Seminario di Analisi Matematica
Dipartimento di Matematica
dell'Università di Bologna

Anno Accademico 2010-2011

Fausto Ferrari

Some relations between fractional Laplace operators and Hessian operators

9 giugno 2011

Sunto

Dopo aver ricordato le numerose rappresentazioni del laplaciano frazionario e alcune sue importanti proprietà, verranno presentati alcuni recenti risultati ottenuti in collaborazione con Bruno Franchi e Igor Verbitsky sulle relazioni esistenti tra l'energia (delle funzioni k-convesse che si annullano all'infinito) associata all'operatore Hessiano di ordine k e l'energia di un opportuno operatore frazionario per la stessa funzione.

Verrà infine richiamata una formula di integrazione per parti del laplaciano frazionario di cui si fornirà una nuova dimostrazione elementare.

Abstract

After recalling the many representations of the fractional Laplace operator and some of its important properties, some recent results (proved in a joint work with Bruno Franchi and Igor Verbitsky) about the relations between the $k-$Hessian energy of the $k-$Hessian operator of a k convex function vanishing at infinity and the fractional energy of a particular fractional operator will be introduced.

Moreover we shall recall an integration by parts formula for the fractional Laplace operator giving a new simpler proof.

1. Introduction

In this note I would like to point out some properties of the fractional Laplace operators and moreover introduce a few recent results, obtained with Bruno Franchi and Igor Verbitsky see [7], about the relations between the fractional Laplace operators family and the Hessian operators family. I also will discuss shortly the nonlocal behavior of the fractional Laplace just to recall its main character.

Indeed, the fractional Laplace operator is not a differentiable operator. Usually it describes phenomena where, roughly speaking, the value of the operator applied to a function does not depend on the local behavior of the function, on the contrary, it depends on the global behavior of the function itself.

Some examples of applications may be found considering: the thin obstacle problem, phase transition problems, quasi-geostrophic flows, conformal geometry and many others subjects; see [16] and [13] for detailed references. Concerning the Hessian operators, we know that they naturally arise from the differential geometry and they are an example of nonlinear operators.

In [7] a first tentative to determine some relations between fractional Laplace operators and Hessian operators has been done. In particular, given positive integer number k, it was proved the existence of positive constants $C_{k,n}$, and $\alpha(k)$, such that for each $k-$convex function vanishing at infinity that satisfies some $\alpha/2-$subharmonic hypotheses the following inequality holds:

$$\int_{\mathbb{R}^n} (-(-\Delta)^{\alpha(k)/2} u)^{k+1} dx \leq C_{n,k} \int_{\mathbb{R}^n} -u F_k[u] dx.$$

The right hand side is the energy associated with the Hessian operator of order k, F_k. For further details see Theorem 6.1 in the Section 6 of this note . In case $k = 1$, then $\alpha(1) = 1$. Thus, for example, the previous inequality reduces to

$$\int_{\mathbb{R}^n} (-(-\Delta)^{\frac{1}{2}} u)^2 dx \leq C_{n,1} \int_{\mathbb{R}^n} -u \Delta u dx.$$

Then integrating by parts, and recalling that u vanishes at infinity it works out

$$\int_{\mathbb{R}^n} (-(-\Delta)^{\frac{1}{2}} u)^2 dx \leq C_{n,1} \int_{\mathbb{R}^n} \mid \nabla u \mid^2 dx.$$

Moreover, if k is a not to large integer, more precisely $1 \leq k \leq \frac{n}{2}$, then there exists a positive constant $c_{k,n}$ such that for every $k-$convex function vanishing at infinity

$$\int_{\mathbb{R}^n} -uF_k[u]dx \leq c_{k,n} \int_{\mathbb{R}^n} \mid (-\Delta)^{\frac{\alpha(k)}{2}} \mid^{k+1} dx, \tag{1}$$

see Theorem 6.2 for the precise statement. As a consequence, see Corollary 6.1, we deduce that for each $k-$convex function u a $\tilde{u}$ function exists such that the $\alpha(k)$ fractional energy of $\tilde{u}$ is equivalent to the $k-$Hessian energy and $c_1 \leq \frac{u}{\tilde{u}} \leq c_2$.

These results can be exploited for example to prove some results in potential theory, see Corollary 6.2, with the help of particular inequality, see Lemma 6.1, formula (24). In Section 7 a simple proof of this inequality, that was first proved in [5] and then in [9], will be showed. In Section 2 some charachterisations of the fractional Laplace operator and a physical motivation will be presented. In Section 3 we remind the recent Caffarelli Silvestre approach to fractional Laplace operators. In Section 4 we revisit some well-known properties of fractional Laplace operators, in particular the strong maximum principle, see [11], and some mean formulas that can be deduced from the simplest representation of fractional Laplace operators. These mean formulas type, as far as I know, can be considered as already known in literature, even if I would not know to cite any explicit reference, see in particular Theorem 4.2 and formula (1). The only one handbook that deals with this argument still remains the Landkof book, see [11]. In Section 5 the main definitions concerning the $k-$ Hessian operators are recalled. Eventually in Section 6 some results proved with Bruno Franchi and Igor Verbitsky are listed, see also [7].

2. somethings about fractional Laplace operators

We recall in short a physical motivation. Following [3], see also [4], a 2D model of the quasi-geostrophic active scalar equations is the following

$$\frac{\partial \theta}{\partial t} + \langle v, \nabla \theta \rangle = 0, \tag{2}$$

where the two-dimensional velocity, $v = (v_1, v_2)$ is determined from θ by a stream function $(v_1, v_2) = (-\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1})$ and the stream function ψ satisfies

$$(-\Delta)^{\frac{1}{2}} \psi = -\theta \tag{3}$$

The variable θ represents the potential temperature, v is the fluid velocity , and the stream function ψ can be identified with the pressure.

Notice that, in general, active scalar are the solutions of advection-diffusion equations with given divergence-free velocities, see [2], that determine their own velocity:

$$\psi = A(\theta),$$

$(v_1, v_2) = (-\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1})$, where the operator A must be nonlocal, otherwise the convective term $v \cdot \nabla\theta$ would vanish, see [2]. The choice to take $A^{-1} = -(-\Delta)^{1/2}$ is just determined by the necessity to deal with a nonlocal operator, as the fractional Laplace one.

These equations are derived from the more general quasigeostrophic approximation [12] for nonhomogeneous fluid flow in a rapidly rotating three-dimensional half-space with small Rossby and Ekman numbers; for the case of special solutions with constant potential vorticity in the interior and constant buoyancy frequency (normalized to one), the general quasigeostrophic equations reduce to the evolution equations for the temperature on the two-dimensional boundary given in (2)-(3).

Indeed, following one more time [3], there are analytic analogies between the $2D$ quasi-geostrophic active scalar equations (2) - (3) and the $3-D$ incompressible Euler equations in vorticity-stream

$$\frac{\partial \omega}{\partial t} + v \cdot \nabla\omega = (\nabla v)\omega, \tag{4}$$

where v is the three dimensional velocity field with $\mathrm{div} v = 0$, and$\omega = \mathrm{rot}(v)$ is the vorticity vector. On the other hand, deriving (2) with respect to the vector field $\nabla^\perp = (-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1})$, we get the system:

$$\frac{\partial \nabla^\perp\theta}{\partial t} + v \cdot \nabla\nabla^\perp\theta = (\nabla v)\nabla^\perp\theta,$$

where $v = \nabla^\perp\psi$ so that $\mathrm{div} v = 0$, that has the same structure of (4).

For this reason it results in some sense natural to identify $\nabla^\perp\theta$ with the vorticity vector ω and study (2) - (3) instead of the system (4).

