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Interpolation inequalities in pattern formation [1]

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Abstract

We prove an interpolation inequality in every dimension d, which let to control the $L^{4/3}$-norm of a function u with the product of the square roots of its BV-norm and H^{-1}-norm. In dimension $d = 2$ and for functions u bounded below, we can improve this result gaining a factor of the scaling $\log^{1/4}|u|$ on the left-hand side. Our two interpolation inequalities are the strong version of two already known estimates in weak form, which play a crucial role in the study of pattern formation in physics.

1. Introduction

In these notes we prove two interpolation inequalities, involving the BV-norm and the H^{-1}-norm of a function u. The first estimate holds in any dimension d and it is established in Proposition 1.1 below. The second inequality (see Proposition 1.2) holds in dimension 2 for functions bounded below, and it improves the result in Proposition 1.1 by a factor $\log^{1/4} u$. Both inequalities are the strong version of two already known estimates in weak form, which play a crucial role in the study of pattern formation in physics.

In many physical problems described by a variational model, in order to understand why certain patterns are preferred, it is natural to study whether these patterns are energy optimal. In several applications (coarsening, domain branching in ferromagnets, superconductors, twin branching in shape memory alloys) the energy is given by the competitions of two main terms: an interfacial energy and a field energy. Our interpolations inequalities are crucial ingredients in the proof of a lower bound for the energy, since they have a natural form which involves a BV-norm (which describes an interfecial energy) and a H^{-1}-norm (which is related to a field energy).

A lot of pioneering work in the exploration of this connection between physical phenomena and interpolation inequalities has been done by Kohn and Otto, a particular starting point is [4].

In [4] Kohn and Otto established an upper bound of the coarsening rate (which corresponds to lower bound of the energy) for two standard model of surface-energy-driven coarsening: a constant-mobility Cahn-Hilliard equation (whose large-time behaviour corresponds to Mullins-Sekerka dynamics) and a degenerate mobility Cahn-Hilliard equation (whose large-time behaviour corresponds to motion by surface diffusion). The basic idea is to use the gradient-flow structure of the Cahn-Hilliard equation.

In the Cahn-Hilliard model for phase transitions we have a scalar order parameter $u : [0, \Lambda^d] \to \mathbb{R}$, periodic with length Λ and that typically takes values in $[0, 1]$. The energy is given by

$$\mathcal{E}(u) = \frac{1}{2} \int_{[0,\Lambda^d]} (|\nabla u|^2 + (1 - u^2)^2) dx.$$

It is easy to see that the Cahn-Hilliard evolution is the gradient flow of $\mathcal{E}$ with respect to the Euclidian structure given by $|||\nabla|^{-1}\cdot||_{L^2}$. We recall that the H^{-1}-norm of a function u with vanishing average is defined as follows

$$\begin{aligned}|||\nabla|^{-1}u||_2^2 &= \inf_j \left\{ \int |j|^2 | \nabla\cdot j = u \right\} \\ &= \int |\nabla\varphi|^2 \quad \text{where} \quad -\Delta\varphi = u.\end{aligned}$$

It can also be defined via Fourier transform:

$$|||\nabla|^{-1}u||_2^2 = \int (|k|^{-1}F(u))^2 dk.$$

If in the Cahn-Hilliard model we choose u to take values only in $\{-1,+1\}$, then the interfacial area density is given by $\frac{1}{2\Lambda^d}\int|\nabla u|$.

In [4] essentially the argument makes use of the following two quantities: the interfacial area density $E := \frac{1}{2\Lambda^d}\int|\nabla u|$, which has the dimension of 1/length and the physical scale $L := \left(\frac{1}{\Lambda^d}\int ||\nabla|^{-1}u|^2\right)^{1/2}$, which has the dimension of length. The proof of the lower bound of the energy relies on two main ingredients. The first one consists of an interpolation inequality involving the quantities E and L, which implies that $EL > C$, for some universal constant C. The second ingredient is given by some differential inequalities that are consequences of the energy-dissipating structure of the dynamic. Using these two ingredients, the lower bound of the energy follows by an ODE argument.

These ideas have been used in several works on coarsening rates. In [3] Conti, Niethammer and Otto studied coarsening of a binary mixture within the Mullins-Sekerka evolution in the regime where one phase has small volume fraction $\Phi \ll 1$. In particular, they gave a lower bound on how the energy decreases depending on Φ. Their main contribution is an interpolation inequality in dimension $d = 2$ which let to gain a term of the scaling $\log^{1/3}\Phi$ (see Proposition 3.1).

In Viehmann's PhD thesis [7] interpolation estimates are used to study branching in micromagnetics. In this problem the magnetic direction (more precisely one component) plays a role similar to that of the order parameter u in the Cahn-Hilliard equation.

