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Determining the source term in an abstract parabolic problem from a time integral of the solution

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Abstract

Consideriamo il problema della ricostruzione del termine di sorgente in un'equazione astratta di tipo parabolico. L'informazione supplementare, necessaria per la determinazione della soluzione del sistema e della parte incognita del termine di sorgente, è data dalla conoscenza di un integrale della soluzione rispetto alla variabile temporale e a una certa misura di Borel. Presento un teorema di esistenza e unicità di una soluzione, che è anche di regolarità massimale. Esamino alcuni casi particolari, assieme al fatto che talvolta il problema gode di una sorta di proprietà dell'alternativa di Fredholm.

The main aim of this paper is to illustrate some results concerning the following inverse problem of determination of the source term:

$$
(1) \qquad \begin{cases} u'(t) = Au(t) + f(t)\xi + g(t), \quad t \in [0,T], \\ \\ u(0) = u_0, \\ \\ \int_{[0,T]} u(t)d\mu(t) = y, \end{cases}
$$

Here A is a sectorial operator in the complex Banach space X (we state in Definition 1 what we mean with the expression "sectorial operator"). The data of the problem are f and g (which are assumed to be continuous with values in $\mathbb{C}$ and X, respectively), u_0 and y, belonging to the domain $D(A)$ of A, and μ, which is a (scalar) complex Borel measure in $[0,T]$. The unknown terms are the function u, together with the element ξ of X. If we consider the system given by the two first equations in (1), we obtain a standard abstract Cauchy problem of parabolic type. As ξ is unknown, which implies that the source term is not completely known, it is necessary to give a supplementary information, in order to determine u. This is given by the last condition in (1). The system (1) is quite general, as it contains, as a particular case, the specification of $u(T)$, corresponding to the case $\mu = \delta_T$. This case is the most treated in the literature, and the unknown ξ is often thought as a control, in order to obtain a prescribed $u(T)$.

I am going to explain some results obtained by myself in the paper [3].

I start by introducing some notation, and quote some previous work, which is related to problem (1).

Let X be a complex Banach space, and let A be a set. We indicate with $\|\cdot\|_X$ the norm in X. We simply write $\|\cdot\|$ if the space is clear from the context. We indicate with $B(A;X)$ the set of bounded functions from A to X, equipped with its natural norm. If A is a topological space, we indicate with $C(A;X)$ the set of continuous functions, equipped with the norm of subspace of $B(A;X)$ in case A is compact. If $X = \mathbb{C}$, we shall simply write $B(A)$ ($C(A)$).

If E and F are Banach spaces, we indicate with $\mathcal{L}(E, F)$ the Banach space of linear bounded operators from E to F. We shall write $\mathcal{L}(E)$ instead of $\mathcal{L}(E, E)$.

A linear operator in the Banach space X is a linear map from $D(A)$ to X, with $D(A)$ linear subspace of X. We indicate with $\rho(A)$ its resolvent set and with $\sigma(A)$ its spectrum. If A is closed, its domain $D(A)$ is a Banach space, if it is equipped with the natural norm

$$\|x\|_{D(A)} := \|x\| + \|Ax\|. \tag{2}$$

If $\theta \in (0, 1)$, we shall indicate with $D_\theta(A)$ the real interpolation space $(X, D(A))_{\theta,\infty}$. We set also

$$D_{1+\theta}(A) := \{x \in D(A) : Ax \in D_\theta(A)\}, \tag{3}$$

always intended as being equipped with a natural norm.

Finally, C will indicate a generic positive constant, that we are not interested to precise, and may be different from time to time.

Now we give the promised definition of "sectorial operator".

Definition 1. *Let A be a linear operator in the complex Banach space X and let $\omega \in \mathbb{R}$, and $\phi \in (0, \pi]$. We shall write $A \in S(\omega, \phi)$ if*

$$\Sigma(\omega, \phi) := \{\lambda \in \mathbb{C} \setminus \{\omega\} : |Arg(\lambda - \omega)| \leq \pi - \phi\} \subseteq \rho(A)\},$$

and, moreover, there exists $M \in \mathbb{R}^+$, such that, for $\lambda \in \Sigma(\omega, \phi)$,

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda - \omega|}.$$

We shall say that A is sectorial if ϕ can be chosen in $(0, \frac{\pi}{2})$.

Now, we describe some previous work, which is connected with (1).

The oldest paper treating this kind of problem is (in my knowledge) [13], with some (simple) results concerning the case $\mu = \delta_T$. The interesting article [11] considers a problem which is more general than (1) (in the case $g(t,x) \equiv 0$), namely

$$
\text{(4)} \qquad \begin{cases} u'(t) = Au(t) + \Phi(t)\xi, \quad t \in [0,T], \\ \\ u(0) = u_0, \\ \\ \int_0^T u(t) d\mu(t) = y, \end{cases}
$$

with A infinitesimal generator of a C_0-semigroup $(e^{tA})_{t\geq 0}$ (not necessarily analytic) in X and $\Phi \in C^1([0,T];\mathcal{L}(E))$, with u and ξ unknown. The Fredholm property of the problem (roughly speaking, existence is equivalent to uniqueness) is put in light in the case that e^{tA} is compact, for $t > 0$. In the case that Φ is scalar valued (which is the one we consider), assumptions on the sign of Φ' and strong conditions on μ imply the well posedness of (4). Further results are obtained, if X is a Banach lattice. Finally, perturbations are considered. The same system (4) is studied in [8], with the conditions $\mu = \delta_T$, or $d\mu = \nu dt$, with $\nu \in L^1(0,T)$, with assumptions of positivity and compactness in a Banach space with a reproducing cone.