It can be useful now to put in evidence the probabilistic interpretation of the fractional Laplace operator. Indeed, it is wellknow the relation between the Laplace operator and the Wiener process. Here we wish to point out as we can deduce the fractional operator starting from a probabilistic approach that partially help to understand the reasons wy this operator is interesting.

We consider a random walk on the lattice $h\mathbb{Z}^n$. In particular for each point $x \in h\mathbb{Z}^n$ we consider all the points $x + he_j$ and $x - he_j$ for $j \in \mathbb{N}$. We denote by $p(x,t)$ the probability that our particle lies at $x \in h\mathbb{Z}^n$ at time $t \in \tau\mathbb{Z}^n$. First we argue in one dimension assuming that the particle, of the random walk, can move at each step, independently to the previous steps, only by one step, then

$$\begin{aligned} p(x,t+\tau) = P(\{\omega :\ (x,t+\tau)\}) &= P(\{\omega : (x-h,t) \to (x,t+\tau)\}) \\ &+ P(\{\omega : (x+h,t) \to (x,t+\tau)\}) \\ &= P(\{\omega : (x,t+\tau)\} \mid \{\omega : (x-h,t)\})P(\{\omega :\ (x-h,t)\}) \\ &+ P(\{\omega : (x,t+\tau)\} \mid \{\omega(x+h,t)\})P(\{\omega :\ (x+h,t)\}) \end{aligned}$$

If, for instance, $P(\{\omega : (x,t+\tau)\} \mid \{\omega(x-h,t)\}) = P(\{\omega : (x,t+\tau)\} \mid \{\omega(x+h,t)\}) = \frac{1}{2}$ then

$$p(x,t+\tau) = P(\{\omega :\ (x,t+\tau)\}) = \frac{1}{2}p(x-h,t) + \frac{1}{2}p(x+h,t). \tag{5}$$

Thus, if we assume that p represents a density of probability regular enough we get

$$p(x,t+\tau) = p(x,t) + \frac{\partial p}{\partial t}(x,t)\tau + o(\tau)$$

as $\tau \to 0$ and

$$p(x \pm h,t) = p(x,t) \pm \frac{\partial p}{\partial x}(x,t)h + \frac{1}{2}\frac{\partial^2 p}{\partial x^2}(x,t)h^2 + o(h^2)$$

as $h \to 0$. Hence, substituting in (5), we get

$$\begin{aligned} &p(x,t) + \frac{\partial p}{\partial t}(x,t)\tau + o(\tau) = \frac{1}{2}\big(p(x,t) - \frac{\partial p}{\partial x}(x,t)h + \frac{1}{2}\frac{\partial^2 p}{\partial x^2}(x,t)h^2\big) \\ &+ \frac{1}{2}\big(p(x,t) + \frac{\partial p}{\partial x}(x,t)h + \frac{1}{2}\frac{\partial^2 p}{\partial x^2}(x,t)h^2\big) + o(h^2) = p(x,t) + \frac{1}{2}\frac{\partial^2 p}{\partial x^2}(x,t)h^2 + o(h^2). \end{aligned}$$

Hence

$$\frac{\partial p}{\partial t}(x,t) = \frac{1}{2}\frac{\partial^2 p}{\partial x^2}(x,t)\frac{h^2}{\tau} + o(\frac{h^2}{\tau}).$$

If we assume that $\frac{h^2}{\tau}$ is a positive constant and we denote $\frac{h^2}{\tau} = 2a$, then we get, when $\tau \to 0$ and $h \to 0$:

$$\frac{\partial p}{\partial t}(x,t) = a\frac{\partial^2 p}{\partial x^2}(x,t).$$

Arguing in dimension n we get that p satisfies the heat equation

$$\frac{\partial p}{\partial t}(x,t) = a\Delta p(x,t).$$

We would like to describe a process where the particle may jumps from a point to any other in the lattice, though with small probability if the new point is far away. Recalling the previous approach, and still arguing in dimension one, when we want to evaluate $p(\{\omega :\ (x,t+\tau)\})$ we have to take in account that the particle may freely jumps from the point x to the points $x+kh$. Hence

$$p(x,t+\tau) = P(\{\omega :\ (x,t+\tau)\}) = \sum_{k\in\mathbb{Z}} P(\{\omega : (x+hk,t) \to (x,t+\tau)\}).$$

This fact explains that we can not localize the random walk just considering a finite sum of terms. Moreover

$$\sum_{k\in\mathbb{Z}} P(\{\omega : (x+hk,t) \to (x,t+\tau)\})$$
$$= \sum_{k\in\mathbb{Z}} P(\{\omega : (x,t+\tau)\} \mid \{\omega : (x+kh,t)\})P(\{\omega :\ (x+kh,t)\}),$$

so that

$$P(\{\omega :\ (x,t+\tau)\}) = \sum_{k\in\mathbb{Z}} P(\{\omega : (x,t+\tau)\} \mid \{\omega : (x+kh,t)\})P(\{\omega :\ (x+kh,t)\}).$$

In order to reduce the difficulty of the problem, we assume that there exists a function $\mu : \mathbb{R} \to [0,+\infty)$ such that $\mu(x) = \mu(-x)$ and

$$\sum_{k\in\mathbb{Z}} \mu(k) = 1.$$

Moreover we assume that $P(\{\omega : (x,t+\tau)\} \mid \{\omega : (x+kh,t)\}) = \mu(k)$. That is probability that a particle that is in $x+kh$ at time t jumps to the point x at time $t+\tau$ is $\mu(k)$. Thus the probability that the particle is at $x \in h\mathbb{Z}^n$ at time $t+\tau$ is given by

$$p(x,t+\tau) = \sum_{k\in\mathbb{Z}} \mu(k)p(x+hk,t).$$

Hence

$$p(x,t+\tau)-p(x,t)=\sum_{k\in\mathbb{Z}}\mu(k)p(x+hk,t)-p(x,t)\sum_{k\in\mathbb{Z}}\mu(k)=\sum_{k\in\mathbb{Z}}\mu(k)(p(x+hk,t)-p(x,t)).$$

Moreover

$$\frac{p(x,t+\tau)-p(x,t)}{\tau}=\sum_{k\in\mathbb{Z}}\frac{\mu(k)}{\tau}(p(x+hk,t)-p(x,t)).$$

Analogously to the case of the random walk that generates the Laplace operator, we need to controle $\frac{\mu(k)}{\tau}$. If we take $\mu(y)=\mid y\mid^{-1-\alpha}$, for $y\neq 0$ and $\mu(0)=0$, for $\alpha\in(0,2)$, then

$$\sum_{k\in\mathbb{Z}}\mu(k)=c(\alpha).$$

Thus by fixing $\tau=h^\alpha$ we get that

$$\frac{\mu(k)}{\tau}=h\mu(hk),$$

so that

$$\frac{p(x,t+\tau)-p(x,t)}{\tau}=\sum_{k\in\mathbb{Z}}h\mu(hk)(p(x+hk,t)-p(x,t)).$$

The right hand side is an approximation of the Riemann sum so that, as $h\to 0$, we get

$$\frac{\partial p}{\partial t}(x,t)=\int_{\mathbb{R}}\frac{p(x+y,t)-p(x,t)}{\mid y\mid^{1+\alpha}}dy.$$