In [7] the crucial interpolation inequality in the proof of the lower bound for the energy in every dimension $d > 2$ reads as follows. Given a periodic function $u : [0, \Lambda^d] \to \mathbb{R}$

satisfying $\int u = 0$, we have

$$||u||_{w-4/3} \leq C||\nabla u||_1^{1/2}|||\nabla|^{-1}u||_2^{1/2}, \tag{1.1}$$

where $||u||_{w-4/3} := \sup_{\mu>0} \mu|\{|u| \geq \mu\}|^{3/4}$ denotes the weak $L^{4/3}-$norm of u. However, in dimension $d = 2$ this inequality does not allow to deduce the optimal scaling of the energy. In this case, the following stronger interpolation inequality is needed:

$$\mu \log^{1/4}\left(\frac{\mu}{\Phi}\right)\{|u| \geq \mu\}|^{3/4} \leq C||\nabla u||_1^{1/2}|||\nabla|^{-1}(u-\Phi)||_2^{1/2}, \tag{1.2}$$

where now $u \geq -1$ is a function bounded below with $\frac{1}{\Lambda^2}\int u = \Phi$ and $\mu \geq 2\Phi + 2$. Following [3], to gain a factor of the scaling $\log^{1/4}(\frac{\mu}{\Phi})$, the author uses a careful choice of the convolution kernel.

The aim of these notes is to establish the corresponding strong versions of inequalities (1.1) and (1.2), that is to replace the weak $L^{4/3}$-norm of u by the strong one. More precisely we prove the following two propositions.

Proposition 1.1. *There exists a constant $C < \infty$ such that for all periodic functions $u : (0,\Lambda)^d \to \mathbb{R}$, with $\int u = 0$, we have*

$$\|u\|_{L^{4/3}} \ \leq \ C||\nabla u||_1^{1/2}|||\nabla|^{-1}u||_2^{1/2}. \tag{1.3}$$

The proof of Proposition 1.1 uses a technique introduced by Ledoux in [5] to give a direct proof of some improved Sobolev inequalities. These inequalities were already been studied by Cohen, Dahmen, Daubechies, and DeVore in [2] using the wavelet analysis of the space BV.

The following proposition is the corresponding strong version of inequality (1.2), which holds in dimension $d = 2$ for functions bounded below.

Proposition 1.2. *There exists a constant $C < \infty$ such that for all periodic functions $u : (0,\Lambda)^2 \to \mathbb{R}$, with $u \geq -1$ and $\frac{1}{\Lambda^2}\int u = 0$, we have*

$$\|u \ln^{\frac{1}{4}} \max\{u, e\}\|_{\frac{4}{3}} \ \leq \ C\|\nabla u\|_1^{\frac{1}{2}} \ \||\nabla|^{-1}u\|_2^{\frac{1}{2}}. \tag{1.4}$$

2. Interpolation inequality in general dimension

In this section we give the proof of Proposition 1.1. We start by recalling the weak version of estimate (1.3).

Lemma 2.1 ([7]). *There exists a constant $C < \infty$ such that for all periodic functions $u : (0,\Lambda)^d \to \mathbb{R}$, with $\int u = 0$, we have*

$$\|u\|_{w-\frac{4}{3}} := \sup_{\mu \geq 0} \mu |\{|u| > \mu\}|^{\frac{3}{4}} \leq C \|\nabla u\|_1^{\frac{1}{2}} \ \||\nabla|^{-1} u\|_2^{\frac{1}{2}}.$$

This Lemma is proven in [7]. Here, for the sake of completeness, we give the proof of this weak estimate, since it is also useful to prove the strong version (1.3).

Proof of Lemma 2.1. For simplicity in the following we will write "$a \lesssim b$" to mean that there exists a positive constant C such that $a \leq Cb$. By a scaling argument in x, it is enough to show

$$\sup_{\mu \geq 0} \mu^{\frac{4}{3}} |\{|u| > \mu\}| \lesssim \|\nabla u\|_1 + \||\nabla|^{-1} u\|_2^2.$$

Indeed, the change of variables $x = L\hat{x}$ yields

$$\sup_{\mu \geq 0} \mu^{\frac{4}{3}} |\{|u| > \mu\}| \lesssim L^{-1} \|\hat{\nabla} u\|_1 + L^2 \||\hat{\nabla}|^{-1} u\|_2^2,$$

where the symbol $\hat{\nabla}$ denotes the gradient with respect to the new variable $\hat{x}$. The choice of $L = \|\hat{\nabla} u\|_1^{\frac{1}{3}} \||\hat{\nabla}|^{-1} u\|_2^{-\frac{2}{3}}$ yields

$$\sup_{\mu \geq 0} \mu^{\frac{4}{3}} |\{|u| > \mu\}| \lesssim \|\hat{\nabla} u\|_1^{\frac{2}{3}} \ \||\hat{\nabla}|^{-1} u\|_2^{\frac{2}{3}}.$$

Raising to the power $\frac{3}{4}$ we get as desired

$$\|u\|_{w-\frac{4}{3}} \lesssim \|\hat{\nabla} u\|_1^{\frac{1}{2}} \ \||\hat{\nabla}|^{-1} u\|_2^{\frac{1}{2}}.$$

For an arbitrary level $\mu \geq 0$ we introduce the signed characteristic function $\chi_\mu(x)$ of the level set of u:

$$\chi_\mu := \left\{ \begin{array}{rlrcrcr} 1 & \text{for} & \mu & < & u & & \\ 0 & \text{for} & -\mu & \leq & u & \leq & \mu \\ -1 & \text{for} & & & u & < & -\mu \end{array} \right\}. \tag{2.1}$$