In [6] the very general system

$$
\begin{cases} u'(t) = Au(t) + \int_0^t B(t-s)u(s)ds = E(t)z + f(t), \quad t \in [0,T], \\ \\ u(0) = u_0, \\ \\ \Phi(u) = g, \end{cases}
$$

is treated, with A generator of an analytic, compact semigroup, $B(t) \in \mathcal{L}(D(A),X)$, $E(t)$ bounded operator, Φ mapping continuous functions into vectors, almost commuting with A. The focus is on the Fredholm property of the system. Applications are given to the case $\Phi(u) := u(T)$.

In Chapter 7 of [9] the abstract two-point inverse problem

$$
\begin{cases}
u'(t) = Au(t) + \Phi(t)\xi + g(t), \quad t \in [0,T], \\
\\
u(0) = u_0, \\
\\
u(T) = y,
\end{cases}
\tag{5}
$$

again corresponding to $\mu = \delta_T$, is treated, with u and ξ unknown in the Banach space space X. Here $\Phi \in C^1([0,T]; \mathcal{L}(X))$. The main assumption is that A is the infinitesimal generator of a strongly continuous semigroup $(e^{tA})_{t \geq 0}$. One of the main results is the following: assume that $\|e^{tA}\|_{\mathcal{L}(X)} \leq Me^{\beta t}$, for some $M \in \mathbb{R}^+$, $\beta \in \mathbb{R}$, for every $t \geq 0$, $\lambda, 0 \in \rho(A)$. Then, if

$$
\int_0^T \|[\Phi'(s) - \lambda\Phi(s)]\Phi(T)^{-1}\|_{\mathcal{L}(X)} e^{\beta(T-s)} ds + \|\Phi(0)\Phi(T)^{-1}\|_{\mathcal{L}(X)} e^{\beta T} \leq M^{-1},
$$

(5) is well posed. The particular cases of A self-adjoint and semibounded in a Hilbert space and X Banach lattice are carefully treated.

In the paper [1], the problem (1) is treated, with A infinitesimal generator of an exponentially decreasing analytic semigroup, and $f \in C^1([0,T])$. The problem is well posed if the operator

$$
\int_{[0,T]} A(\int_0^t f(s) e^{(t-s)A} ds) d\mu(t)
$$

is invertible. Sufficient conditions assuring this are given, which are connected with the signs of f, f' and $\int_{[0,T]} f(t) d\mu(t)$. Finally, a representation formula of ξ as a sum of a series is supplied.

In [16], the problem of uniqueness of a solution to system (1) in case $\mu = \delta_T$ is considered. If A is the infinitesimal generator of a strongly continuous semigroup $(e^{tA})_{t \geq 0}$, $f \in C([0,T])$, $\int_{T-\epsilon}^T |f(t)| dt > 0$ $\forall \epsilon \in (0,T)$, or $ker(e^{tA}) = \{0\}$ $\forall t \in \mathbb{R}^+$ and $\int_0^T |\phi(t)| dt > 0$, it is proved that the solution (if existing) is unique.

A similar problem in the less general case that $f(t) \equiv 1$ is treated in [15]. Here necessary and sufficient conditions for uniqueness are found.

Approximation schemes, convergence and discretization methods connected with (1) are treated in [10].

Other authors considered only concrete systems of partial differential equations.

In the book [9], systems of the form

$$\begin{cases} u_t(x,t) - (Lu)(t,x) = \xi(x)h(x,t) + g(x,t), \quad (x,t) \in \Omega \times (0,T), \\ \\ (\mathcal{B}u)(x,t) = b(x,t), \quad (x,t) \in \partial\Omega \times (0,T), \\ \\ u(x,0) = a(x), \quad x \in \Omega, \\ \\ (lu)(x) = \chi(x), \quad x \in \Omega, \end{cases}$$

are carefully treated. Here L is a second order strongly elliptic operator, $\mathcal{B}$ is, either the identity, or a first order linear operator, $(lu)(x) = u(x,t_1)$, with $0 < t_1 \leq T$, or

$$(lu)(x) = \int_0^T u(x,\tau)\omega(\tau)d\tau,$$

with ω given and u and ξ unknown. Results of existence and uniqueness in an L^2 setting, and also in spaces of Hölder continuous functions are given. Even the Fredholm property of the system is emphasized. Mixed parabolic equations of the second order are treated also in [13] in the case of $\mu = \delta_T$.