Arguing in $\mathbb{R}^n$ we define $\mu:\mathbb{R}^n\to[0,+\infty)$ such that $\mu(x)=\mu(-x)$,

$$\sum_{k\in\mathbb{Z}^n}\mu(k)=1.$$

Hence

$$p(x,t+\tau)=\sum_{k\in\mathbb{Z}^n}\mu(k)p(x+hk,t).$$

Taking $\mu(y)=\mid y\mid^{-n-\alpha}$, for $y\neq 0$ and $\mu(0)=0$, for $\alpha\in(0,2)$, then

$$\sum_{k\in\mathbb{Z}^n}\mu(k)=c(n,\alpha).$$

Thus by fixing $\tau=h^\alpha$ we get that

$$\frac{\mu(k)}{\tau}=h^n\mu(hk),$$

so that

$$\frac{p(x,t+\tau)-p(x,t)}{\tau}=\sum_{k\in\mathbb{Z}^n} h^n\mu(hk)(p(x+hk,t)-p(x,t)).$$

The right hand side is an approximation of the Riemann sum in $\mathbb{R}^n$ so that, as $h\to 0$, we get

$$\frac{\partial p}{\partial t}(x,t)=\int_{\mathbb{R}^n}\frac{p(x+y,t)-p(x,t)}{\mid y\mid^{n+\alpha}}dy.$$

We can define for every continuous function $f:\mathbb{R}^n\to\mathbb{R}$ such that

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^n\setminus\{|y|\le\epsilon\}}\frac{f(x+y)-f(x)}{\mid y\mid^{n+\alpha}}dy\in\mathbb{R}$$

and

$$\int_{\mathbb{R}^n}\frac{\mid f(y)\mid}{(1+\mid y\mid^2)^{\frac{n+\alpha}{2}}}dy$$

converges for $\alpha\in(0,2)$, the operator

$$\mathcal{L}_\alpha f(x)=\int_{\mathbb{R}^n}\frac{f(y)-f(x)}{\mid x-y\mid^{n+\alpha}}dy.$$

On the other hand, the fractional operator $(-\Delta)^{\alpha/2}$, $(\alpha\in(0,2))$ defined using the Fourier transform as follows, see [11], [14]

$$(-\Delta)^{\alpha/2}=\mathcal{F}^{-1}(\mid\xi\mid^\alpha\mathcal{F}f)$$

can be also represented as the principal value of the singular integral

$$(-\Delta)^{\alpha/2}f=\int_{\mathbb{R}^n}\frac{f(x)-f(y)}{\mid x-y\mid^{n+\alpha}}dy,$$

see e.g. [13] and [17] for a proof. In particular $\mathcal{L}_\alpha f=-(-\Delta)^{\alpha/2}f$. It is interesting to note the relation of $(-\Delta)^{\alpha/2}f$ with the Hessian matrix and the Laplace operator too. Indeed, assume that $u\in C^2(\mathbb{R}^n)$, then

$$\int_{B_R(x)\setminus\{|x-y|\le\epsilon\}}\frac{f(x)-f(y)}{\mid x-y\mid^{n+\alpha}}dy=\int_{B_R(x)\setminus\{|x-y|\le\epsilon\}}\frac{f(x)-f(y)-\langle\nabla u(x),y-x\rangle}{\mid x-y\mid^{n+\alpha}}dy,$$

because

$$\int_{B_R(x)\setminus\{|x-y|\le\epsilon\}}\frac{\langle\nabla u(x),y-x\rangle}{\mid x-y\mid^{n+\alpha}}dy=0.$$

Moreover

$$f(x)-f(y)-\langle\nabla u(x),y-x\rangle=\frac{1}{2}\langle D^2u(\eta)(y-x),(y-x)\rangle,$$

$\eta=x+\theta(y-x)$, for some $\theta\in[0,1]$.

Hence

$$\mid f(x)-f(y)-\langle\nabla u(x),y-x\rangle\mid=\frac{1}{2}\mid\langle D^2u(\eta)(y-x),(y-x)\rangle\mid\leq\frac{1}{2}\mid D^2u\mid_{B_1(x)}\mid y-x\mid^2,$$

and

$$\int_{B_R(x)\setminus\{|x-y|\leq\epsilon\}}\frac{\mid f(x)-f(y)-\langle\nabla u(x),y-x\rangle\mid}{\mid x-y\mid^{n+\alpha}}dy$$
$$\leq C\mid D^2u\mid_{B_1(x)}\int_{B_R(x)\setminus\{|x-y|\leq\epsilon\}}\mid x-y\mid^{2-n-\alpha}dy<+\infty.$$

More precisely, let us consider

$$-\int_{B_R(x)\setminus\{|x-y|\leq\epsilon\}}\frac{\Delta f(y)}{\mid x-y\mid^{n-2+\alpha}}dy=-\int_{B_R(x)\setminus\{|x-y|\leq\epsilon\}}\frac{\Delta(f(x)-f(y))}{\mid x-y\mid^{n-2+\alpha}}dy$$

and integrate by parts

$$-\int_{B_R(x)\setminus\{|x-y|\leq\epsilon\}}\frac{\Delta(f(x)-f(y))}{\mid x-y\mid^{n+2-\alpha}}dy=\int_{B_R(x)\setminus\{|x-y|\leq\epsilon\}}\langle\nabla(f(x)-f(y)),\nabla\mid x-y\mid^{-n+2-\alpha}\rangle dy$$
$$-\int_{\partial(B_R(x)\setminus\{|x-y|\leq\epsilon\})}\langle\nabla(f(x)-f(y)),n\rangle\mid x-y\mid^{-n+2-\alpha}d\mathcal{H}^{n-1}$$
$$=-\int_{B_R(x)\setminus\{|x-y|\leq\epsilon\}}(f(x)-f(y))\Delta(\mid x-y\mid^{-n+2-\alpha})dy$$
$$+\int_{\partial(B_R(x)\setminus\{|x-y|\leq\epsilon\})}\langle\nabla\mid x-y\mid^{-n+2-\alpha},n\rangle(f(x)-f(y))d\mathcal{H}^{n-1}$$
$$-\int_{\partial(B_R(x)\setminus\{|x-y|\leq\epsilon\})}\langle\nabla(f(x)-f(y)),n\rangle\mid x-y\mid^{-n+2-\alpha}d\mathcal{H}^{n-1}$$
$$=-\int_{B_R(x)\setminus\{|x-y|\leq\epsilon\}}(f(x)-f(y))\Delta(\mid x-y\mid^{-n+2-\alpha})dy$$
$$-(-n+2-\alpha)\int_{\partial(B_R(x)\setminus\{|x-y|\leq\epsilon\})}\mid x-y\mid^{-n+2-\alpha}\langle\frac{x-y}{\mid x-y\mid},n\rangle\frac{f(x)-f(y)}{\mid x-y\mid}d\mathcal{H}^{n-1}$$
$$+\int_{\partial(B_R(x)\setminus\{|x-y|\leq\epsilon\})}\langle\nabla f(y),n\rangle\mid x-y\mid^{-n+2-\alpha}d\mathcal{H}^{n-1}$$