We select a smooth symmetric $\psi(\hat{x}) \geq 0$ supported in $\{|\hat{x}| \leq 1\}$ with $\int \psi d\hat{x} = 1$ and define the Dirac sequence $\psi_R(x) = \frac{1}{R^d}\psi(\frac{x}{R})$. Consider the mollification of a function v on scale R: $v_R := \psi_R * v$. We have the identity

$$\int \chi_\mu u = \int \chi_\mu (u - u_R) + \int \chi_{\mu,R} u.$$

We get the inequality

$$\mu \int \chi_\mu \leq \int \chi_\mu u \leq \|u - u_R\|_1 + \|\nabla \chi_{\mu,R}\|_2 \; \||\nabla|^{-1} u\|_2. \tag{2.2}$$

On the one hand, since ψ_R is supported in $\{|x| \leq R\}$, we have

$$\|u - u_R\|_1 \leq R\|\nabla u\|_1. \tag{2.3}$$

On the other hand, we have

$$\|\nabla \chi_{\mu,R}\|_2 \leq \|\nabla \psi_R\|_1 \|\chi_\mu\|_2 = R^{-1}\|\hat{\nabla}\psi\|_1 \left(\int \chi_\mu\right)^{\frac{1}{2}}. \tag{2.4}$$

Plugging (2.3) and (2.4) into (2.2), we get

$$\mu \int \chi_\mu \leq R\|\nabla u\|_1 + R^{-1}\|\hat{\nabla}\psi\|_1 \left(\int \chi_\mu\right)^{\frac{1}{2}} \||\nabla|^{-1} u\|_2.$$

The choice of $R = \mu^{-\frac{1}{3}}$ thus yields after multiplication with $\mu^{\frac{1}{3}}$:

$$\mu^{\frac{4}{3}} \int \chi_\mu \leq \|\nabla u\|_1 + \|\hat{\nabla}\psi\|_1 \left(\mu^{\frac{4}{3}} \int \chi_\mu\right)^{\frac{1}{2}} \||\nabla|^{-1} u\|_2.$$

With help of Young's inequality, we may absorb the first factor of the second term on the right-hand side and obtain the desired estimate.

□

We give now the proof of Proposition 1.1. The interpolation estimate (1.3) was first established by Cohen, Dahmen, Daubechies, and Devore [2] by wavelet methods. We give here an elementary proof.

Proof of Proposition 1.1. Again, by scaling in x as in the proof of Lemma 2.1 it is enough to prove

$$\int |u|^{\frac{4}{3}} \lesssim \|\nabla u\|_1 + \||\nabla|^{-1} u\|_2^2.$$

For arbitrary level $\mu > 0$ we use the signed characteristic function χ_μ defined in (2.1). Following an idea of Ledoux [5] for the proof a similar interpolation inequality we introduce a factor $M \gg 1$ to be adjusted later. We have:

$$\begin{aligned}\int \chi_\mu u &= \int (\chi_\mu - \chi_{\mu,R})u + \int \chi_{\mu,R} u \\ &= \int_{|u|\leq M\mu} (\chi_\mu - \chi_{\mu,R})u + \int_{|u|>M\mu} (\chi_\mu - \chi_{\mu,R})u + \int \chi_{\mu,R} u.\end{aligned}$$

Using that $\|\chi_\mu - \chi_{\mu,R}\|_\infty \leq 2$, we obtain the inequality

$$\begin{aligned}\int_{|u|>\mu} |u| &\leq M\mu \int |\chi_\mu - \chi_{\mu,R}| + 2\int_{|u|>M\mu} |u| + \int \chi_{\mu,R} u \\ &\leq M\mu R \int |\nabla \chi_\mu| + 2\int_{|u|>M\mu} |u| + \int \chi_{\mu,R} u.\end{aligned}$$

We multiply with $\mu^{-\frac{2}{3}}$ and choose $R = \mu^{-\frac{1}{3}}$ as in the proof of Lemma 2.1. Integrating over $\mu \in (0,\infty)$, we get

$$\begin{aligned}&\int_0^\infty \mu^{-\frac{2}{3}} \int_{|u|>\mu} |u| dx d\mu \\ \leq\ & M \int_0^\infty \int |\nabla \chi_\mu| dx d\mu + 2\int_0^\infty \mu^{-\frac{2}{3}} \int_{|u|>M\mu} |u| dx d\mu \\ &+ \int \int_0^\infty \mu^{-\frac{2}{3}} \chi_{\mu,R} d\mu\, u dx \\ \leq\ & M \int_0^\infty \int |\nabla \chi_\mu| dx d\mu + 2\int_0^\infty \mu^{-\frac{2}{3}} \int_{|u|>M\mu} |u| dx d\mu \\ &+ \|\nabla (\int_0^\infty \mu^{-\frac{2}{3}} \chi_{\mu,R} d\mu)\|_2 \ \||\nabla|^{-1} u\|_2,\end{aligned}$$

where we keep the abbreviation $R = \mu^{-\frac{1}{3}}$.