Finally, specific one-dimensional parabolic problems of determination of the source term (independent of time) together with a certain scalar function are considered in [4].

We pass to the results in [3]. The basic one is the following general

Theorem 1. *Let X be a Banach space, A a sectorial, injective operator in X, $T \in \mathbb{R}^+$, μ a complex Borel measure in $[0,T]$. Consider the system (1), with u and ξ unknown. Assume that*

(a) $f \in C([0,T])$, $g \in C([0,T];X)$, $u_0, y \in D(A)$;

(b) $\int_{[0,T]} f(t)\chi_{(0,T]}(t)d\mu(t) \neq 0$.

Then:

(I) for every $\phi \in (0, \pi/2)$, there exists $\omega(\phi) \in \mathbb{R}$, depending only on T, f, μ and ϕ, such that, if $A \in S(\omega, \phi)$, with $\omega < \omega(\phi)$, (1) has, at most, one solution (u, ξ) in $[C^1([0,T]; X) \cap C([0,T]; D(A))] \times X$;

(II) if, moreover, $g \in B([0,T]; D_\theta(A))$ for some $\theta \in (0,1)$, and $u_0, y \in D_{1+\theta}(A)$, such solution exists;

(III) in this case, u' and Au belong to $B([0,T]; D_\theta(A))$, while $\xi \in D_\theta(A)$.

From the proof of Theorem 1, one can see that a value of $\omega(\phi)$ can be determined in the way we are going to explain. We start by introducing the Borel measure ν in $[0,T]$ defined as follows:

$$\nu(\Gamma) := \mu(\Gamma \cap (0,T]),$$

for every Γ Borel subset of $[0,T]$. We introduce also the following function Φ:

$$\Phi(t) := -(\int_{[0,T]} f(s) d\nu(s))^{-1} f(t), \quad t \in [0,T],$$

(we observe that $\int_{[0,T]} f(s) d\nu(s) = \int_{[0,T]} f(s) \chi_{(0,T]}(s) d\mu(s) \neq 0$) and the following operator B_Φ in $C([0,T]; X)$:

$$(6) \qquad \begin{cases} D(B_\Phi) = \{u \in C^1([0,T]; X) : u(0) = 0\}, \\ \\ B_\Phi u(t) = -u'(t) - \Phi(t) \int_{[0,T]} u'(s) d\nu(s). \end{cases}$$

Then, one can show that, $\forall \phi_1 \in (\frac{\pi}{2}, \pi]$, there exists $R(\phi_1) \in \mathbb{R}^+$, such that $B_\Phi \in S(R(\phi_1), \phi_1)$. We fix ϕ_1, such that $\phi + \phi_1 < \pi$, and consider $R(\phi_1)$, such that $B_\Phi \in S(R(\phi_1), \phi_1)$. Then, we can take $\omega(\phi) = -R(\phi_1)$. One can show also that 0 always belongs to the spectrum of B_Φ. So, necessarily, $R(\phi_1) \geq 0$, which implies $-R(\phi_1) \leq 0$.

We examine an example: We consider the following system

$$(7) \qquad \begin{cases} u'(t) = Au(t) + \xi + g(t), \quad t \in [0,T], \\ \\ u(0) = u_0, \\ \\ u(T) = y, \end{cases}$$

with u and ξ unknown, and A sectorial in the Banach spaces X. We assume that $g \in C([0,T];X)$, $u_0, y \in D(A)$. In this case, $f(t) \equiv 1$ and $\mu = \delta_T$, so that

$$\int_{[0,T]} \chi_{(0,T]}(t) f(t) d\mu(t) = 1.$$

In this case, we have $\Phi(t) \equiv -1$. By simple calculations, one can see that

$$\sigma(B_\Phi) = \{\frac{2k\pi i}{T} : k \in \mathbb{Z}\}$$

and $B_\Phi \in S(0,\phi)$, $\forall \phi \in (\pi/2, \pi]$. So, if $A \in S(\omega, \phi)$, for some $\omega < 0$ and $\phi \in (0, \pi/2)$, Theorem 1 is applicable. We observe that this result is, in some sense, optimal. In fact, consider the simple case $X = \mathbb{C}$, $Au = \frac{2k\pi i}{T} u$, with $k \in \mathbb{Z} \setminus \{0\}$, that is, consider the system

$$(8) \qquad \begin{cases} u'(t) = \frac{2k\pi i}{T} u(t) + \xi + g(t), \quad t \in [0,T], \\ \\ u(0) = u_0, \\ \\ u(T) = y, \end{cases}$$

with $u_0, y \in \mathbb{C}$, $g \in C([0,T])$. Then one can easily verify that (8) is not well posed. In fact, if we take, for example, $g(t) \equiv 0$ and $u_0 = y = 0$, we obtain that, $\forall \xi \in \mathbb{R}$, (8) has the solution (u, ξ), with

$$u(t) = \frac{T(e^{\frac{2k\pi i t}{T}} - 1)}{2k\pi i} \xi.$$

We observe that, $\forall \omega \in \mathbb{R}^+$, $A \in S(\omega, \phi)$, for some $\phi \in (0, \pi/2)$, but this does not happen if $\omega \leq 0$.

