, then $\overline{S_u}$ is the union of (at most) countably many injective C^0 arcs $\gamma_i : [a_i, b_i] \to \Omega$ which are C^1 on $]a_i, b_i[$ and satisfy the two conditions of Conjecture 1.4.*

Our refinement of the result in [2] is contained in the following proposition (see [16, Proposition 1.5]).

Proposition 1.10 (De Lellis & Focardi [16])**.** *The Conjecture 1.9 is true for $u \in \mathcal{M}(\Omega)$ if and only if $\nabla u \in L^{4,\infty}_{loc}(\Omega)$.*

The if direction of Proposition 1.10 is achieved by first proving that $\overline{S_u}$ has locally finitely many connected components and then invoking the regularity theory developed by Bonnet [5]. In turn, the proof that the connected components are locally finite is a fairly simple application of David's ε-regularity theory [11]. The subtle difference between Conjecture 1.4 and Conjecture 1.9 is in the following point: assuming Conjecture 1.9 holds, if $p = \gamma_i(a_i)$ is a "loose end" of the arc γ_i, i.e. does not belong to any other arc, then the techniques in [5] show that any blow-up is a cracktip, but do not give the uniqueness. In particular, Bonnet is not able to exclude the possibility that γ_i "spirals" around p infinitely many times (compare with the discussion at the end of [5, Section 1]). As far as we know this point is still open.

We have come to the conclusion of this long introduction to the motivations of our researches, along which we have introduced several notations and definitions necessary in the sequel as well. In the rest of the paper we shall go into more details on the results we proved in [16] and [17] in Sections 2 and 3, respectively.

2. The Density Lower Bound Estimate

First, we introduce some useful notation: Given a function $u \in \mathcal{M}(\Omega)$, a point $z \in \Omega$, and a radius $r \in (0, \mathrm{dist}(z, \partial\Omega))$, let

$$e_z(r) := \int_{B_r(z)} |\nabla u|^2 dx, \quad \ell_z(r) := \mathcal{H}^1(S_u \cap B_r(z)), \quad \text{and} \quad m_z(r) := \mathrm{MS}(u, B_r(z)).$$

The quantity $m_z(\cdot)$ in Theorem 1.3 allows us to take advantage of a suitable monotonicity formula, discovered independently by David and Léger in [12] and Maddalena and Solimini in [21].

Lemma 2.1. *Let $u \in \mathcal{M}(\Omega)$, then for every $z \in \Omega$ and for $\mathcal{L}^1$ a.e. $r \in (0, \mathrm{dist}(z, \partial\Omega))$*

$$\int_{\partial B_r(z)} \left(\left(\frac{\partial u}{\partial \nu}\right)^2 - \left(\frac{\partial u}{\partial \tau}\right)^2 \right) d\mathcal{H}^1 + \frac{\ell_z(r)}{r} = \frac{1}{r} \int_{S_u \cap \partial B_r(z)} |\langle \nu_u^\perp(x), x\rangle| d\mathcal{H}^0(x),$$

$\frac{\partial u}{\partial \nu}$ and $\frac{\partial u}{\partial \tau}$ being the projections of ∇u in the normal and tangential directions to $\partial B_r(z)$, respectively.

In [16, Appendix A] we give an alternative proof of Lemma 2.1 above, by exploiting directly the Euler-Lagrange equation tested on special radial inner variations.

A simple iteration of Theorem 1.3 gives a density lower bound as in (6) with an explicit constant θ_0 (see [16, Corollary 1.2]).

Corollary 2.1 (De Lellis & Focardi [16])**.** *If* $u \in \mathcal{M}(\Omega)$, *then*

$$\frac{\ell_z(r)}{2r} \geq \frac{\pi}{2^{24}} \qquad \text{for all } z \in \overline{S_u} \text{ and all } r \in (0, \mathrm{dist}(z, \partial\Omega)). \tag{9}$$

Let us now skecth the proof of Theorem 1.3.

Sketch of the proof of Theorem 1.3. The proof is based upon a direct variational argument exploting the following geometrical fact: if $\ell_z(r) < r$ for some $r \in (0, \mathrm{dist}(z, \partial\Omega))$, then for a set I of positive measure in $(0, r)$ we have

$$\mathcal{H}^0(S_u \cap \partial B_\rho(z)) = 0 \quad \text{for all } \rho \in I.$$

By taking into account that if $z \in \Omega_u$ then $m_z(r) < r$ for some $r \in (0, \mathrm{dist}(z, \partial\Omega))$, radii $\rho \in I$ can be chosen to satisfy some additional conditions, in a way that testing the minimality of u with the harmonic competitor having the same boundary value on $\partial B_\rho(z)$, one infers that $m_z(\rho) < \rho$. To deduce this, we employ the monotonicity Lemma 2.1.