We remark that if f is bounded

$$\Big| \int_{\partial B_R(x)} \mid x-y \mid^{-n+1-\alpha} \langle \frac{x-y}{\mid x-y \mid}, n \rangle (f(x)-f(y)) d\mathcal{H}^{n-1} \Big|$$
$$\leq R^{-n+1-\alpha} \int_{\partial B_R(x)} (\mid f(y) \mid + \mid f(x) \mid) d\mathcal{H}^{n-1} \leq C \mid\mid f \mid\mid_{L^\infty(\mathbb{R}^n)} R^{-\alpha} \to 0$$

as $R \to 0$. Moreover if $\frac{|\nabla f(x-y)|}{|y|} \to 0$, as $\mid x-y \mid \to \infty$, then

$$\int_{\partial B_R(x)} \langle \nabla f(y), n \rangle \mid x-y \mid^{-n+2-\alpha} d\mathcal{H}^{n-1} \to 0,$$

as $R \to \infty$. As a consequence

$$- \int_{\mathbb{R}^n \setminus \{|x-y| \leq \epsilon\}} \frac{\Delta(f(x)-f(y))}{\mid x-y \mid^{n+2-\alpha}} dy = -c(n,\alpha) \int_{\mathbb{R}^n \setminus \{|x-y| \leq \epsilon\}} \frac{f(x)-f(y)}{\mid x-y \mid^{n+\alpha}} dy$$
$$- (-n+2-\alpha) \int_{|x-y|=\epsilon} \mid x-y \mid^{-n+2-\alpha} \langle \frac{x-y}{\mid x-y \mid}, n \rangle \frac{f(x)-f(y)}{\mid x-y \mid} d\mathcal{H}^{n-1}$$
$$+ \int_{|x-y|=\epsilon} \langle \nabla f(y), n \rangle \mid x-y \mid^{-n+2-\alpha} d\mathcal{H}^{n-1}$$

On the other hand

$$- (-n+2-\alpha) \int_{|x-y|=\epsilon} \mid x-y \mid^{-n+2-\alpha} \langle \frac{x-y}{\mid x-y \mid}, n \rangle \frac{f(x)-f(y)}{\mid x-y \mid} d\mathcal{H}^{n-1}$$
$$+ \int_{|x-y|=\epsilon} \langle \nabla f(y), n \rangle \mid x-y \mid^{-n+2-\alpha} d\mathcal{H}^{n-1}$$
$$= (-n+2-\alpha) \int_{|x-y|=\epsilon} \mid x-y \mid^{-n+2-\alpha} \langle \frac{x-y}{\mid x-y \mid}, n \rangle \frac{\langle \nabla f(y), y-x \rangle}{\mid x-y \mid} d\mathcal{H}^{n-1}$$
$$+ (-n+2-\alpha) \int_{|x-y|=\epsilon} \mid x-y \mid^{-n+2-\alpha} \langle \frac{x-y}{\mid x-y \mid}, n \rangle \frac{\langle D^2 f(\eta)(y-x), (y-x) \rangle}{\mid x-y \mid} d\mathcal{H}^{n-1}$$
$$+ \int_{|x-y|=\epsilon} \langle \nabla f(y), n \rangle \mid x-y \mid^{-n+2-\alpha} d\mathcal{H}^{n-1},$$

where $\eta = x + \theta(y-x)$, $\theta \in (0,1)$ and

$$\Big| \int_{|x-y|=\epsilon} \mid x-y \mid^{-n+2-\alpha} \langle \frac{x-y}{\mid x-y \mid}, n \rangle \frac{\langle D^2 f(\eta)(y-x), (y-x) \rangle}{\mid x-y \mid} d\mathcal{H}^{n-1} \Big| \leq C\epsilon^{2-\alpha} \to 0,$$

as $\epsilon \to 0$ and recalling that ∇f is bounded

$$\Big| \int_{|x-y|=\epsilon} \langle \nabla f(y), n \rangle \mid x-y \mid^{-n+2-\alpha} d\mathcal{H}^{n-1} \Big| \leq C\epsilon^{1-\alpha} \to 0$$

as $\epsilon \to 0$.

Hence we can conclude that if f has a suitable behavior at infinity, then:

$$\int_{\mathbb{R}^n} \frac{\Delta f(y)}{\mid x-y \mid^{n+2-\alpha}} dy = -c(n,\alpha) \int_{\mathbb{R}^n} \frac{f(x)-f(y)}{\mid x-y \mid^{n+\alpha}} dy.$$

Indeed the fractional Laplace operator can be introduced using the Riesz potential, see [14] and [11]. Let μ a measure. We define for $0<\alpha<n$ the Riesz potential

$$I_\alpha \mu(x) = \gamma(n,\alpha) \int \frac{\mu(y)}{\mid x-y \mid^{n-\alpha}}.$$

Let f be subharmonic function vanishing at ∞. Then solution of $-\Delta f = \mu \geq 0$. Then $f(x) = -(\Delta)^{-1}\mu = -I_2\mu$. Hence we can define $(-\Delta)^{\alpha/2}$ as that operator that for every $f \in \Phi_0(\mathbb{R}^n)$

$$-(-\Delta)^{\alpha/2} f = (-\Delta)^{\alpha/2}(\Delta)^{-1}\mu = I_{2-\alpha}\mu \geq 0.$$

Hence

$$(-\Delta)^{\alpha/2} f = -I_{2-\alpha}\mu = -\gamma(n,\alpha) \int_{\mathbb{R}^n} \frac{\Delta f(y)}{\mid x-y \mid^{n-\alpha}} dy.$$

3. The Caffarelli-Silvestre approach to the fractional Laplace operator

In [13] and [1] it has been described an interesting definition of the fractional Laplace operator. In particular it was proved that any fractional Laplace operator can be defined as a weighted normal derivative of a weighted divergence operator in larger dimension. More precisely. Let f be a given C^2 function in $\mathbb{R}^n$. Let v a solution of the following Dirichlet problem in the semi-hyperspace $\mathbb{R}^n \times \mathbb{R}^+$ for a suitable $a \in \mathbb{R}$, where $x \in \mathbb{R}^n$ and $y > 0$:

$$\begin{cases} \operatorname{div}_{x,y}(y^a \nabla v(x,y)) = 0, \;\; \mathbb{R}^n \times \mathbb{R}^+ \\ v(x,0) = f(x), \end{cases} \tag{6}$$

$f \in C^2(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|^2)^{\frac{n+2s}{2}}} dx < \infty$, $0 < s \leq 1$. Let us define the following map T_a

$$f \to -\lim_{y\to 0} y^a \frac{\partial v(x,y)}{\partial y}.$$

It can be proved that, up to a multiplicative constant:

$$(-\Delta)^s f = T_a f,$$

where $s = \frac{1-a}{2}$.

There are several aspect to be verified, however heuristically and formally it is easy to recognize, for example, that

$$(T_0)^2 = -\Delta.$$

Indeed, if v satisfies (6) when $a = 0$, then

$$\frac{\partial^2 v}{\partial y^2}(x, y) = -\Delta_n v(x, y). \tag{7}$$

Then

$$T_0 f(x) = -\frac{\partial v}{\partial y}(x, 0),$$

and since $\frac{\partial v}{\partial y}$ is still a solution of (6) with Dirichlet datum $-\frac{\partial v}{\partial y}(x, 0)$, it follows from (7) that

$$\frac{\partial}{\partial y}(\frac{\partial v}{\partial y}(x, y)) = -\Delta_n v(x, y),$$

that is

$$\lim_{y\to 0} \frac{\partial^2 v}{\partial y^2}(x, y) = T_0^2 f(x) = -\Delta_n v(x, 0) = -\Delta f(x).$$

4. Maximum principle for fractional Laplace operators

We say that u is $s-$fractional subharmonic if $u \in C^2(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \frac{|u(x)|}{(1+|x|^2)^{\frac{n+2s}{2}}} < \infty$ and for every $x \in \mathbb{R}^n$

$$-(-\Delta)^s u \geq 0.$$

Analogously, we say that u is $s-$fractional superharmonic if $u \in C^2(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \frac{|u(x)|}{(1+|x|^2)^{\frac{n+2s}{2}}} < \infty$ and for every $x \in \mathbb{R}^n$

$$-(-\Delta)^s u \leq 0.$$

The operator $(-\Delta)^s u$ satisfies the following strong maximum principle.