We examine another simple example, where we can compare the limitations of ω previuosly indicated with the effective ones. We consider the problem

$$
\begin{cases}
D_t u(t,x) = (\Delta_x - \lambda)u(t,x) + \xi(x), \quad t \in [0,1], x \in \mathbb{R}^n, \\[2ex]
u(0,x) = 0, \quad x \in \mathbb{R}^n, \\[2ex]
u(1,x) - \alpha u(1/2,x) = y(x), \quad x \in \mathbb{R}^n,
\end{cases} \tag{9}
$$

with $\alpha > 1$ and u and ξ unknown. Here $\lambda \in \mathbb{R}$ and $y \in H^2(\mathbb{R}^n)$. We introduce the following operator A_0 in the Hilbert space $X := L^2(\mathbb{R}^n)$:

$$
\begin{cases}
D(A_0) := H^2(\mathbb{R}^n), \\[2ex]
A_0 u := \Delta u, \quad u \in D(A).
\end{cases}
$$

It is well known that A_0 is a sectorial operator in X. We consider the case $A := A_0 - \lambda$. Employing the Fourier transform with respect to x, (9) can be written in the form

$$
S(\lambda + |\eta|^2)\hat{\xi}(\eta) = \hat{y}(\eta), \quad \eta \in \mathbb{R}^n, \tag{10}
$$

with

$$
S(\mu) := \frac{\alpha(e^{-\mu/2} - 1) - (e^{-\mu} - 1)}{\mu}, \quad \mu \neq 0, \quad S(0) = 1 - \frac{\alpha}{2}.
$$

We observe that

$$
S(\lambda + |\eta|^2)^{-1} \sim (1-\alpha)^{-1}(\lambda + |\eta|^2) \quad (|\eta| \to \infty).
$$

So, as $y \in H^2(\mathbb{R}^n)$, (10) has a unique solution ξ in $L^2(\mathbb{R}^n)$ if and only if $S(\lambda + |\eta|^2) \neq 0$ $\forall \xi \in \mathbb{R}^n$. This happens if and only if $\lambda > -2\ln(\alpha - 1)$. In this case, (9) has a unique solution (u, ξ) in $[C^1([0,T]; X) \cap C([0,T]; D(A))] \times X$ $\forall y \in H^2(\mathbb{R}^n)$.

Now we try to apply Theorem 1. We observe that A is injective, because $-|\eta|^2 - \lambda = 0$ only in a subset of $\mathbb{R}^n$ of measure 0 and we are working in the space $L^2(\mathbb{R}^n)$. Moreover, $\mu = \nu = \delta_1 - \alpha\delta_{1/2}$, $f(t) \equiv 1$, so that

$$
\int_{[0,T]} f(t) d\nu(t) = 1 - \alpha \neq 0.
$$

We have also that $\sigma(B_\Phi) = \{\alpha + 4k\pi i : \alpha \in \{0, -2\ln(\alpha - 1)\}, k \in \mathbb{Z}\}$. So, if $R > \max\{0, -2\ln(\alpha-1)\}$, $B_\Phi \in S(R, \phi_1)$, $\forall \phi_1 \in (\pi/2, \pi)$. We conclude that, if $A \in S(\omega, \phi)$, for some $\phi \in (0, \pi/2)$, and $\omega < \min\{0, 2\ln(\alpha-1)\}$, the conclusions of Theorem 1 hold. It is an easy application of the Fourier transform that $A \in S(\omega, \phi)$, $\forall \omega \leq -\lambda$, $\forall \phi \in (0, \pi/2)$. So, we have to take $\lambda > \max\{0, -2\ln(\alpha-1)\}$. This is optimal, in case $\max\{0, -2\ln(\alpha-1)\} = -2\ln(\alpha-1)$, that is, $1 < \alpha \leq 2$.

Suppose that A is a sectorial operator, that is, $A \in S(\omega, \phi)$, for some $\omega \in \mathbb{R}$, and $\phi \in (0, \pi/2)$. Consider the real number $\omega(\phi)$ defined in the statement of Theorem 1. Then the set

$$\sigma_1 := \sigma(A) \cap \overline{\Sigma(\omega(\phi) \wedge 0, \phi)} \tag{11}$$

is a closed subset of $\mathbb{C}$. We shall be interested in the case that σ_1 is a spectral set of A, according to the following definition:

Definition 2. *Let X be a complex Banach space, A a closed operator in X and let $\sigma \subseteq \sigma(A)$. We shall say that σ is a spectral set of A if both σ and $\sigma(A) \setminus \sigma$ are closed in $\mathbb{C}$.*

In case σ_1 is a compact spectral set for A, we indicate with γ the boundary of a bounded open subset of $\mathbb{C}$ containing σ_1, whose closure is disjoint from $\sigma_2 := \sigma(A) \setminus \sigma_1$, such that γ consists of a finite number of rectifiable closed Jordan curves, oriented counterclockwise. We define a bounded linear operator P by

$$P := \frac{1}{2\pi i} \int_\gamma (z - A)^{-1} dz. \tag{12}$$

Then we have the following (see [7], Appendix 1, or [14], Chapter 5):