Actually, we need to propagate the estimates in a quantitative way: if $m_z(r) \leq (1-\varepsilon)r$ for some $r \in (0, \mathrm{dist}(z, \partial\Omega))$ and $\varepsilon \in (0, 1)$, then we are able to show that

$$m_z(\rho) < (1 - \varepsilon)\rho \quad \text{for some } \rho \in I.$$

An iteration of the previous argument gives that

$$\theta^1_*(S_u, z) := \liminf_{\rho \downarrow 0^+} \frac{\ell_z(\rho)}{2\rho} \leq \frac{1 - \varepsilon}{2}.$$

From this, it turns out that the set $\Omega_u := \{x \in \Omega : m_x(r) < r\}$ is open and satisfying

$$\Omega_u \cap \{x \in S_u : \theta^1_*(S_u, x) = 1\} = \emptyset.$$

As $\mathcal{H}^1(S_u \setminus \{x \in S_u : \theta^1_*(S_u, x) = 1\}) = 0$, the latter equality implies straightforwardly the inclusion $\overline{S_u} \subseteq \Omega \setminus \Omega_u$.

Finally, as u is harmonic on $\Omega \setminus \overline{S_u}$ by minimality, it is elementary to check that $\Omega \setminus \overline{S_u} \subseteq \Omega_u$, and thus we are done. □

Let us remark that the geometric argument used in Theorem 1.3 has no direct analogue in dimension greater than 2 as simple examples show. In spite of this, Bucur & Luckhaus have used a similar idea independently from us (see [7]). Furthermore, they were able to improve remarkably upon this idea and establish results in the spirit of Theorem 1.3 and Corollary 2.1 without our dimensional limitation.

A natural question is the sharpness of the estimates (7) and (9). The analysis performed by Bonnet [5] suggests that $\pi/2^{24}$ in (9) should be replaced by 1/2 and 1 in (7) by 2. Note that the square root function $u(\rho,\theta)=\sqrt{\frac{2}{\pi}\rho}\cdot\sin(\theta/2)$ satisfies $\ell_0(\rho)=e_0(\rho)=\rho$ for all $\rho>0$. Thus both the constants conjectured above would be sharp by [11, Section 62]. Unfortunately, we cannot prove any of them.

A similar result can be established for quasi-minimizers of the Mumford & Shah energy, the most prominent examples being minimizers of the functional in equation (1). More precisely, a quasi-minimizer is any function v in $SBV(\Omega)$ with $\mathrm{MS}(v)<+\infty$ and satisfying for some $\omega\geq 0$ and $\alpha>0$ and for all balls $B_\rho(z)\subset\Omega$

$$\mathrm{MS}(v,B_\rho(z))\leq \mathrm{MS}(w,B_\rho(z))+\omega\,\rho^{1+\alpha}\qquad \text{whenever } \{w\neq v\}\subset\subset B_\rho(z).$$

We can then prove the ensuing infinitesimal version of (7) (cp. with [16, Corollary 1.3]).

Corollary 2.2 (De Lellis & Focardi [16]). *Let v be a quasi-minimizers of the Mumford & Shah energy, then*

$$\overline{S_v}=\left\{z\in\Omega:\ \liminf_{r\downarrow 0^+}\frac{m_z(r)}{r}\geq\frac{2}{3}\right\}. \tag{10}$$

The proof of this corollary, though, needs a blow-up analysis and a new SBV Poincaré-Wirtinger type inequality of independent interest, obtained by improving upon some ideas contained in [19] (cp. with [16, Theorem B.6]); it is, therefore, much more technical.

Let us finally remark that Theorem 1.3 can be slightly improved, by combining the ideas of its proof hinted to above with the SBV Poincaré-Wirtinger type inequality in [16, Theorem B.6]. Indeed, [16, Remark 2.3] shows that for all $u\in\mathcal{M}(\Omega)$

$$\overline{S_u}=\{x\in\Omega:\ m_x(r)>r\quad \text{for all } r\in(0,\mathrm{dist}(x,\partial\Omega))\}.$$

3. The Higher Integrability Result

Following a classical path, the key ingredient to establish Theorem 1.8 is a reverse Hölder inequality for the gradient, which we state independently (see [17, Theorem 1.3]).