On the left-hand side we have

$$\int_0^\infty \mu^{-\frac{2}{3}} \int_{|u(x)|>\mu} |u(x)| dx d\mu = \int |u(x)| \int_0^{|u(x)|} \mu^{-\frac{2}{3}} d\mu dx = 3\int |u|^{\frac{4}{3}}.$$

We address the three terms on the right-hand side one by one. We start by the second one:

$$\begin{aligned}
&\int_0^\infty \mu^{-\frac{2}{3}} \int_{|u(x)|>M\mu} |u(x)| dx d\mu \\
=& \int |u(x)| \int_0^{M^{-1}|u(x)|} \mu^{-\frac{2}{3}} d\mu dx = 3M^{-\frac{1}{3}} \int |u|^{\frac{4}{3}}.
\end{aligned}$$

We now address the first term. By the coarea formula we get

$$\int_0^\infty \int |\nabla \chi_\mu| dx d\mu = \int_0^\infty (\mathrm{Per}(\{u > \mu\}) + \mathrm{Per}(\{u < -\mu\}))d\mu = \|\nabla u\|_1.$$

Finally we consider the last term (with $R' := \mu'^{-\frac{1}{3}}$):

$$\begin{aligned}
&\|\nabla(\int_0^\infty \mu^{-\frac{2}{3}} \chi_{\mu,R} d\mu)\|_2^2 \\
=& \int_0^\infty \int_0^\infty \mu^{-\frac{2}{3}} \mu'^{-\frac{2}{3}} \int \nabla\chi_{\mu,R} \cdot \nabla\chi_{\mu',R'} dx d\mu' d\mu \\
=& \; 2\int_0^\infty \int_0^\mu \mu^{-\frac{2}{3}} \mu'^{-\frac{2}{3}} \int (-\triangle)\chi_{\mu,R} \; \chi_{\mu',R'} dx d\mu' d\mu \\
=& \; 2\int_0^\infty \int_0^\mu \mu^{-\frac{2}{3}} \mu'^{-\frac{2}{3}} \int \psi_{R'} * (-\triangle\psi_R) * \chi_\mu \; \chi_{\mu'} dx d\mu' d\mu \\
\leq& \; 2\int_0^\infty \int_0^\mu \mu^{-\frac{2}{3}} \mu'^{-\frac{2}{3}} \|\psi_{R'}\|_1 \; \|\triangle\psi_R\|_1 \; \|\chi_\mu\|_1 \; \|\chi_{\mu'}\|_\infty d\mu' d\mu \\
=& \; 2\|\hat{\triangle}\psi\|_1 \int_0^\infty \int_0^\mu \mu^{-\frac{2}{3}} \mu'^{-\frac{2}{3}} R^{-2} \; \|\chi_\mu\|_1 d\mu' d\mu \\
=& \; 2\|\hat{\triangle}\psi\|_1 \int_0^\infty \int_0^\mu \mu'^{-\frac{2}{3}} d\mu' \; \|\chi_\mu\|_1 d\mu \\
=& \; 6\|\hat{\triangle}\psi\|_1 \int_0^\infty \mu^{\frac{1}{3}} \; |\{|u| > \mu\}| d\mu \; = \; 6\|\hat{\triangle}\psi\|_1 \int \int_0^{|u(x)|} \mu^{\frac{1}{3}} d\mu dx \\
=& \; \frac{9}{2}\|\hat{\triangle}\psi\|_1 \int |u|^{\frac{4}{3}}.
\end{aligned}$$

These inequalities combine to

$$\begin{aligned}
&3\int |u|^{\frac{4}{3}} \\
\leq& \; M\|\nabla u\|_1 + 6M^{-\frac{1}{3}} \int |u|^{\frac{4}{3}} + \left(\frac{9}{2}\|\hat{\triangle}\psi\|_1 \int |u|^{\frac{4}{3}}\right)^{\frac{1}{2}} \; \||\nabla|^{-1}u\|_2.
\end{aligned}$$

We obtain the desired estimate by absorbing the middle right-hand side term for $M \gg 1$ and absorbing the first factor of the last right-hand side term by Young's inequality. □

3. Interpolation inequality in dimension 2

In this section we prove Proposition 1.2. We begin by recalling a geometric version of estimate (1.4), which was established by Conti, Niethammer, and Otto in [3].

Lemma 3.1 ([3]). *For $d = 2$ and a characteristic function $\chi(x) \in \{0,1\}$ with volume fraction $\Phi := \Lambda^{-2}\int \chi \ll 1$ we have*

$$\Phi \ln^{\frac{1}{3}} \frac{1}{\Phi} \lesssim \left(\Lambda^{-2}\int |\nabla\chi|\right)^{\frac{2}{3}} \left(\Lambda^{-2}\int ||\nabla|^{-1}(\chi-\Phi)|^2\right)^{\frac{1}{3}}. \tag{3.1}$$

The proof of this Lemma made use of the following geometric construction, that plays a crucial role also in the proof of the weak estimate (1.2) and of the strong one (1.4).