(I) P is a projection, and $P(X) \subseteq D(A^n)$, $\forall n \in \mathbb{N}$;

(II) if we set

$$X_1 := P(X), \quad X_2 := (1 - P)(X), \tag{13}$$

X_1 and X_2 are invariant with respect to A. Defining

$$\begin{cases} A_1 : X_1 \to X_1, \quad A_1 x = Ax \quad \forall x \in X_1, \\ \\ A_2 : D(A_2) = D(A) \cap X_2 \to X_2, \quad A_2 x = Ax \quad \forall x \in D(A_2), \end{cases} \tag{14}$$

we have that $A_1 \in \mathcal{L}(X_1)$, and

$$\sigma(A_1) = \sigma_1, \quad \sigma(A_2) = \sigma_2,$$

$$(\lambda - A_1)^{-1} = (\lambda - A)^{-1}_{|X_1}, \quad (\lambda - A_2)^{-1} = (\lambda - A)^{-1}_{|X_2}, \quad \forall \lambda \in \rho(A).$$

Applying the operators P and $1 - P$ to (1), and setting

$$u_1(t) := Pu(t), \quad u_2(t) := (1-P)u(t),$$

$$v_1 := Pu_0, \qquad v_2 := (1-P)u_0,$$

$$\xi_1 := P\xi, \qquad \xi_2 := (1-P)\xi,$$

$$y_1 := Py, \qquad y_2 := (1-P)y,$$

we obtain the two separated systems

$$\begin{cases} u_j'(t) = A_j u_j(t) + f(t)\xi_j, \quad t \in [0,T], \\ \\ u_j(0) = v_j, \\ \\ \int_{[0,T]} u_j(t) d\mu(t) = y_j, \quad j \in \{1,2\}. \end{cases} \tag{15}$$

It is clear that, if we solve (15) for both $j \in \{1,2\}$, and take

$$u := u_1 + u_2, \quad \xi := \xi_1 + \xi_2,$$

(u, ξ) solves (1). The advantage in considering these two systems lies in the fact that Theorem 1 is applicable to A_2, while A_1 is bounded. Concerning the case that A is bounded, the following fact holds:

Lemma 1. *Let X be a Banach space, $A \in \mathcal{L}(X)$, $T \in \mathbb{R}^+$, μ a complex Borel measure in $[0,T]$, $f \in C([0,T])$. We indicate with $(e^{tA})_{t\in\mathbb{R}}$ the group in $\mathcal{L}(X)$ generated by A. We set*

$$Q := \int_{[0,T]} (\int_0^t f(s) e^{(t-s)A} ds) d\mu(t). \tag{16}$$

Then the following conditions are equivalent:

(I) for every $g \in C([0,T];X)$, $\forall u_0, y \in X$, (1) has a unique solution (u,ξ) in $C^1([0,T];X) \times X$;

(II) Q is invertible in $\mathcal{L}(X)$.

Proof. Assume that (II) holds. If a solution (u,ξ) of (1) exists,

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}g(s)ds + \int_0^t f(s)e^{(t-s)A}ds\,\xi, \quad t \in [0,T].$$

Imposing the last condition in (1), we obtain

$$\xi = Q^{-1}\{y - \int_{[0,T]}[e^{tA}u_0 + \int_0^t e^{(t-s)A}g(s)ds]d\mu(t)\}.$$

On the other hand, assume that (I) holds. Taking $g \equiv 0$ and $u_0 = 0$, we obtain

$$u(t) = \int_0^t f(s)e^{(t-s)A}ds\,\xi, \quad t \in [0,T], \tag{17}$$

and $Q\xi = y$, so that Q is onto X. Moreover, if $\xi \in X$, $Q\xi = 0$, and u is as in (17), (u,ξ) solves (1) with $g \equiv 0$, $u_0 = y = 0$. This implies that $\xi = 0$. □

As we shall see, in some significant cases, the space X_1 has finite dimension. Then, using the fact that a bounded operator in a finite dimensional space is surjective if and only if it is injective, one can show the following property of Fredholm type:

Theorem 2. *Let X be a complex Banach space, $A \in S(\omega,\phi)$, for some $\omega \in \mathbb{R}$ and $\phi \in (0,\pi/2)$, let $T \in \mathbb{R}^+$, μ a complex Borel measure in $[0,T]$ and let $f \in C([0,T])$. We assume the following:*

(a) $\int_{[0,T]} f(t)\chi_{(0,T]}(t)d\mu(t) \neq 0$.

(b) Let $\omega(\phi)$ be a real number as in Theorem 1 (I) (depending on ϕ, T, μ and f). We set

$$\sigma_1 := \sigma(A) \cap \overline{\Sigma(\omega(\phi) \wedge 0, \phi)}$$

and assume that σ_1 is a bounded spectral set for A.