Theorem 4.1. *Let $u \in C^2(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \frac{|u(x)|}{(1+|x|^2)^{\frac{n+2s}{2}}} < \infty$. If $-(-\Delta)^s u \geq 0$ then u can not assume a maximum in $\mathbb{R}^n$, unless u is constant in $\mathbb{R}^n$.*

Proof. Notice that

$$
\begin{aligned}
(-\Delta)^s u &= \int_{B_\epsilon(x)} \frac{u(x)-u(y)}{\mid x-y \mid^{n+2s}}dy + \int_{\mathbb{R}^n\setminus B_\epsilon(x)} \frac{u(x)-u(y)}{\mid x-y \mid^{n+2s}}dy \\
&= \int_{B_\epsilon(x)} \frac{u(x)-u(y)}{\mid x-y \mid^{n+2s}}dy + c(n,s)\epsilon^{-2s}u(x) - \int_{\mathbb{R}^n\setminus B_\epsilon(x)} \frac{u(y)}{\mid x-y \mid^{n+2s}}dy.
\end{aligned} \tag{8}
$$

Assume that there exists a point $x_0 \in \mathbb{R}^n$ such that $\sup_{\mathbb{R}^n} u = u(x_0)$. Then the function $u - u(x_0)$ is still a s-fractional sub-harmonic function. In particular for $x = x_0$ we get:

$$
0 \geq (-\Delta)^s(u-u(x_0))(x_0) = \int_{B_\epsilon(x)} \frac{u(x_0)-u(y)}{\mid x_0-y \mid^{n+2s}}dy - \int_{\mathbb{R}^n\setminus B_\epsilon(x_0)} \frac{u(y)-u(x_0)}{\mid x_0-y \mid^{n+2s}}dy \tag{9}
$$

Hence

$$
\int_{\mathbb{R}^n\setminus B_\epsilon(0)} \frac{u(y)-u(x_0)}{\mid x_0-y \mid^{n+2s}}dy \geq \int_{B_\epsilon(x)} \frac{u(x_0)-u(y)}{\mid x_0-y \mid^{n+2s}}dy.
$$

Thus, recalling that $u(x) \leq u(x_0)$, we get that

$$
0 \geq \int_{\mathbb{R}^n\setminus B_\epsilon(x_0)} \frac{u(y)-u(x_0)}{\mid x_0-y \mid^{n+2s}}dy \geq \int_{B_\epsilon(x_0)} \frac{u(x_0)-u(y)}{\mid x_0-y \mid^{n+2s}}dy \geq 0,
$$

that is $u \equiv u(x_0)$.

□

Theorem 4.2. *Let $u \in C^2(\mathbb{R}^n)$ be a $s-$ (sub, super) harmonic function in $\mathbb{R}^n$, then there exist a positive constant $c = c(c, s)$*

$$
u(x) = (\leq, \geq) \lim_{\epsilon \to 0} c^{-1}\epsilon^{2s} \int_{\mathbb{R}^n\setminus B_\epsilon(x)} \frac{u(y)}{\mid x-y \mid^{n+2s}}dy.
$$

Proof. We shall prove only the case when u is $s-$harmonic. We recall that

$$
\begin{aligned}
(-\Delta)^s u(x) &= \int_{B_\epsilon(x)} \frac{u(x)-u(y)}{\mid x-y \mid^{n+2s}}dy + \int_{\mathbb{R}^n\setminus B_\epsilon(x)} \frac{u(x)-u(y)}{\mid x-y \mid^{n+2s}}dy \\
&= \int_{B_\epsilon(x)} \frac{u(x)-u(y)}{\mid x-y \mid^{n+2s}}dy + C(n,s)\epsilon^{-2s}u(x) - \int_{\mathbb{R}^n\setminus B_\epsilon(x)} \frac{u(y)}{\mid x-y \mid^{n+2s}}dy.
\end{aligned} \tag{10}
$$

Thus

$$
\begin{aligned}
C(n,s)u(x) &= \epsilon^{2s} \int_{\mathbb{R}^n\setminus B_\epsilon(x)} \frac{u(y)}{\mid x-y \mid^{n+2s}}dy - \epsilon^{2s} \int_{B_\epsilon(x)} \frac{u(x)-u(y)}{\mid x-y \mid^{n+2s}}dy \\
&= \epsilon^{2s} \int_{\mathbb{R}^n\setminus B_\epsilon(x)} \frac{u(y)}{\mid x-y \mid^{n+2s}}dy + \epsilon^{2s} \int_{B_\epsilon(x)} \sum_{i=1}^n \partial_i^2 u(x) \frac{(x_i-y_i)^2}{\mid x-y \mid^{n+2s}}dy + o(\epsilon^2).
\end{aligned} \tag{11}
$$

Notice that

$$k(\epsilon)=\int_{B_\epsilon(x)}\frac{(x_i-y_i)^2}{\mid x-y\mid^{n+2s}}dy=\frac{1}{n}\int_{B_\epsilon(x)}\frac{\sum_{i=1}^n(x_i-y_i)^2}{\mid x-y\mid^{n+2s}}dy.$$

Moreover

$$\begin{aligned}\int_{B_\epsilon(x)}\frac{\sum_{i=1}^n(x_i-y_i)^2}{\mid x-y\mid^{n+2s}}dy&=\int_{B_\epsilon(x)}\mid x-y\mid^{-n-2s+2}dy\\ &=c_n'\int_0^\epsilon t^{-1-2s+2}dt=c_n\epsilon^{2-2s}.\end{aligned}\tag{12}$$

As a consequence

$$C(n,s)u(x)=\epsilon^{2s}\int_{\mathbb{R}^n\setminus B_\epsilon(x)}\frac{u(y)}{\mid x-y\mid^{n+2s}}dy+\epsilon^{2s}k(\epsilon)\Delta u(x)+o(\epsilon^2).\tag{13}$$

We remark, recalling (12), that

$$\epsilon^{2s}k(\epsilon)=c_n\epsilon^2.$$

Hence, as $\epsilon\to 0$ we get

$$u(x)=\frac{1}{C(n,s)}\lim_{\epsilon\to 0}\epsilon^{2s}\int_{\mathbb{R}^n\setminus B_\epsilon(x)}\frac{u(y)}{\mid x-y\mid^{n+2s}}dy,$$

because $\epsilon^{2s}k(\epsilon)\Delta u(x)\to 0$, as $\epsilon\to 0$. Notice that, by a re-scaling argument,

$$\int_{\mathbb{R}^n\setminus B_\epsilon(x)}\frac{1}{\mid x-y\mid^{n+2s}}dy=C(n,s)\epsilon^{-2s}.$$

It is worth to say, in any case, that

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_\epsilon(x)}\frac{1}{\mid x-y\mid^{n+2s}}dy&\equiv\int_{|x-y|>\epsilon}\frac{1}{\mid x-y\mid^{n+2s}}dy\\ &=\int_\epsilon^{+\infty}\int_{|x-y|=t}\frac{1}{t^{n+2s}}d\mathcal{H}^{n-1}dt\equiv C\int_\epsilon^{+\infty}\frac{t^{n-1}}{t^{n+2s}}dt=C(n,s)\epsilon^{-2s}.\end{aligned}\tag{14}$$