Lemma 3.2 ([3]). *For $\chi(x) \in \{0,1\}$ and $R \ll L$ there exists a potential $\phi_{R,L}(x) \in [0,1]$ such that*

$$\int \chi \lesssim R\int |\nabla\chi| + \int \chi\phi_{R,L}, \tag{3.2}$$

$$\int \max\{-\triangle\phi_{R,L}, 0\} \lesssim R^{-2}(\ln^{-1}\frac{L}{R})\int \chi, \tag{3.3}$$

$$\int \phi_{R,L} \lesssim L^2R^{-2}\int \chi. \tag{3.4}$$

We note that for $L = R$ we could just choose $\phi_{R,L} = \psi_R * \chi = \chi_R$; the interest here is the logarithmic gain $\ln^{-1}\frac{L}{R}$ for $L \gg R$.

Remark 3.3. We observe, for later reference, that for any function $\phi'(x) \in [0,1]$ we have

$$\int \nabla\phi_{R,L}\cdot\nabla\phi' \lesssim R^{-2}\ln^{-1}\frac{L}{R}\int \chi.$$

Indeed, we have

$$\begin{aligned} &\int \nabla\phi_{R,L}\cdot\nabla\phi' \\ &= \int(-\triangle\phi_{R,L})\phi' \le \int \max\{-\triangle\phi_{R,L}, 0\}\phi' \\ &\le \int \max\{-\triangle\phi_{R,L}, 0\} \le R^{-2}\ln^{-1}\frac{L}{R}\int \chi, \end{aligned}$$

where in the first two inequailities we have used $\phi' \geq 0$ and $\phi' \leq 1$ respectively. The last inequality follows by applying (3.3).

In particular, we obtain for $\phi' = \phi_{R,L}$

$$\int |\nabla \phi_{R,L}|^2 \lesssim R^{-2}(\ln^{-1} \frac{L}{R}) \int \chi. \tag{3.5}$$

This type of geometric construction was first used by Choksi, Conti, Kohn, and Otto in [1] in the context of branched patterns in superconductors, but its main ingredient goes back to De Giorgi.

Proof of Lemma 3.2. In a first step, we construct a set Ω_R that covers most of $\{\chi = 1\}$ (*Claim 1*) and has radius of curvature $\lesssim R$ (*Claim 2*). As before, let $\chi_R = \psi_R * \chi$ denote the mollification of χ on scale R. We define

$$\Omega_R := \{\chi_R > \frac{1}{2}\}.$$

This time, we take the non-smooth "Dirac sequence"

$$\psi_R = \left\{ \begin{array}{ccc} \frac{4}{\pi R^2} & \text{for} & |x| < \frac{R}{2} \\ 0 & \text{for} & |x| \geq \frac{R}{2} \end{array} \right\},$$

so that Ω_R can be characterized via the density of $\{\chi = 1\}$ in balls of radius $\frac{R}{2}$ as follows

$$\Omega_R = \{x \mid |\{\chi = 1\} \cap B_{\frac{R}{2}}(x)| > \tfrac{1}{2}|B_{\frac{R}{2}}(x)|\}.$$

Claim 1: We have

$$\int \chi \ \lesssim \ R \int |\nabla \chi| + \int_{\Omega_R} \chi.$$

Indeed, $\int \chi - \int_{\Omega_R} \chi = |\{\chi = 1\} \cap \{\chi_R \leq \frac{1}{2}\}| \leq 2\|\chi - \chi_R\| \leq 2R \int |\nabla \chi|$.

Claim 2: There exists a finite subset $C \subset \Omega_R$ such that $\Omega_R \subset \bigcup_{y \in C} B_R(y)$ while $R^2 \# C \lesssim \int \chi$, where $\# C$ denotes the cardinality of the set C.

Indeed, let $C \subset \Omega_R$ be maximal with the property that $B_{\frac{R}{2}}(y) \cap B_{\frac{R}{2}}(y') = \emptyset$ for any distinct $y, y' \in C$. The first part of the claim follows from the maximality of C: if there were an $y_0 \in \Omega_R$ with $y_0 \notin B_R(y)$ and thus $B_{\frac{R}{2}}(y_0) \cap B_{\frac{R}{2}}(y) = \emptyset$ for all $y \in C$, also the strictly larger set $\{y_0\} \cup C$ would be admissible. The second part of the claim can be seen as follows:

$$\#C\frac{\pi}{4}R^2 = \sum_{y\in C} |B_{\frac{R}{2}}(y)| < 2\sum_{y\in C} |\{\chi = 1\} \cap B_{\frac{R}{2}}(y)| \leq 2|\{\chi = 1\}|,$$

where in the first inequality, we have used that by definition of Ω_R, we have for $y \in C \subset \Omega_R$ that $|\{\chi = 1\} \cap B_{\frac{R}{2}}(y)| > \frac{1}{2}|B_{\frac{R}{2}}(y)|$. In the last inequality we have used the pairwise disjointness of $\{B_{\frac{R}{2}}(y)\}_{y\in C}$.

