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Hypoellipticity and Non Hypoellipticity for Sums of Squares of Complex Vector Fields

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Abstract

In this talk we consider the analogue of Kohn's operator but with a point singularity,

$$P = BB^* + B^*(t^{2\ell} + x^{2k})B, \qquad B = D_x + ix^{q-1}D_t.$$

We show that this operator is hypoelliptic and Gevrey hypoelliptic in a certain range, namely $k < \ell q$, with Gevrey index $\frac{\ell q}{\ell q - k} = 1 + \frac{k}{\ell q - k}$. Outside the above range of the parameters, i.e. when $k \geq \ell q$, the operator is not even hypoelliptic.

1. Introduction

In J. J. Kohn's recent paper [11] (see also [6]) the operator

$$E_{m,k} = L_m \overline{L_m} + \overline{L_m}|z|^{2k} L_m, \quad L_m = \frac{\partial}{\partial z} - i\bar{z}|z|^{2(m-1)}\frac{\partial}{\partial t}$$

was introduced and shown to be hypoelliptic, yet to lose $2 + \frac{k-1}{m}$ derivatives in L^2 Sobolev norms. Christ [7] showed that the addition of one more variable destroyed hypoellipticity altogether.

In a recent volume, dedicated to J.J.Kohn, A.Bove and D.S.Tartakoff, [5], showed that Kohn's operator with an added Oleinik-type singularity, of the form studied in [4],

$$E_{m,k} + |z|^{2(p-1)} D_s^2$$

is s–Gevrey hypoelliptic for any $s \geq \frac{2m}{p-k}$, (here $2m > p > k$). A related result is that for the 'real' version, with $X = D_x + ix^{q-1}D_t$, where $D_x = i^{-1}\partial_x$,

$$R_{q,k} + x^{2(p-1)} D_s^2 = XX^* + (x^k X)^*(x^k X) + x^{2(p-1)} D_s^2$$

is sharply s–Gevrey hypoelliptic for any $s \geq \frac{q}{p-k}$, where $q > p > k$ and q is an even integer.

Here we consider the operator

$$P = BB^* + B^*(t^{2\ell} + x^{2k})B, \qquad B = D_x + ix^{q-1}D_t, \tag{1.1}$$

where k, ℓ and q are positive integers, q even.

Observe that P is a sum of three squares of complex vector fields, but, with a small change not altering the results, we might make P a sum of two squares of complex vector fields in two variables, depending on the same parameters, e.g. $BB^* + B^*(t^{2\ell} + x^{2k})^2 B$.

Let us also note that the characteristic variety of P is $\{x = 0, \xi = 0\}$, i.e. a codimension two analytic symplectic submanifold of $T^*\mathbb{R}^2 \setminus 0$, as in the case of Kohn's operator. Moreover the Poisson-Treves stratification for P has a single stratum thus coinciding with the characteristic manifold of P.

We want to analyze the hypoellipticity of P, both in C^∞ and in Gevrey classes. As we shall see the Gevrey classes play an important role. Here are our results:

Theorem 1.1. *Let P be as in* (1.1), q even.

(i) Suppose that

$$\ell > \frac{k}{q}. \tag{1.2}$$

Then P is C^∞ hypoelliptic (in a neighborhood of the origin) with a loss of $2\frac{q-1+k}{q}$ derivatives.

(ii) Assume that the same condition as above is satisfied by the parameters ℓ, k and q. Then P is s–Gevrey hypoelliptic for any s, with

$$s \geq \frac{\ell q}{\ell q - k}. \tag{1.3}$$

(iii) The value in (1.3) for the Gevrey hypoellipticity of P is optimal, i.e. P is **not** s–Gevrey hypoelliptic for any

$$1 \leq s < \frac{\ell q}{\ell q - k}.$$

(iv) Assume now that

$$\ell \leq \frac{k}{q}. \tag{1.4}$$

Then P is not C^∞ hypoelliptic.

The proof of the above theorem is lengthy and will appear in the forthcoming paper [3]. We refer to [3] for greater details, comments as well as further references.

In this paper we give a sketch of the proofs of items (i) and (ii) of the theorem using two different methods.