Hence

$$u(x)=\lim_{\epsilon\to 0}⨍_{\mathbb{R}^n\setminus B_\epsilon(x)}u(y)d\mu^x(y)=\lim_{\epsilon\to 0}\frac{1}{\mu^x(\mathbb{R}^n\setminus B_\epsilon(x))}\int_{\mathbb{R}^n\setminus B_\epsilon(x)}u(y)d\mu^x(y)=\int_{\mathbb{R}^n\setminus B_\epsilon(x)}u(y)d\nu^{x,\epsilon}(y),$$

where

$$d\mu^x=\frac{1}{\mid x-y\mid^{n+2s}}dy,$$

and for every measurable set $\Omega\subseteq\mathbb{R}^n$

$$\int_{\Omega\setminus B_\epsilon(x)}u(y)d\mu^x(y)=\int_{\Omega\setminus B_\epsilon(x)}\frac{u(y)}{\mid x-y\mid^{n+2s}}dy,$$

$$\nu^{x,\epsilon} = C(n,s)^{-1}\epsilon^{2s}\mu^x = \frac{\mu^x}{\mu^x(\mathbb{R}^n \setminus B_\epsilon(x))},$$

$$\int_{\mathbb{R}^n\setminus B_\epsilon(x)} d\nu^x(y) = 1.$$

□

It can be checked moreover that, if $u : \mathbb{R}^n \to \mathbb{R}$ is a continuous function such that $\int_{\mathbb{R}^n} \frac{|u|}{(1+|x|^2)^{\frac{n+2s}{2}}} < \infty$ and for every $x \in \mathbb{R}^n$

$$\lim_{\epsilon\to 0^+} \frac{u(x) - \frac{1}{C(n)\epsilon^{-2s}}\int_{\mathbb{R}^n\setminus B_\epsilon(x)} u dx}{\epsilon^{2s}} = 0, \tag{15}$$

then $(-\Delta)^s u = 0$ in $\mathbb{R}^n$.

Indeed

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus B_\epsilon(x)} \frac{u(x)-u(y)}{\mid x-y\mid^{n+2s}}dy = \int_{\mathbb{R}^n\setminus B_\epsilon(x)} \frac{u(x)}{\mid x-y\mid^{n+2s}}dy - \int_{\mathbb{R}^n\setminus B_\epsilon(x)} \frac{u(y)}{\mid x-y\mid^{n+2s}}dy\\
&= u(x)\mu^x(\mathbb{R}^n\setminus B_\epsilon(x)) - \int_{\mathbb{R}^n\setminus B_\epsilon(x)} \frac{u(y)}{\mid x-y\mid^{n+2s}}dy\\
&= \mu^x(\mathbb{R}^n\setminus B_\epsilon(x))\left(u(x) - \frac{1}{\mu^x(\mathbb{R}^n\setminus B_\epsilon(x))}\int_{\mathbb{R}^n\setminus B_\epsilon(x)} u(y)d\mu^x(y)\right)\\
&= C(n,s)\epsilon^{-2s}\left(u(x) - \frac{1}{\mu^x(\mathbb{R}^n\setminus B_\epsilon(x))}\int_{\mathbb{R}^n\setminus B_\epsilon(x)} u(y)d\mu^x(y)\right).
\end{aligned}$$

Thus recalling (15) we get that

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^n\setminus B_\epsilon(x)} \frac{u(x)-u(y)}{\mid x-y\mid^{n+2s}}dy = 0 = (-\Delta)^s u(x).$$

5. $k-$Hessian operators

Let $1 \le k \le n$. The $k-$th Hessian operators can be defined as the symmetric elementary functions of the eigenvalues of the Hessian matrix of a C^2 function u. In particular, denoting by $\lambda_i(x)$ the eigenvalues, $i = 1,\dots,n$, of the matrix $D^2u(x)$,

(1) $F_1[D^2u](x) = \sum_{i=1}^n \lambda_i(x) = \mathrm{Tr}(D^2u) = \Delta u$;
(2) $F_2[D^2u(x)] = \sum_{1\le i_1<i_2\le n} \lambda_{i_1}(x)\lambda_{i_2}(x)$;
- ...
(k) $F_k[D^2u(x)] = \sum_{1\le i_1<i_2<\cdots<i_k\le n} \lambda_{i_1}(x)\cdots\lambda_{i_k}(x)$;

- ...

(n) $F_n[D^2u(x)] = \sum_{1\le i_1<i_2<\cdots<i_n\le n} \lambda_{i_1}(x)\cdots\lambda_{i_n}(x) = \det(D^2u(x))$.

In 2008 Verbitsky, see [17], proved that the following Hessian Schwarz inequality, for every $k-$ convex functions u, v with zero boundary values and $F_k[u]$, $F_k[v]$ are Hessian measures, $k = 1, 2, \ldots, n$

$$\int_\Omega \mid v \mid F_k[u] \le \left(\int_\Omega \mid u \mid F_k[u]\right)^{\frac{k}{k+1}} \left(\int_\Omega \mid v \mid F_k[v]\right)^{\frac{1}{k+1}}. \tag{16}$$

We recall that an upper semicontinuous function $u : \Omega \to [-\infty, +\infty)$ is $k-$convex in Ω, see [15], if $F_k[q] \ge 0$ for any quadratic polynomial q such that $u - q$ has a local finite maximum in Ω $(1 \le k \le n)$.

A function $C^2_{loc}(\Omega)$ is k convex if and only if

$$F_j[u] \ge 0$$

in Ω, $j = 1, \ldots, k$. We recall that $C^2_{loc}(\Omega)$ means that the C^2 norms could not be finite in Ω. We denote by $\Phi^k(\Omega)$ the class of all $k-$convex functions in Ω not identically equal to $-\infty$ in each component of Ω).

$$\Phi^n(\Omega) \subset \Phi^{n-1}(\Omega) \subset \cdots \subset \Phi^1(\Omega).$$

Notice that $\Phi^1(\Omega)$ are the classical subharmonic functions in Ω, while $\Phi^n(\Omega)$ are the convex functions.

We say that a bounded set Ω is a uniformly $k - 1$ convex in $\mathbb{R}^n$, if $H_j(\partial\Omega) > 0$, for $j = 1, \ldots, k - 1$; where $H_j(\partial\Omega)$ denotes the j-mean curvature of the boundary $\partial\Omega$, that is, $H_j(\partial\Omega) = c(n, j) \sum_{1\le i_1<i_2<\cdots<i_j\le n-1} \kappa_{i_1}(x)\cdots\kappa_{i_j}(x)$ where κ_i, $i = 1, \ldots, n - 1$ are the principal curvatures of $\partial\Omega$.

It can be proved that (Trudinger-Wang) that for each $u \in \Phi^k(\Omega)$, there exists a non-negative Borel measure $\mu_k[u]$ in Ω such that

- $\mu_k[u] = F_k[u]$ for $u \in C^2(\Omega)$,
- if $\{u_m\}_{m\in\mathbb{N}}$ is a sequence in $\Phi^k(\Omega)$ converging in L^1_{loc} to $u \in \Phi^k(\Omega)$, then the correspondeing measures $\mu_k[u_m]$ converge weakly to $\mu_k[u]$. The measure $\mu_k[u]$ is called the $k-$Hessian measure associated with $u \in \Phi^k(\Omega)$.

For a positive measure μ on $\mathbb{R}^n$, $p>1$, $\alpha>0$, Wolff's potential is defined as

$$W_{\alpha,p}\mu(x)=\int_0^{+\infty}\left[\frac{\mu(B_r(x))}{r^{n-\alpha p}}\right]^{\frac{1}{p-1}}\frac{dr}{r},\ \ x\in\mathbb{R}^n.$$

This potential can be defined also in bounded domains, for $0<R\leq 2\mathrm{diam}(\Omega)$:

$$W^R_{\alpha,p}\mu(x)=\int_0^{R}\left[\frac{\mu(B_r(x))}{r^{n-\alpha p}}\right]^{\frac{1}{p-1}}\frac{dr}{r},\ \ x\in\Omega.$$

The Wolff's inequality states that there exist $C_1,C_2>0$ such that

$$C_1\int_{\mathbb{R}^n}W_{\alpha,p}\mu d\mu\leq\int_{\mathbb{R}^n}\mid(-\Delta)^{-\frac{\alpha}{2}}\mid^{p'}dx\leq C_2\int_{\mathbb{R}^n}W_{\alpha,p}\mu d\mu. \tag{17}$$

For Hessian operators: $\alpha=\frac{2k}{k+1}$, $p=k+1$.