In the second step, we construct the potential $\phi_{R,L}$. We introduce the capacity potential $\hat{\phi}_{R,L}$ of $B_R(0)$ in $B_L(0)$ given by

$$\hat{\phi}_{R,L}(\hat{x}) := \left\{ \begin{array}{ccccccc} 1 & \text{for} & & & |\hat{x}| & \leq & R \\ \frac{\ln\frac{L}{|\hat{x}|}}{\ln\frac{L}{R}} & \text{for} & R & \leq & |\hat{x}| & \leq & L \\ 0 & \text{for} & L & \leq & |\hat{x}| & & \end{array} \right\} \in [0,1].$$

We define

$$\phi_{R,L}(x) := \max_{y\in C} \hat{\phi}_{R,L}(x-y) \ \in \ [0,1].$$

Claim 3: we claim that

$$\int \chi \lesssim R\int |\nabla\chi| + \int \chi\phi_{R,L}.$$

This follows from the first part of *Claim 1* and the fact that $\phi_{R,L} = 1$ on Ω_R. The latter follows from the first part of *Claim 2* and the fact that $\hat{\phi}_{R,L} = 1$ on $B_R(0)$.

Claim 4:

$$\int \phi_{R,L} \lesssim L^2R^{-2}\int \chi.$$

Indeed,

$$\int \phi_{R,L} \leq \#C\int \hat{\phi}_{R,L} \lesssim \#CL^2 \lesssim L^2R^{-2}\int \chi,$$

where we have used the second part of *Claim 2* in the last estimate.

Claim 5:

$$\int \max\{-\triangle\phi_{R,L}, 0\} \lesssim R^{-2}(\ln^{-1}\frac{L}{R})\int \chi.$$

Indeed, using the well-known fact that the singular part of $(-\triangle)\max\{\phi_1,\phi_2\}$ is negative, we conclude similarly to the previous step:

$$\begin{aligned}\int \max\{-\triangle\phi_{R,L},0\} &\le \#C\int\max\{-\triangle\hat{\phi}_{R,L},0\}\\ &= \#C 2\pi\ln^{-1}\frac{L}{R}\lesssim R^{-2}(\ln^{-1}\frac{L}{R})\int\chi.\end{aligned}$$

This concludes the proof of Lemma 3.2. $\square$

We can now give the proof of the geometric estimate (3.1).

Proof of Proposition 3.1. By the three properties of the geometric construction we have

$$\begin{aligned}&\int\chi\\ \lesssim\;& R\int|\nabla\chi|+\int\phi_{R,L}\chi\\ =\;& R\int|\nabla\chi|+\int\phi_{R,L}(\chi-\Phi)+\Phi\int\phi_{R,L}\\ \le\;& R\int|\nabla\chi|+\left(\int|\nabla\phi_{R,L}|^2\int||\nabla|^{-1}(\chi-\Phi)|^2\right)^{\frac12}+\Phi\int\phi_{R,L}\\ \lesssim\;& R\int|\nabla\chi|+\left(R^{-2}(\ln^{-1}\frac{L}{R})\int\chi\int||\nabla|^{-1}(\chi-\Phi)|^2\right)^{\frac12}+\Phi(\frac{L}{R})^2\int\chi.\end{aligned}$$

We first absorb the first factor of the middle right-hand side term by Young's inequality

$$\int\chi\;\lesssim\;R\int|\nabla\chi|+R^{-2}(\ln^{-1}\frac{L}{R})\int||\nabla|^{-1}(\chi-\Phi)|^2+\Phi(\frac{L}{R})^2\int\chi.$$

In order to absorb the last right-hand side term, we choose L to be a small but order one multiple of $\Phi^{-\frac12}R$. Since L is a small multiple of $\Phi^{-\frac12}R$, we have $\Phi(\frac{L}{R})^2\ll 1$ so that indeed we can absorb; since it is an order one multiple of $\Phi^{-\frac12}R$ and $\Phi\ll1$, we have $L\gg R$ and $\ln\frac{L}{R}\sim\ln\frac{1}{\Phi}$. Hence we obtain:

$$\int\chi\;\lesssim\;R\int|\nabla\chi|+R^{-2}(\ln^{-1}\frac{1}{\Phi})\int||\nabla|^{-1}(\chi-\Phi)|^2.$$

We finally optimize in R by choosing and we get $R=(\int|\nabla\chi|)^{-\frac13}((\ln^{-1}\frac{1}{\Phi})\int||\nabla|^{-1}(\chi-\Phi)|^2)^{\frac13}$:

$$\int\chi\;\lesssim\;(\ln^{-\frac13}\frac{1}{\Phi})\left(\int|\nabla\chi|\right)^{\frac23}\left(\int||\nabla|^{-1}(\chi-\Phi)|^2\right)^{\frac13}.$$

Dividing by Λ^2 and multiplying by $\ln^{\frac{1}{3}}\frac{1}{\Phi}$ yields the desired estimate. □

As in the previous section, we recall here the weak version of our interpolation inequality (1.4) in dimension 2.

Proposition 3.4 ([7]). *For $d = 2$ and $u(x) \geq -1$ we have*

$$\sup_{\mu\geq e} \mu(\ln^{\frac{1}{4}}\mu)\,|\{|u|>\mu\}|^{\frac{3}{4}} \lesssim \|\nabla u\|_1^{\frac{1}{2}}\ \||\nabla|^{-1}u\|_2^{\frac{1}{2}}.$$

This estimate was first proved in the PhD thesis of Viehmann [7].