We adopt the notations (13)-(14), and consider the following operator Q in the space X_1:

$$Q := \int_{[0,T]} (\int_0^t f(s)e^{(t-s)A_1}ds)d\mu(t).$$

Consider the three conditions:

(α) Q is invertible in $\mathcal{L}(X_1)$;

(β) consider the system

$$
\begin{cases}
u'(t) = Au(t) + f(t)\xi, \quad t \in [0,T], \\
\\
u(0) = 0, \\
\\
\int_{[0,T]} u(t)d\mu(t) = 0.
\end{cases} \tag{18}
$$

Then, there exists $n \in \mathbb{N}$, such that the system (18) has only the trivial solution in $C^1([0,T];\, D(A^n)) \times D(A^n)$;

(γ) consider the system

$$
\begin{cases}
u'(t) = Au(t) + f(t)\xi, \quad t \in [0,T], \\
\\
u(0) = 0, \\
\\
\int_{[0,T]} u(t)d\mu(t) = y.
\end{cases} \tag{19}
$$

Then, there exists $n \in \mathbb{N}$, such that, if $y \in D(A^n)$, (19) has (at least) one solution (u,ξ) in $[C^1([0,T];X) \cap C([0,T];D(A))] \times X$.

Then:

(I) if (α) holds, (1) has, at most, one solution (u,ξ) in $[C^1([0,T];X) \cap C([0,T];D(A))] \times X$, $\forall g \in C([0,T];X)$, $\forall u_0, y \in D(A)$; if, for some $\theta \in (0,1)$, $g \in C([0,T];X) \cap B([0,T];D_\theta(A))$, and $u_0, y \in D_{1+\theta}(A)$, such solution (u,ξ) exists, u' and Au belong to $B([0,T];D_\theta(A))$, and $\xi \in D_\theta(A)$.

(II) if X_1 is finite dimensional, (α), (β) and (γ) are equivalent.

If the assumptions of Theorem 2 are satisfied and $D(A)$ is compactly embedded into X, then X_1 is finite dimensional. In fact, $\forall \lambda \in \rho(A)$, $(\lambda - A)^{-1}$ is a compact operator in X, so that the projection P onto X_1 is compact. Therefore, if $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence in X_1, it admits a convergent subsequence, because $x_n = Px_n$, $\forall n \in \mathbb{N}$. See, also, for example, [5], chapter 5.

In order to illustrate Theorem 2, We consider the system

$$
(20) \qquad \begin{cases} D_t u(t,x) = D_x^2 u(t,x) + f(t)\xi(x) + g(t,x), \quad t \in [0,T], x \in [0,\pi] \\ \\ u(t,0) = u(t,\pi) = 0, \quad t \in [0,T], \\ \\ u(0,x) = u_0(x), \\ \\ \int_{[0,T]} u(t,x) d\mu(t) = y(x), \quad x \in [0,\pi], \end{cases}
$$

with u and ξ unknown, $f \in C([0,T])$ and μ complex Borel measure in $[0,T]$. We put

$$
(21) \qquad X := L^p(0,\pi),
$$

with $p \in [1,\infty)$. We consider the following operator

$$
\begin{cases} D(A) := W^{2,p}(0,\pi) \cap W_0^{1,p}(0,\pi), \\ \\ Au := D_x^2. \end{cases}
$$

It is easily seen that

$$
\sigma(A) = \{-n^2 : n \in \mathbb{N}\},
$$

and, for each $n \in \mathbb{N}$,

$$
(22) \qquad \begin{cases} Ker(n^2 + A) = \{c \sin(n\cdot) : c \in \mathbb{C}\}, \\ \\ (n^2 + A)(D(A)) = \{f \in X : \int_0^\pi f(x)\sin(nx)dx = 0\}. \end{cases}
$$

so that

$$
(23) \qquad Ker(n^2 + A) \oplus (n^2 + A)(D(A)) = X.
$$

Moreover, $A \in S(-1,\phi)$, $\forall \phi \in (0,\pi]$. If $\theta \in (0, 1/(2p))$, one has also

$$
D_\theta(A) = B^{2\theta}_{p,\infty}(0,\pi),
$$

and

$$D_{1+\theta}(A) = \{u \in B^{2+2\theta}_{p,\infty}(0,\pi) : u(0) = u(\pi) = 0\} = B^{2+2\theta}_{p,\infty}(0,\pi) \cap W^{1,p}_0(0,\pi).$$

(see [2], Theorem 3.5). Then, Theorem 2 has the following consequence:

Proposition 1. *We deal with system (20). Assume that* $f \in C([0,T])$ *and*

$$\int_{[0,T]} f(t)\chi_{(0,T]}(t)d\mu(t) \neq 0.$$

Consider the three conditions:

(α_1) $\forall n \in \mathbb{N}$, $\int_{[0,T]}(\int_0^t f(s)e^{-n^2(t-s)}ds)d\mu(t) \neq 0$;

(β_1) consider the system (20), with $g(t,\cdot) \equiv 0$ $\forall t \in [0,T]$, $u_0 = y = 0$. *Then, there exists* $n \in \mathbb{N}$, *such that, if* $u \in C^1([0,T]; W^{2n,p}(0,\pi))$, $u^{(2k)}(0) = u^{(2k)}(\pi) = 0$, *for each* $k = 0, ..., n-1$, $\xi \in W^{2n,p}(0,\pi)$, $\xi^{(2k)}(0) = \xi^{(2k)}(\pi) = 0$, *for each* $k = 0, ..., n-1$, *and* (u,ξ) *is a solution, then* $u(t,\cdot) = 0$ $\forall t \in [0,T]$ *and* $\xi = 0$.