Theorem 3.1 (De Lellis & Focardi [17]). *For all $q \in (1,2)$ there exist $\rho \in (0,1)$ and $C > 0$ such that*

$$\|\nabla u\|_{L^2(B_\rho)} \leq C \|\nabla u\|_{L^q(B_1)} \qquad \text{for any } u \in \mathcal{M}(B_1). \tag{11}$$

Using the obvious scaling invariance of (3), Theorem 3.1 yields a corresponding reverse Hölder inequality for balls of arbitrary radius: Theorem 1.8 is then a consequence of (by now) classical arguments (see for instance [20]). The exponent p could be explicitely estimated in terms of q, C and ρ. However, since our argument for Theorem 3.1 is indirect, we do not have any explicit estimate for C (ρ can instead be computed). Hence, combining Theorem 1.8 with [2] we can only conclude that the dimension of the singular set of $\overline{S_u}$ is strictly smaller than 1. This was already proved in [11] using different arguments and, though not stated there, Guy David pointed out to us that the corresponding dimension estimate could be made explicit. In fact, after discussing the present result, he suggested to us that also the constant C in Theorem 3.1 might be estimated: a viable strategy would combine the core argument of this paper with some ideas from [11] (the proof of Theorem 3.1 given here makes already a fundamental use of the paper [11], but depends only on the ε-regularity theorem for "triple junctions" and "segments"). However, the resulting estimate would give an extremely small number, whereas the proof would very likely become much more complicated. Since we do not see any way to make further progress, we have decided not to pursue this issue here.

In addition, our indirect proof has some interesting side results that we shall highligt in what follows. Indeed, in this section we shall give a rapid sketch of the proof of Theorem 3.1, and rather than discussing all the details we shall mainly focus on a compactness result, Theorem 3.2 below, that is one of the most important ingredients to establish Theorem 3.1, and on the related consequences. We strongly believe that Theorem 3.2 has some interest in its own.

Sketch of the proof of Theorem 3.1. We fix an exponent $q \in (1,2)$ and a suitable radius ρ (whose choice will be specified later) for which (11) is false, that is

$$\|\nabla u_k\|_{L^2(B_\rho)} \geq k\|\nabla u_k\|_{L^q(B_1)} \quad \text{for a sequence } (u_k)_{k\in\mathbb{N}} \in \mathcal{M}(B_1). \tag{12}$$

Since the Mumford & Shah energy of any $u \in \mathcal{M}(B_1)$ can be easily bounded apriori by 2π, we have $\|\nabla u_k\|_{L^q(B_1)} \to 0$. A suitable competitor argument then shows that:

(a) The L^2 energy of the gradients of u_k converge to 0;
(b) The discontinuity set S_{u_k} of u_k converges in the local Hausdorff metric to a set J which is a (locally finite) union of minimal connections.

Though this last statement is, intuitively, quite clear, it is technically demanding, because we do not have any apriori control of the norms $\|u_k\|_{L^1}$, thus preventing the use of Ambrosio's SBV compactness theorem, as well as of its generalized version in $GSBV$. We can not even employ De Giorgi's SBV Poincaré-Wirtinger inequality, since it holds true in a regime of small jumps rather than of small gradients as the current one.

A very similar issue is investigated in [2, Proposition 5.3, Theorem 5.4] under the stronger assumption that $\|\nabla u_k\|_{L^2}$ converges to 0. Such results hinge upon the notion of Almgren's area minimizing sets, and thus need a delicate study of the behaviour of the composition of SBV functions with Lipschitz deformations that are not necessarily one-to-one, and some specifications on the regularity theory for those sets. Instead, in [17, Proposition 5.1] (see Proposition 3.2 below) we set the analysis into the more natural framework of Caccioppoli partitions. Because of this, as pointed out in item (a) above, the fact that the Dirichlet energy of u_k is infinitesimal turns out to be a consequence of (12) and of the energy upper bound for functions in $\mathcal{M}(B_1)$.

Having established (a) and (b), an elementary argument shows the existence of a universal constant ρ such that the intersection of J with $B_{2\rho}$ is:

(i) either empty;
(ii) or a straight segment;
(iii) or three segments meeting at a common point with equal angles.