Proof of Proposition 3.4. By Proposition 2.1 and by a scaling argument, it is enough to show

$$\sup_{\mu\gg 1} \mu^{\frac{4}{3}}(\ln^{\frac{1}{3}}\mu)\,|\{|u|>\mu\}| \lesssim \|\nabla u\|_1 + \||\nabla|^{-1}u\|_2^2.$$

For a given level $\mu \gg 1$ we consider the characteristic function $\chi_\mu(x) \in \{0,1\}$ of the corresponding level set of u, that is

$$\{\chi_\mu = 1\} = \{u > \mu\}.$$

For given length scales $L \ll R$ (to be chosen later) let $\phi_{\mu,R,L}$ be the potential constructed in Lemma 3.2 based on χ_μ. According to Lemma 3.2 we have

$$\int \chi_\mu \ \lesssim\ R\int |\nabla\chi_\mu| + \int \chi_\mu\phi_{\mu,R,L}.$$

Crucially using the assumption $u \geq -1$ we rewrite this as

$$
\begin{aligned}
& R\int|\nabla\chi_\mu| + \int \chi_\mu \phi_{\mu,R,L} \\
\overset{\phi_{\mu,R,L}\geq 0}{\leq} \quad & R\int|\nabla\chi_\mu| + \mu^{-1}\int \chi_\mu\phi_{\mu,R,L}u \\
= \quad & R\int|\nabla\chi_\mu| + \mu^{-1}\int \chi_\mu\phi_{\mu,R,L}(u+1) - \mu^{-1}\int \chi_\mu\phi_{\mu,R,L} \\
\overset{u\geq -1}{\leq} \quad & R\int|\nabla\chi_\mu| + \mu^{-1}\int \phi_{\mu,R,L}(u+1) - \mu^{-1}\int \chi_\mu\phi_{\mu,R,L} \\
= \quad & R\int|\nabla\chi_\mu| + \mu^{-1}\int \phi_{\mu,R,L}u + \mu^{-1}\int (1-\chi_\mu)\phi_{\mu,R,L} \\
\overset{\phi_{\mu,R,L}\geq 0}{\leq} \quad & R\int|\nabla\chi_\mu| + \mu^{-1}\int \phi_{\mu,R,L}u + \mu^{-1}\int \phi_{\mu,R,L} \\
\leq \quad & R\int|\nabla\chi_\mu| + \mu^{-1}\left(\int|\nabla\phi_{\mu,R,L}|^2\int||\nabla|^{-1}u|^2\right)^{\frac{1}{2}} + \mu^{-1}\int\phi_{\mu,R,L}.
\end{aligned}
$$

We now insert estimates (3.4) and (3.5) from Lemma 3.2 to obtain

$$
\begin{aligned}
\int\chi_\mu \quad \lesssim \quad & R\int|\nabla\chi_\mu| \\
& +\mu^{-1}\left(R^{-2}(\ln^{-1}\frac{L}{R})\int\chi_\mu\int||\nabla|^{-1}u|^2\right)^{\frac{1}{2}} + \mu^{-1}(\frac{L}{R})^2\int\chi_\mu.
\end{aligned}
$$

In order to absorb the last right-hand side term, we choose L to be a small but order one multiple of $\mu^{\frac{1}{2}}R$. Since L is a small multiple of $\mu^{\frac{1}{2}}R$, we have $\mu^{-1}(\frac{L}{R})^2 \ll 1$ so that indeed we can absorb; since it is an order one multiple of $\mu^{\frac{1}{2}}R$ and $\mu \gg 1$, we have $L \gg R$ and $\ln\frac{L}{R} \sim \ln\mu$. Hence we obtain:

$$
\int\chi_\mu \quad \lesssim \quad R\int|\nabla\chi_\mu| + \mu^{-1}\left(R^{-2}(\ln^{-1}\mu)\int\chi_\mu\int||\nabla|^{-1}u|^2\right)^{\frac{1}{2}}.
$$

In order to absorb the first factor of the last remaining right-hand side term, we use Young's inequality

$$
\begin{aligned}
|\{u>\mu\}| = & \int\chi_\mu \\
\lesssim \quad & R\int|\nabla\chi_\mu| + \mu^{-2}R^{-2}(\ln^{-1}\mu)\int||\nabla|^{-1}u|^2.
\end{aligned}
$$

By the coarea formula, we have $\int_{\frac{\mu}{2}}^{\mu} \int |\nabla \chi_\mu| d\mu \leq \int |\nabla u|$ so that there exists a $\mu' \in [\frac{\mu}{2}, \mu]$ with $\mu \int |\nabla \chi_{\mu'}| \leq 2 \int |\nabla u|$. Using the above for μ replaced by μ' we thus have

$$|\{u > \mu'\}| \lesssim R\mu^{-1} \int |\nabla u| + \mu'^{-2} R^{-2} (\ln^{-1} \mu') \int ||\nabla|^{-1} u|^2,$$

which because of $\mu' \in [\frac{\mu}{2}, \mu]$ turns into

$$|\{u > \mu\}| \lesssim R\mu^{-1} \int |\nabla u| + \mu^{-2} R^{-2} (\ln^{-1} \mu) \int ||\nabla|^{-1} u|^2.$$

We multiply with $\mu^{\frac{4}{3}} \ln^{\frac{1}{3}} \mu$ and we get

$$\begin{aligned} &\mu^{\frac{4}{3}} \ln^{\frac{1}{3}} \mu |\{u > \mu\}| \\ &\lesssim R(\mu \ln \mu)^{\frac{1}{3}} \int |\nabla u| + R^{-2} (\mu \ln \mu)^{-\frac{2}{3}} \int ||\nabla|^{-1} u|^2. \end{aligned}$$

The choice of $R = (\mu \ln \mu)^{-\frac{1}{3}}$ yields the desired estimate. □

We can now give the proof of the strong interpolation inequality in dimension 2.