It is worth noting that the operator P satisfies the complex Hörmander condition, i.e. the brackets of the fields of length up to $k+q$ generate the two dimensional complex Lie algebra $\mathbb{C}^2$. Note that in the present case the vector fields involved are B^*, $x^k B$ and $t^\ell B$, but only the first two enter in the brackets spanning $\mathbb{C}^2$.

A couple of remarks are in order. The above theorem seems to us to suggest strongly that Treves conjecture cannot be extended to the case of sums of squares of complex vector fields, since lacking C^∞ hypoellipticity we believe that P is not analytic hypoelliptic for any choice of the parameters. We will address this point further in the subsequent paper.

The second and trivial remark is that, even in two variables, there are examples of sums of squares of complex vector fields, satisfying the Hörmander condition, that are *not* hypoelliptic. In this case the characteristic variety is a symplectic manifold. In our opinion this is due to the point singularity exhibited by the second and third vector field, or by $(t^{2\ell}+x^{2k})B$ in the two-fields version.

Restricting ourselves to the case q even is no loss of generality, since the operator (1.1) corresponding to an odd integer q is plainly hypoelliptic and actually subelliptic, meaning by that term that there is a loss of less than two derivatives. This fact is due to special circumstances, i.e. that the operator B^* has a trivial kernel in that case. We stress the fact that the original Kohn's operator, in the complex variable z, automatically has an even q, while in the "real case" the parity of q matters.

We also want to stress microlocal aspects of the theorem: the characteristic manifold of P is symplectic in $T^*\mathbb{R}^2$ of codimension 2 and as such it may be identified with $T^*\mathbb{R}\setminus 0 \sim \{(t,\tau) \mid \tau \neq 0\}$ (leaving aside the origin in the τ variable). On the other hand, the operator $P(x,t,D_x,\tau)$, thought of as a differential operator in the x-variable depending on (t,τ) as parameters, for $\tau>0$ has an eigenvalue of the form $\tau^{2/q}(t^{2\ell}+a(t,\tau))$, modulo a non zero function of t. Here $a(t,\tau)$ denotes a (non-classical) symbol of order -1 defined for $\tau>0$ and such that $a(0,\tau)\sim\tau^{-\frac{2k}{q}}$. Thus we may consider the pseudodifferential operator $\Lambda(t,D_t)=\operatorname{Op}\left(\tau^{2/q}(t^{2\ell}+a(t,\tau))\right)$ as defined in a microlocal neighborhood of our base point in the characteristic manifold of P. One can show that the hypoellipticity properties of P are shared by Λ, e.g. P is C^∞ hypoelliptic iff Λ is.

The last section of this paper includes a computation of the symbol of Λ as well as the proof that P is hypoelliptic if Λ is hypoelliptic. This is done following ideas of Boutet de Monvel, Helffer and Sjöstrand.

2. The operator P is C^∞ hypoelliptic

Theorem 2.1. *Under the restriction that $k < \ell q$, q even, the operator P is hypoelliptic.*

Denoting by $W_j, j = 1, 2, 3, 4$ the operators

$$W_1 = B^*, \quad W_2 = t^\ell B, \quad W_3 = x^k B, \quad \textit{and} \quad W_4 = \langle D_t\rangle^{-\frac{k-1}{q}},$$

$(\sigma(\langle D_t\rangle) = (1 + |\tau|^2)^{1/2})$ then for $v \in C_0^\infty$ and of small support near $(0, 0)$ we have the estimate, following [11] and [6],

$$\sum_{j=1}^{4} \|W_j v\|^2 \lesssim (Pv, v) + \|v\|^2_{-\infty}$$

where, unless otherwise noted, norms and inner products are in $L^2(R^2)$. Here the last norm indicates a Sobolev norm of arbitrarily negative order. This estimate was established in [11] without the norm of $t^\ell B$ (and without the term $B^* t^{2\ell} B$ in the operator) and our estimate follows at once in our setting.

A first observation is that we may work microlocally near the τ axis, since away from that axis (conically) the operator is elliptic.

A second observation is that no localization in space is necessary, since away from the origin $(0, 0)$, we have estimates on both $\|Bv\|^2$ and $\|B^* v\|^2$, and hence the usual subellipticity (since $q - 1$ brackets of B and B^* generate the 'missing' vector field $\dfrac{\partial}{\partial t}$).