We use then the regularity theory developed by David (see [11]) to conclude that, if k is large enough, $\overline{S_{u_k}} \cap B_{2\rho}$ is diffeomorphic to (and a small perturbation of) one of these three cases. Finally a variational argument (based on a simple "Fubini and competitor" trick) shows the existence of a constant C (independent of k) with the property that

$$\|\nabla u_k\|_{L^2(B_\rho)} \leq C\|\nabla u_k\|_{L^q(B_1)} \tag{13}$$

which contradicts (12). □

To state the compactness result alluded to in item (b) above we need to introduce Caccioppoli partitions.

Definition 3.1. *A Caccioppoli partition of Ω is a countable partition $\mathscr{E} = \{E_i\}_{i=1}^{\infty}$ of Ω in sets of (positive Lebesgue measure and) finite perimeter with $\sum_{i=1}^{\infty} \mathrm{Per}(E_i, \Omega) < \infty$.*

For each Caccioppoli partition $\mathscr{E}$ consider its set of interfaces

$$J_{\mathscr{E}} := \bigcup_i \partial^* E_i,$$

$\partial^* E_i$ being the essential boundary of E_i (recall the notation introduced in example (ii) after Definition 1.1). The partition $\mathscr{E}$ is said to be minimal if

$$\mathcal{H}^1(J_{\mathscr{E}}) \leq \mathcal{H}^1(J_{\mathscr{F}})$$

for all Caccioppoli partitions $\mathscr{F}$ for which there exists an open subset $\Omega' \subset\subset \Omega$ with

$$\sum_{i=1}^{\infty} \mathcal{L}^2\left((F_i \triangle E_i) \cap (\Omega \setminus \Omega')\right) = 0.$$

There is an important correspondance between Caccioppoli partitions and the subspace of "piecewise constant" SBV functions (see [4, Theorems 4.23, 4.25 and 4.39]), in a way that minimizing the Mumford & Shah energy over such a subspace corresponds exactly to the minimal area problem for Caccioppoli partitions.

Below we state Theorem 3.2 only in the 2-dimensional case of interest here. In spite of this, the analogous result in any dimension can be obtained only with straightforward notational changes in the statement below (and also in the corresponding proof).

Nevertheless, dimension 2 enters dramatically in the proof of Theorem 3.1 as we heavily exploit the structure of minimal Caccioppoli partitions in $\mathbb{R}^2$, described precisely via minimal connections (cp. with item (b) above and [17, Proposition 3.2]).

Theorem 3.2 (De Lellis & Focardi [17]). *Let* $(u_k)_{k\in\mathbb{N}} \subset \mathcal{M}(B_1)$ *be such that*

$$\lim_k \|\nabla u_k\|_{L^1(B_1)} = 0.$$

Then, (up to the extraction of a subsequence not relabeled for convenience) there exists a minimal Caccioppoli partition $\mathscr{E} = \{E_i\}_{i\in\mathbb{N}}$ *such that* $(\overline{S_{u_k}})_{k\in\mathbb{N}}$ *converges locally in the Hausdorff distance to* $\overline{J_{\mathscr{E}}}$*, and*

$$\lim_k \mathrm{MS}(u_k, A) = \lim_k \mathcal{H}^1(S_{u_k} \cap A) = \mathcal{H}^1(J_{\mathscr{E}} \cap A) \qquad \text{for all open sets } A \subset B_1. \tag{14}$$

In particular, (14) *implies that*

$$\mathcal{H}^1 \llcorner S_{u_k} \stackrel{*}{\to} \mathcal{H}^1 \llcorner J_{\mathscr{E}} \text{ in the sense of measures, and } \lim_k \|\nabla u_k\|_{L^2(B_1)} = 0.$$

Let us finally discuss some interesting consequences of Theorem 3.2:

(i) Blow-up limits in singular points of a minimizer in the regime of small gradients, i.e. points $x \in \Sigma_u^1$, are minimal Caccioppoli partitions (in any dimension!).

In particular, thanks to the structure of minimal Caccioppoli partitions in 2-dimensions mentioned above, the blow-up limits in singular points are (locally) the union of three segments meeting in a commont point with equal angles;

(ii) A more elementary proof of the estimate (and of its analogue in any dimension!)

$$\dim_{\mathcal{H}} \Sigma_u^1 = 0 \qquad (\text{recall that } \Sigma_u^1 = \{x \in \Sigma_u : \lim_{\rho\downarrow 0} \mathscr{D}(x,\rho) = 0\}),$$

follows from Theorem 3.2, the regularity theory for minimal Caccioppoli partitions by Massari & Tamanini [22], and standard blow-up arguments (see [18]).

No use of Almgren's area minimizing sets and of the corresponding regularity theory is then needed.

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