Associate to the $k-$Hessian operators for a k convex function $u\in C^2(\Omega)$ the $k-$Hessian energy is defined as follows

$$\mathcal{E}_k[u]=\int_\Omega -uF_k[u]dx.$$

For example, when $k=1$, $F_1[u]=\Delta u$. Hence integrating by parts we get

$$\mathcal{E}_1[u]=\int_\Omega\mid\nabla u\mid^2dx,$$

for $u\in W^{1,2}_0(\Omega)$. We shall denote with $\Phi^k_0(\Omega)$ the cone of the $k-$convex functions with zero boundary values. We want to study relations between the Hessian energy $\mathcal{E}_k[u]$ and the fractional Sobolev energy

$$E_{\alpha,p}[u]=\int_{\mathbb{R}^n}\mid(-\Delta)^{\alpha/2}u\mid^p dx,$$

where $\alpha=\frac{2k}{k+1}$ and $p=k+1$. In this case we simply write

$$E_k[u]=E_{\frac{2k}{k+1},k+1}[u].$$

6. MAIN RESULTS

In this paragraph we list some recent results obtained with Bruno Franchi and Igor Verbitsky, see [7] for the proofs.

Theorem 6.1. *Let $u\in C^2(\mathbb{R}^n)$ be a $k-$convex function on $\mathbb{R}^n$ vanishing at ∞, where $1\leq k<\frac{n}{2}$. Let $\alpha=\frac{2k}{k+1}$. If*

(i) $-(-\Delta)^{\alpha/2}u\geq 0$,

(ii) $(-\Delta)^{\alpha/2}[-(-\Delta)^{\alpha/2}u]^k \geq 0$,

then there exists a positive constant $C_{k,n}$ *such that*

$$\int_{\mathbb{R}^n} \left(-(-\Delta)^{\alpha/2}u\right)^{k+1} dx \leq C_{k,n} \int_{\mathbb{R}^n} -u\, F_k[u]\, dx. \tag{18}$$

The proof of the Theorem 6.1 is based on a duality argument that reduces the problem to prove the following inequality:

$$\int_{\mathbb{R}^n} (-\Delta)^{\alpha/2}(-u)\, \phi dx \leq C_{k,n} \|\phi\|_{L^{1+\frac{1}{k}}} \cdot \left(\int_{\mathbb{R}^n} -u\, F_k[u]\, dx\right)^{\frac{1}{k+1}} \tag{19}$$

when $\phi = [-(-\Delta)^{\alpha/2}u]^k$. In order to prove this fact we solve the equation

$$F_k[v] = (-\Delta)^{\alpha/2}\phi,$$

in the viscosity sense, where v is a k-convex function vanishing at ∞ (see [15]). Now recalling that the fractional Laplace is self-adjoint we can apply the the Hessian-Schwarz inequality proved in [17], see also inequality (16). As a consequence we can apply the inequalities in term of Wolff's potentials, see e.g. (17), achieving the thesis.

Theorem 6.2. *Let* $u \in C^2(\mathbb{R}^n)$ *be a* $k-$*convex function vanishing at* ∞. *Let* $\alpha = \frac{2k}{k+1}$, *where* $1 \leq k < \frac{n}{2}$. *Then there exists a positive constant* $c_{k,n}$ *such that*

$$\int_{\mathbb{R}^n} -u\, F_k[u]\, dx \leq c_{k,n} \int_{\mathbb{R}^n} \left|(-\Delta)^{\frac{\alpha}{2}} u\right|^{k+1} dx. \tag{20}$$

The proof of Theorem 6.2 is based on a duality argument, similar to the one applied in Theorem 6.1, for fractional Sobolev spaces.

Corollary 6.1. *Let* $u \in C^2(\mathbb{R}^n)$ *be a* $k-$*convex function vanishing at* ∞, *where* $1 \leq k < \frac{n}{2}$. *Then there exists* $\tilde{u}$ *such that* $c_1 \leq u/\tilde{u} \leq c_2$, *and*

$$C_1 \int_{\mathbb{R}^n} \left|(-\Delta)^{\frac{\alpha}{2}} \tilde{u}\right|^{k+1} dx \leq \int_{\mathbb{R}^n} -u\, F_k[u]\, dx \leq C_2 \int_{\mathbb{R}^n} \left|(-\Delta)^{\frac{\alpha}{2}} \tilde{u}\right|^{k+1} dx, \tag{21}$$

where the constants of equivalence c_i, C_i $(i = 1, 2)$ *depend only on* k *and* n.

The proof of Corollary 6.1 stems by taking $\tilde{u} = -I_\alpha (I_\alpha v)^{\frac{1}{k}}$, where $v = F_k[u]$ is the k-Hessian measure associated with u, and apply Theorems 6.1 and 6.2.

Let us recalling that the Riesz potential of a positive Borel measure μ of oreder $\alpha \in (0, n)$ is defined as follows:

$$I_\alpha \mu(x) = a_{\alpha,n} \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^{n-\alpha}}, \qquad x \in \mathbb{R}^n. \tag{22}$$

Lemma 6.1. *Let $0 < \alpha < 2$, $a = 1 - \alpha$, and $1 \le p < \infty$. Suppose $f \in C^2(\mathbb{R}^n) \cap L^p(w)$ where $w(x) = (1+|x|)^{-(n+\alpha)}$. Let u and v be respectively the Caffarelli-Silvestre extensions of f and f^p to the upper half-space $\mathbb{R}^{n+1}_+$. If $f \ge 0$, or p is an even integer, then*

$$\frac{1}{1-a} \lim_{y \to 0} y^a (v_y(x,y) - (u^p)_y(x,y)) = p\, f^{p-1} \cdot (-\Delta)^{\alpha/2} f - (-\Delta)^{\alpha/2} (f^p) \ge 0 \quad a.e. \tag{23}$$

Consequently, if $f \ge 0$, then

$$(-\Delta)^{\alpha/2}(f^p) \le p\, f^{p-1} \cdot (-\Delta)^{\alpha/2} f \quad \text{a.e.} \tag{24}$$

The proof of Lemma 6.1 is based on the characterization given by Caffarelli and Silvestre, [13], of the fractional Laplace operator. However the inequality (24) can be deduced from a sort of integration by parts formula that I will prove in the next section.

Corollary 6.2. *Let $1 \le p < \infty$ and $0 < \alpha \le 2$. Suppose μ is a positive Borel measure on $\mathbb{R}^n$. Let $I_\alpha \mu = (-\Delta)^{-\alpha/2} \mu$ be the Riesz potential of μ defined by (22). Then*

$$(I_\alpha \mu)^p \le p\, I_\alpha [(I_\alpha \mu)^{p-1}\, d\mu] \quad a.e. \tag{25}$$

This last result is a consequence of Lemma 6.1 via an approximation argument.

I conclude this section with a couple of remarks about Theorem 6.1.

Remark 6.1. *The condition (ii) in Theorem 6.1 descends from (i) when $k = 1$. Indeed if $k = 1$, then $\alpha = 1$ because $\alpha(k) = \frac{2k}{k+1}$. Hence if u is $1-$convex, that is u is subharmonic, and vanishing at infinity, and moreover (i) is fulfilled, that is $-(-\Delta)^{\frac{1}{2}} u \ge 0$, then condition (ii) is*

$$(-\Delta)^{\frac{1}{2}}[-(-\Delta)^{\frac{1}{2}} u] = \Delta u \ge 0,$$

because u is subharmonic. So the question is the following one: whenever u is $k-$ convex, $k \ge 1$, vanishing at infinity, and (i) is satisfied, can we conclude that condition (ii) is fulfilled for $\alpha = \frac{2k}{k+1}$? When $k = 1$ the answer is positive. Is this conjecture still true for $k > 1$?