Our aim will be to show that for a solution u of $Pu = f \in C^\infty$ and arbitrary N,

$$(\frac{\partial}{\partial t})^N u \in L^2_{loc}.$$

To do this, we pick a Sobolev space to which the solution belongs, i.e., in view of the ellipticity of P away from the τ axis, we pick s_0 such that $\langle D_t\rangle^{-s_0} u \in L^2_{loc}$ and from now on all indices on norms will be in the variable t only.

Actually we will change our point of view somewhat and assume that the left hand side of the *a priori* estimate is finite locally for u with norms reduced by s_0 and show that this is true with the norms reduced by only $s_0 - \delta$ for some (fixed) $\delta > 0$.

Taking $s_0 = 0$ for simplicity, we will assume that $\sum_1^4 \|W_j u\|_0 < \infty$ and show that in fact $\sum_1^4 \|W_j u\|_\delta < \infty$ for some positive δ. Iterating this‘bootstrap operation will prove that the solution is indeed smooth.

The main new ingredient in proving hypoellipticity is the presence of the term $t^\ell B$, which will result in new brackets. As in Kohn's work and ours, the solution u will initially be smoothed out in t so that the estimate may be applied freely, and at the end the smoothing will be allowed to tend suitably to the identity and we will be able to apply a Lebesgue bounded convergence theorem to show that the $\sum_1^4 \|W_j u\|_\delta$ are also finite, leading to hypoellipticity.

Without loss of generality, as observed above, we may assume that the solution u to $Pu = f \in C_0^\infty$ has small support near the origin (to be more thorough, we could take a localizing function of small support, ζ, and write $P\zeta u = \zeta P u + [P, \zeta]u = \zeta f \mod C_0^\infty$ so that $P\zeta u \in C_0^\infty$ since we have already seen that u will be smooth in the support of derivatives of ζ by the hypoellipticity of P away from the origin.)

In order to smooth out the solution in the variable t, we introduce a standard cut-off function $\chi(\tau) \in C_0^\infty(|\tau| \leq 2), \chi(\tau) \equiv 1, |\tau| \leq 1$, and set $\chi_M(\tau) = \chi(\tau/M)$. Thus $\chi_M(D)$ is infinitely smoothing (in t) and, in supp $\chi_M', \tau \sim M$ and $|\chi_M^{(j)}| \sim M^{-j}$. Further, as $M \to \infty$, $\chi_M(D) \to Id$ in such a way that it suffices to show $\|\chi_M(D)w\|_r \leq C$ independent of M to conclude that $w \in H^r$.

Introducing of χ_M, however, destroys compact support, so we shall introduce $v = \psi(x,t)\langle D_t\rangle^\delta \chi_M(D)u$ into the *a priori* estimate and show that the left hand remains bounded uniformly in M as $M \to \infty$.

For clarity, we restate the estimate in the form in which we will use it, suppressing the spatial localization now as discussed above:

$$\|B^*\langle D_t\rangle^\delta \chi_M u\|^2 + \|t^\ell B\langle D_t\rangle^\delta \chi_M u\|^2 + \|x^k B\langle D_t\rangle^\delta \chi_M u\|^2 + \|\langle D_t\rangle^{-\frac{k-1}{q}}\langle D_t\rangle^\delta \chi_M u\|^2$$

$$\lesssim (P\langle D_t\rangle^\delta \chi_M u, \langle D_t\rangle^\delta \chi_M u) + \|\langle D_t\rangle^\delta \chi_M u\|_{-\infty}^2.$$

Clearly the most interesting bracket which will enter in bringing $\langle D_t\rangle^\delta \chi_M$ past the operator P, and the only term which has not been handled in the two papers cited above, is when t^l is differentiated, as in

$$([B^* t^{2\ell} B, \langle D_t\rangle^\delta \chi_M]u, \langle D_t\rangle^\delta \chi_M u) \sim (B^*[t^{2\ell}, \langle D_t\rangle^\delta \chi_M]Bu, \langle D_t\rangle^\delta \chi_M u)$$
$$\sim \sum (t^{2\ell-j}(\langle D_t\rangle^\delta \chi_M)^{(j)} Bu, B\langle D_t\rangle^\delta \chi_M u)$$

in obvious notation. The derivatives on the symbol of $\langle D_t\rangle^\delta \chi_M$ are denoted $(\langle D_t\rangle^\delta \chi_M)^{(j)}$.