Proof of Proposition 1.2. By a scaling argument and the result in Proposition 1.1, it is enough to show for $M \gg 1$:

$$\int_{u \geq M} u^{\frac{4}{3}} \ln^{\frac{1}{3}} u \lesssim \|\nabla u\|_1 + \||\nabla|^{-1} u\|_2^2.$$

We consider an arbitrary level $\mu \geq M \gg 1$ and start as in the proof of Proposition 3.4, considering the potential $\phi_{\mu,R,L}$. For L chosen such that $(\frac{L}{R})^2 \sim \mu$, we get

$$\int \chi_\mu \lesssim R \int |\nabla \chi_\mu| + \mu^{-1} \int \phi_{\mu,R,L} u.$$

But we now rather proceed as in Proposition 1.1. We multiply with $(\mu \ln \mu)^{\frac{1}{3}}$, choose $R = (\mu \ln \mu)^{-\frac{1}{3}}$ and integrate in $\mu \in (M, \infty)$ for $M \gg 1$:

$$\begin{aligned} &\int_M^\infty (\mu \ln \mu)^{\frac{1}{3}} \int \chi_\mu dx \, d\mu \\ &\lesssim \int_M^\infty \int |\nabla \chi_\mu| dx d\mu + \int \int_M^\infty \frac{\ln^{\frac{1}{3}} \mu}{\mu^{\frac{2}{3}}} \phi_{\mu,R,L} d\mu u dx \\ &\leq \|\nabla u\|_1 + \|\nabla \int_M^\infty \frac{\ln^{\frac{1}{3}} \mu}{\mu^{\frac{2}{3}}} \phi_{\mu,R,L} d\mu\|_2 \, \||\nabla|^{-1} u\|_2. \end{aligned}$$

On the last right-hand side term we argue along the lines of Proposition 1.1, now using the property of our geometric construction that $\int \nabla\phi_{\mu,R,L}\cdot\nabla\phi_{\mu',R',L'}dx \lesssim \frac{1}{R^2}\frac{1}{\ln\frac{L}{R}}\int\chi_\mu dx$ (we use the short-hand notation $R' := \mu'^{-\frac{1}{3}}$, $(\frac{L'}{R'})^2\sim\mu'$):

$$
\begin{aligned}
&\|\nabla(\int_M^\infty \frac{\ln^{\frac{1}{3}}\mu}{\mu^{\frac{2}{3}}}\phi_{\mu,R,L}d\mu)\|_2^2\\
&= \int_M^\infty\int_M^\infty \frac{\ln^{\frac{1}{3}}\mu}{\mu^{\frac{2}{3}}}\frac{\ln^{\frac{1}{3}}\mu'}{\mu'^{\frac{2}{3}}}\int\nabla\phi_{\mu,R,L}\cdot\nabla\phi_{\mu',R',L'}dxd\mu'd\mu\\
&= 2\int_M^\infty\int_M^\mu \frac{\ln^{\frac{1}{3}}\mu}{\mu^{\frac{2}{3}}}\frac{\ln^{\frac{1}{3}}\mu'}{\mu'^{\frac{2}{3}}}\int\nabla\phi_{\mu,R,L}\cdot\nabla\phi_{\mu',R',L'}dxd\mu'd\mu\\
&\lesssim \int_M^\infty\int_M^\mu \frac{\ln^{\frac{1}{3}}\mu}{\mu^{\frac{2}{3}}}\frac{\ln^{\frac{1}{3}}\mu'}{\mu'^{\frac{2}{3}}}\frac{1}{R^2}\frac{1}{\ln\frac{L}{R}}\int\chi_\mu dxd\mu'd\mu\\
&\overset{\text{choice } R,L}{\sim} \int_M^\infty\int_M^\mu \frac{\ln^{\frac{1}{3}}\mu'}{\mu'^{\frac{2}{3}}}d\mu'\int\chi_\mu dxd\mu\\
&\lesssim \int_M^\infty(\mu\ln\mu)^{\frac{1}{3}}\int\chi_\mu dxd\mu.
\end{aligned}
$$

Hence we can absorb this term by Young's inequality and obtain

$$\int_M^\infty(\mu\ln\mu)^{\frac{1}{3}}\int\chi_\mu dxd\mu\lesssim\|\nabla u\|_1+\||\nabla|^{-1}u\|_2^2.$$

We conclude by observing that

$$\int_M^\infty(\mu\ln\mu)^{\frac{1}{3}}\int\chi_\mu dx\,d\mu=\int_{u>M}\int_M^{u(x)}(\mu\ln\mu)^{\frac{1}{3}}d\mu\,dx\overset{M\gg 1}{\gtrsim}\int_{u>2M}u^{\frac{4}{3}}\ln^{\frac{1}{3}}u.$$

□

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