Remark 6.2. *Is the set of the k convex functions vanishing at infinity and such that (i) and (ii) are fulfilled nontrivial? In* [8] *we proved that if $k = 1$ then there exist nontrivial functions satisfying all these hypotheses. Is this conjecture true even when $k \geq 2$?*

7. An interesting formula

Working on the definition of the fractional Laplace operator given as singular integral:

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{\mid x - y \mid^{n+\alpha}},$$

it can be proved the following result

Lemma 7.1. *Let u, v be C^2 functions in $\mathbb{R}^n$ such that*

$$\int_{\mathbb{R}^n} \frac{\mid u \mid}{(1 + \mid x \mid^2)^{\frac{n+\alpha}{2}}} dx < \infty$$

and

$$\int_{\mathbb{R}^n} \frac{\mid v \mid}{(1 + \mid x \mid^2)^{\frac{n+\alpha}{2}}} dx < \infty.$$

Then

$$- (-\Delta)^{\frac{\alpha}{2}} (uv)(x) = u(x)[-(-\Delta)^{\frac{\alpha}{2}} v(x)] + v(x)[-(-\Delta)^{\frac{\alpha}{2}} u(x)]$$
$$+ \int_{\mathbb{R}^n} \frac{(u(x+z) - u(x))(v(x+z) - v(x)))}{\mid z \mid^{n+\alpha}} dz$$

Proof.

$$(-\Delta)^{\frac{\alpha}{2}} u(x)v(x) = \int_{\mathbb{R}^n} \frac{u(x)v(x) - u(x+z)v(x+z)}{\mid z \mid^{n+\alpha}} dz$$
$$= u(x) \int_{\mathbb{R}^n} \frac{v(x) - v(x+z)}{\mid z \mid^{n+\alpha}} dz + \int_{\mathbb{R}^n} v(x+z) \frac{u(x) - u(x+z)}{\mid z \mid^{n+\alpha}} dz$$
$$= u(x) \int_{\mathbb{R}^n} \frac{v(x) - v(x+z)}{\mid z \mid^{n+\alpha}} dz + v(x) \int_{\mathbb{R}^n} \frac{u(x) - u(x+z)}{\mid z \mid^{n+\alpha}} dz$$
$$+ \int_{\mathbb{R}^n} \frac{u(x) - u(x+z)}{\mid z \mid^{n+\alpha}} (v(x+z) - v(x)) dz$$
$$= u(x)(-\Delta)^{\frac{\alpha}{2}} v + v(x)(-\Delta)^{\frac{\alpha}{2}} u - \int_{\mathbb{R}^n} \frac{(u(x) - u(x+z))(v(x) - v(x+z))}{\mid z \mid^{n+\alpha}} dz.$$

□

This sort of derivation formula (or equivalently, such integration by parts formula) for fractional operators is not unknown in literature. Indeed,I wish to thank Igor Verbitsky who pointed out to me, Lemma 7.1 was already proved in [5] and also re-proved in [10] for the regional Laplacian and used for the application in [9]. It is worth to say that in both papers [5] and [10] the proof of Lemma 7.1 seems to be proved applying much nonelementary arguments.

As a corollary of Lemma 7.1, whenever $u = v$ we get that

$$-(-\Delta)^{\frac{\alpha}{2}}u^2(x) = 2u(x)[-(-\Delta)^{\frac{\alpha}{2}}u(x)] + \int_{\mathbb{R}^n} \frac{(u(x+z)-u(x))^2}{\mid z \mid^{n+\alpha}}dz$$

Remark 7.1. *Moreover if ϕ is differentiable and convex, then:*

$$(-\Delta)^{\frac{\alpha}{2}}\phi(u(x)) = \int_{\mathbb{R}^n} \frac{\phi(u(x))-\phi(u(x+z))}{\mid z \mid^{n+\alpha}}dz \leq -\int_{\mathbb{R}^n} \frac{\phi'(u(x))(u(x+z)-u(x))}{\mid z \mid^{n+\alpha}}dz$$
$$= -\phi'(u(x))\int_{\mathbb{R}^n} \frac{u(x+z)-u(x)}{\mid z \mid^{n+\alpha}}dz = \phi'(u(x))(-\Delta)^{\frac{\alpha}{2}}u(x),$$

that is

$$-\phi'(u(x))(-\Delta)^{\frac{\alpha}{2}}\phi(u(x)) \leq -(-\Delta)^{\frac{\alpha}{2}}u(x).$$

In particular if p is even we get

$$-pu^{p-1}(x)(-\Delta)^{\frac{\alpha}{2}}u^p(x) \leq -(-\Delta)^{\frac{\alpha}{2}}u(x)$$

obtaining one more time the inequality (24) of Lemma 6.1

References

[1] L. Caffarelli, L. Silvestre, *An extension problem related to the fractional Laplacian,* Comm. Partial Differential Equations 32 (2007), 12451260.

[2] P. Constantin, *Statistics in turbolence*, SIAM Review, 36, 73-98 (1994).

[3] P. Constantin, A. Majda and E. Tabak, *Formation of strong fronts in the $2-D$ quasi-geostrophic thermal active scalar,* Nonlinearity 7 (1994), 1495-1533.

[4] D. Cordoba, *Nonexistence of simple hyperbolic blow-up for the quasi-geostrophic equation*, Ann. of Math. 148 (1998), 11351152.

[5] S. Eilertsen, *On weighted positivity and the Wiener regularity of aboundary point for the fractional Laplacian*, Ark. Mat., 38 (2000), 53-75.

[6] F. Ferrari, B. Franchi, *Harnack inequality for fractional operators in Carnot groups*, in preparation.

[7] F. Ferrari, B. Franchi, I. Verbitsky, *Hessian inequalities and the fractional Laplacian,* accepted paper: to appear in J. Reine Angew. Math.

[8] F. Ferrari, I. Verbitsky, in preparation.

[9] Q.-Y. Guan, Z.-M. Ma, *Boundary problems for fractional Laplacian* in Stoch. Dyn. 5 (2005), 385-424.

[10] Q.-Y. Guan, Z.-M. Ma, Reflected symmetric $\alpha-$stable processes and regional fractional Laplacian, Probab. Theory Relat. Fields 134, 649-694 (2006)

[11] N. S. Landkof, *Foundations of Modern Potential Theory*, Grundlehren der mathematischen Wissenschaften **180**, Springer–Verlag, New York–Heidelberg, 1972.

[12] J. Pedlosky, *Geophysical fluid dynamics*, New York: Springer 1987

[13] L. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*. PhD thesis, University of Texas at Austin, 2005.

[14] E. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, New Jersey 1970

[15] *N. S. Trudinger and X.-J. Wang, Hessian measures II*, Ann. Math. **150** (1999), 579–604.

[16] E. Valdinoci, *From the long jump random walk to the fractional Laplacian*, Bol. Soc. Esp. Mat. Apl. SeMA No. 49 (2009), 3344.

[17] I. Verbitsky, *The Hessian Sobolev inequality and its extensions*, preprint (2008).

Fausto Ferrari

Dipartimento di Matematica dell'Università

Piazza di Porta S. Donato, 5, 40126 Bologna, Italy

E-mail: `ferrari@dm.unibo.it`