So a typical term would lead, after using a weighted Schwarz inequality and absorbing a term on the left hand side of the estimate, to the need to estimate a constant times the norm

$$\|t^{\ell-j}(\langle D_t\rangle^\delta \chi_M)^{(j)} Bu\|^2.$$

Now we are familiar with handling such terms, although in the above cited works it was powers of x (or z in the complex case) instead of powers of t. The method employed is to 'raise and lower' powers of t and of τ on one side of an inner product and lower them on the other. That is, if we denote by A the operator

$$A = t\langle D_t\rangle^\rho,$$

we have

$$\|A^r w\|^2 = |(A^\rho w, A^\rho w)| \lesssim_N \|w\|^2 + \|A^N w\|^2$$

for any desired positive $N \geq r$ (repeated integrations by parts or by interpolation, since the non-self-adjointness of A is of lower order), together with the observation that a small constant may be placed in front of either term on the right, and the notation $\lesssim_N$ means that the constants involved may depend on N, but N will always be bounded.

In our situation, looking first at the case $j = 1$,

$$\begin{aligned}
&\|t^{\ell-1}B(\langle D_t\rangle^\delta \chi_M)'u\|^2 \\
&\qquad = |(A^{\ell-1}\langle D_t\rangle^{-(\ell-1)\rho}B(\langle D_t\rangle^\delta \chi_M)'u, A^{\ell-1}\langle D_t\rangle^{-(\ell-1)\rho}B(\langle D_t\rangle^\delta \chi_M)'u)| \\
&\qquad \leq \|A^{\ell}\langle D_t\rangle^{-(\ell-1)\rho}B(\langle D_t\rangle^\delta \chi_M)'u\|^2 + \|\langle D_t\rangle^{-(\ell-1)\rho}B(\langle D_t\rangle^\delta \chi_M)'u\|^2 \\
&\qquad = \|t^{\ell}\langle D_t\rangle^{\rho}B(\langle D_t\rangle^\delta \chi_M)'u\|^2 + \|\langle D_t\rangle^{-(\ell-1)\rho}B(\langle D_t\rangle^\delta \chi_M)'u\|^2 \\
&\qquad\qquad \sim \|t^\ell B\tilde{\chi}_M u\|^2_{\rho+\delta-1} + \|B\tilde{\chi}_M u\|^2_{-(\ell-1)\rho+\delta-1}
\end{aligned}$$

modulo further brackets, where $\tilde{\chi}_M$ is another function of τ such as $\langle D_t\rangle\chi'_M$, with symbol uniformly bounded in τ independently of M and of compact support. $\tilde{\chi}_M$ will play the same role as χ_M in future iterations of the *a priori* estimate.

We are not yet done - the first term on the right will be handled inductively provided $\rho - 1 < 0$, but the second contains just B without the essential powers of t.

However, as in [11], we may integrate by parts, thereby converting B to B^* which is maximally controlled in the estimate, but modulo a term arising from the bracket of B and B^*.

As in [6] or [11], or by direct computation, we have

$$\|Bw\|_r^2 \lesssim \|B^*w\|_r^2 + \|x^{\frac{q-2}{2}}w\|_{r+1/2}$$

and while this power of x may not be directly useful, we confronted the same issue in [6] (in the complex form - the 'real' one is analogous). In that context, the exponent $q-2/2$ was denoted $m-1$, but the term was well estimated in norm $-\frac{1}{2m}+\frac{1}{2}-\frac{k-1}{q}$, which in this context reads $-\frac{1}{q}+\frac{1}{2}-\frac{k-1}{q} = \frac{1}{2}-\frac{k}{q}$. We have $-(\ell-1)\rho-1+\frac{1}{2}$, and under our hypothesis that $\ell > k/q$ our norm is less than $1/2 - k/q$ for any choice of $\rho \leq 1$ as desired.

Finally, the terms with $j > 1$ work out similarly.