(γ_1) Consider the system (20), with $g(t,\cdot) = 0$, $\forall t \in [0,T]$ *and* $u_0 = 0$. *Then, there exists* $n \in \mathbb{N}$, *such that, if* $y \in W^{2n,p}(0,\pi)$, $y^{(2k)}(0) = y^{(2k)}(\pi) = 0$, *for each* $k = 0, ..., n-1$, *there is (at least) one solution* (u,ξ) *in* $[C^1([0,T]; L^p(0,\pi)) \cap C([0,T]; W^{2p}(0,\pi))] \times L^p(0,\pi)$.

Then:

(I) if (α_1) *holds, (20) has, at most, one solution*

$$(u,\xi) \in [C^1([0,T]; L^p(0,\pi)) \cap C([0,T]; W^{2,p}(0,\pi))] \times L^p(0,\pi),$$

$\forall g \in C([0,T]; L^p(0,\pi))$, $\forall u_0, y \in W^{2,p}(0,T) \cap W^{1,p}_0(0,T)$; *if, for some* $\theta \in (0, 1/(2p))$, $g \in C([0,T]; L^p(0,\pi)) \cap B([0,T]; B^{2\theta}_{p,\infty}(0,\pi))$, *and* $u_0, y \in B^{2(1+\theta)}_{p,\infty}(0,\pi) \cap W^{1,p}_0(0,T)$, *such solution* (u,ξ) *exists,* u' *and* Au *belong to* $B([0,T]; B^{2\theta}_{p,\infty}(0,\pi))$, *and* $\xi \in B^{2\theta}_{p,\infty}(0,\pi)$.

(II) (α_1), (β_1) *and* (γ_1) *are equivalent.*

Proof We check that Theorem 2 is applicable.

We fix $\phi \in (0,\pi/2)$ and take $\omega(\phi)$ as in Theorem 1(I). We may assume that $\omega(\phi) \leq -1$ and we set

$$\sigma_1 := \{-j^2 : j \in \mathbb{N}, -j^2 \geq \omega(\phi)\} = \{-j_0^2, ..., -1\}. \tag{24}$$

σ_1 is a spectral set of A. If we put (as usual) $X_1 := P(X)$, with P as in (12), in force of (22)-(23) and Proposition A.2.2 in [7], we have that

$$X_1 = \{\sum_{j=1}^{j_0} c_j sin(j\cdot) : c_j \in \mathbb{C} \quad \forall j \in \{1, ..., j_0\}\}.$$

and the corresponding operator A_1 is such that

$$A_1(\sum_{j=1}^{j_0} c_j sin(j\cdot)) = -\sum_{j=1}^{j_0} j^2 c_j sin(j\cdot),$$

so that

$$Q(\sum_{j=1}^{j_0} c_j sin(j\cdot)) = \sum_{j=1}^{j_0} c_j \int_{[0,T]} (\int_0^t f(s)e^{-j^2(t-s)}ds)d\mu(t)sin(j\cdot).$$

So (α_1) implies condition (α) in Theorem 2. In fact, in this case, (α) and (α_1) are equivalent: if (α) holds, necessarily $\int_{[0,T]}(\int_0^t f(s)e^{-j^2(t-s)}ds)d\mu(t) \neq 0$, whenever $-j^2 \geq \omega(\phi)$. In case $-j^2 < \omega(\phi)$, we can apply Theorem 1, taking $X = \{c\sin(j\cdot) : c \in \mathbb{C}\}$, $A(c\sin(j\cdot)) = -j^2 c\sin(j\cdot)$, and Lemma 1, together implying that $\int_{[0,T]}(\int_0^t f(s)e^{-j^2(t-s)}ds)d\mu(t) \neq 0$.

Finally, we observe that (β_1) and (γ_1) are exactly (β) and (γ) in the specific case we are considering.

So the conclusion follows from Theorem 2.

We conclude with a result of approximation. We assume the following:

(H1) X is a complex Hilbert space with scalar product $(\cdot|\cdot)$, A is a self-adjoint operator in X, there exists $\nu \in \mathbb{R}$, such that $(Ax|x) \leq \nu\|x\|^2$ for every $x \in D(A)$ and $D(A)$ is compactly embedded in X.