This means that we do indeed have a weaker norm so that with a different cut off in τ, which we have denoted $\tilde{\chi}_M$, there is a gain, and that as $M \to \infty$ this term will remain bounded.

3. Gevrey hypoellipticity

Again we write the example as

$$P=\sum_1^3 W_j^* W_j,$$

with

$$W_1=B^*,\quad W_2=t^\ell B,\quad W_3=x^k B,\quad B=D_x+ix^{q-1}D_t$$

and omit localization as discussed above, and set $v=T^p u$, the *a priori* estimate we have is

$$\sum_1^4 \|W_j v\|_0^2 \lesssim |(Pv,v)|,\quad W_4=\langle D_t\rangle^{-\frac{k-1}{q}}.$$

The principal (bracketing) errors come from $[W_j,T^p]v, j=1,2,3,$ and the worst case occurs when $j=2:$

$$[W_2,T^p]v=p\ell t^{\ell-1}BT^{p-1}v.$$

Raising and lowering powers of t as above,

$$\begin{aligned}\|t^{\ell-1}BT^{p-1}u\| &\lesssim \|t^\ell BT^{p-1+\delta}u\|+\|BT^{p-1-(\ell-1)\delta}u\|\\ &\lesssim \|W_2T^{p-1+\delta}u\|+\|B^*T^{p-1-(\ell-1)\delta}u\|+\|x^{q-1}T^{p-(\ell-1)\delta}u\|\end{aligned}$$

using the fact that $B-B^*=\pm ix^{q-1}T$. Again we raise and lower powers of x to obtain

$$\|x^{q-1}T^{p-(\ell-1)\delta}u\|\lesssim\|T^{p-(\ell-1)\delta-(q-1)\rho}u\|+\|\{x^{k+q-1}T\}\,T^{p-1-(\ell-1)\delta+k\rho}\,u\|$$

since the term in braces is a linear combination of x^kB and B^*, both of which are optimally estimated. The result is that

$$\begin{aligned}\|t^{\ell-1}BT^{p-1}u\| &\lesssim \|t^\ell BT^{p-1+\delta}u\|+\|BT^{p-1-(\ell-1)\delta}u\|\\ &\lesssim \|W_2T^{p-1+\delta}u\|+\|W_1T^{p-1-(\ell-1)\delta}u\|+\|W_1T^{p-1-(\ell-1)\delta+k\rho}\,u\|\\ &\quad+\|W_4T^{p-(\ell-1)\delta-(q-1)\rho+\frac{k-1}{\ell}}u\|+\|W_3T^{p-1-(\ell-1)\delta+k\rho}\,u\|\end{aligned}$$

where the third term on the right clearly dominates the second. In all, then,

$$\|t^{\ell-1}BT^{p-1}u\|\lesssim\sum_{j=1}^4\|X_jT^{p-\sigma}u\|$$

where

$$\sigma = \min_i \sup_{\substack{0<\rho<1 \\ 0<\delta<1}} s_i$$

with

$$\begin{aligned} s_1 &= 1 - \delta \\ s_2 &= (\ell - 1)\delta + (q-1)\rho - \frac{l-1}{q} \\ s_3 &= 1 + (\ell - 1)\delta - k\rho. \end{aligned}$$

The desired value of σ is achieved when all three are equal by a standard minimax argument, and this occurs when

$$\delta = \frac{k}{q\ell}, \qquad \rho = \frac{1}{q}$$

resulting in $\sigma = \dfrac{\ell q - k}{\ell q}$, which yields $G^{\frac{\ell q}{\ell q - k}} = G^{1+\frac{k}{\ell q - k}}$ hypoellipticity.

The restriction that $k < \ell q$ for hypoellipticity at all takes on greater meaning given this result.

4. Computing Λ

4.1. **q-Pseudodifferential calculus.** The idea, attributed by Sjöstrand and Zworski, [12], to Schur, is essentially a linear algebra remark: assume that the $n \times n$ matrix A has zero in its spectrum with multiplicity one. Then of course A is not invertible, but, denoting by e_0 the zero eigenvector of A, the matrix (in block form)

$$\begin{bmatrix} A & e_0 \\ {}^t e_0 & 0 \end{bmatrix}$$

is invertible as a $(n+1) \times (n+1)$ matrix in $\mathbb{C}^{n+1}$. Here ${}^t e_0$ denotes the row vector e_0.