If $(H1)$ holds, A is sectorial; in fact, one can easily verify that $A \in S(\omega, \phi)$, $\forall \omega > \nu$, $\forall \phi \in (0, \pi/2)$. Now, we fix λ_0 in (ν, ∞), and set

$$K := (\lambda_0 - A)^{-1}.$$

Then, K is a compact, self adjoint, injective operator. Owing to the classical theory concerning this class of operators (see [12], 93), in case X is infinite dimensional,

$$\sigma(K) \setminus \{0\} = \{\mu_j : j \in \mathbb{N}\},$$

with $\mu_j \in \mathbb{R}$ for each $j \in \mathbb{N}$ and $\lim_{j\to\infty}\mu_j = 0$. Moreover, each μ_j is an eigenvalue of K and the corresponding eigenspace Y_j is finite dimensional. These subspaces of X are pairwise orthogonal, and, if we indicate with P_j the orthogonal projection of X onto Y_j,

$$\lim_{n\to\infty} \| \sum_{j=1}^{n} P_j x - x\| = 0, \quad \forall x \in X.$$

It is also easily seen that

$$\sigma(A) = \{\lambda_j := \lambda_0 - \mu_j^{-1} : j \in \mathbb{N}\},$$

and, as A is sectorial,

$$\lim_{j\to\infty} \lambda_j = -\infty.$$

Now we consider system (1).

Theorem 3. *Suppose that (H1) holds. Again, fix $\phi \in (0, \pi/2)$. Let σ_1 be as in (11) and X_1 the corresponding subspace. Then:*

(I) X_1 is finite dimensional.

(II) Condition (α) in Theorem 2 is satisfied if and only if

$$q_j := \int_{[0,T]} (\int_0^t f(s)e^{\lambda_j(t-s)}ds)d\mu(t) \neq 0, \quad \forall j \in \mathbb{N}. \tag{25}$$

(III) Assume that (25) holds. Let $\theta \in (0,1)$ and consider problem (1), in case $g \in C([0,T];X) \cap B([0,T];D_\theta(A))$, and $u_0, y \in D_{1+\theta}(A)$. Then, there exists a unique solution (u,ξ) in $[C^1([0,T];X) \cap C([0,T];D(A))] \times X$. Moreover, u' and Au belong to $B([0,T];D_\theta(A))$, while $\xi \in D_\theta(A)$.

(IV) Assume that the assumptions of (III) are fulfilled. For every $n \in \mathbb{N}$, $\forall t \in [0,T]$, define

$$\xi_n := \sum_{j=1}^{n} q_j^{-1} P_j\{y - \int_{[0,T]} [e^{\lambda_j\tau}u_0 + \int_0^\tau e^{\lambda_j(\tau-s)}g(s)ds]d\mu(\tau)\}, \tag{26}$$

$$\begin{aligned} u_n(t) := \sum_{j=1}^{n} P_j\{e^{\lambda_j t}u_0 + \int_0^t e^{\lambda_j(t-s)}g(s)ds + q_j^{-1}\int_0^t f(s)e^{\lambda_j(t-s)}ds \\ \times\{y - \int_{[0,T]}[e^{\lambda_j\tau}u_0 + \int_0^\tau e^{\lambda_j(\tau-s)}g(s)ds]d\mu(\tau)\}\}. \end{aligned} \tag{27}$$

Then, $\forall \theta' \in (0, \theta)$,

$$\|u_n' - u'\|_{B([0,T];D_{\theta'}(A))} + \|Au_n - Au\|_{B([0,T];D_{\theta'}(A))} + \|\xi_n - \xi\|_{D_{\theta'}(A)} \to 0 \quad (n \to \infty).$$

We observe that we can apply Theorem 3 to the problem considered in Example 2.1, if we take $p = 2$, $\lambda_j = -j^2$ $(j \in \mathbb{N})$, $Y_j = \{c \sin(j\cdot) : c \in \mathbb{C}\}$, and

$$P_j f = \frac{2}{\pi} \int_0^{\pi} f(y) \sin(jy) dy \sin(j\cdot), \quad f \in L^2(0, \pi).$$

so that, if

$$q_j := \int_{[0,T]} (\int_0^t f(s) e^{-j^2(t-s)} ds) d\mu(t) \neq 0, \quad \forall j \in \mathbb{N},$$

$g \in C([0,T]; L^2(0,\pi)) \cap B([0,T]; B^{2\theta}_{2,\infty}(0,\pi))$ $(0 < \theta < 1/4)$, and $u_0, y \in B^{2(1+\theta)}_{2,\infty}(0,\pi)) \cap W^{1,2}_0(0,\pi)$, (u, ξ) is the solution of (20), ξ_n and u_n are as in (26)-(27), $\forall \theta' \in (0, \theta)$,

$$\|D_t u_n - D_t u\|_{B([0,T];B^{2\theta'}_{2,\infty}(0,\pi))} + \|D_x^2 u_n - D_x^2 u\|_{B([0,T];B^{2\theta'}_{2,\infty}(0,\pi))} + \|\xi_n - \xi\|_{B^{2\theta'}_{2,\infty}(0,\pi)} \to 0,$$

as $n \to \infty$.

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