All we want to do is to apply this remark to the operator P whose part BB^* has the same problem as the matrix A, i.e. a zero simple eigenvalue. This occurs since q is even. Note that in the case of odd q, P may easily be seen to be hypoelliptic .

It is convenient to use self-adjoint derivatives from now on, so that the vector field $B^* = D_x - ix^{q-1}D_t$, where $D_x = i^{-1}\partial_x$. It will also be convenient to write $B(x, \xi, \tau)$ for the symbol of the vector field B, i.e. $B(x, \xi, \tau) = \xi + ix^{q-1}\tau$ and analogously for the other vector fields involved. The symbol of P can be written as

$$P(x, t, \xi, \tau) = P_0(x, t, \xi, \tau) + P_{-q}(t, x, \xi, \tau) + P_{-2k}(x, t, \xi, \tau), \tag{4.1.1}$$

where

$$P_0(x, t, \xi, \tau) = (1 + t^{2\ell})(\xi^2 + x^{2(q-1)}\tau^2) + (-1 + t^{2\ell})(q - 1)x^{q-2}\tau;$$

$$P_{-q}(x, t, \xi, \tau) = -2\ell t^{2\ell-1}x^{q-1}(\xi + ix^{q-1}\tau);$$

$$P_{-2k}(x, t, \xi, \tau) = x^{2k}(\xi^2 + x^{2(q-1)}\tau^2) - i2kx^{2k-1}(\xi + ix^{q-1}\tau) + (q - 1)x^{2k+q-2}\tau.$$

It is evident at a glance that the different pieces into which P has been decomposed include terms of different order and vanishing speed. We thus need to say something about the adopted criteria for the above decomposition.

Let μ be a positive number and consider the following canonical dilation in the variables (x, t, ξ, τ):

$$x \to \mu^{-1/q}x, \qquad t \to t, \qquad \xi \to \mu^{1/q}\xi, \qquad \tau \to \mu\tau.$$

It is then evident that P_0 has the following homogeneity property

$$P_0(\mu^{-1/q}x, t, \mu^{1/q}\xi, \mu\tau) = \mu^{2/q}P_0(x, t, \xi, \tau). \tag{4.1.2}$$

Analogously

$$P_{-q}(\mu^{-1/q}x, t, \mu^{1/q}\xi, \mu\tau) = \mu^{2/q-1}P_q(x, t, \xi, \tau) \tag{4.1.3}$$

and

$$P_{-2k}(\mu^{-1/q}x, t, \mu^{1/q}\xi, \mu\tau) = \mu^{2/q-(2k)/q}P_{2k}(x, t, \xi, \tau). \tag{4.1.4}$$

Now the above homogeneity properties help us in identifying some symbol classes suitable for P. Following the ideas of [1] and [2] we define the following class of symbols

Definition 4.1.1. *We define the class of symbols* $S_q^{m,k}(\Omega, \Sigma)$ *where* Ω *is a conic neighborhood of the point* $(0, e_2)$ *and* Σ *denotes the characteristic manifold* $\{x = 0, \xi = 0\}$*, as the set of all* C^∞ *functions such that, on any conic subset of* Ω *with compact base,*

$$\left|\partial_t^\alpha \partial_\tau^\beta \partial_x^\gamma \partial_\xi^\delta a(x, t, \xi, \tau)\right| \lesssim (1 + |\tau|)^{m-\beta-\delta} \left(\frac{|\xi|}{|\tau|} + |x|^{q-1} + \frac{1}{|\tau|^{\frac{q-1}{q}}}\right)^{k-\frac{\gamma}{q-1}-\delta}. \tag{4.1.5}$$

We write $S_q^{m,k}$ *for* $S_q^{m,k}(\mathbb{R}^2 \times \mathbb{R}^2, \Sigma)$.

By a straightforward computation, see e.g. [2], we have $S_q^{m,k} \subset S_q^{m',k'}$ iff $m \leq m'$ and $m - \frac{q-1}{q}k \leq m' - \frac{q-1}{q}k'$. $S_q^{m,k}$ can be embedded in the Hörmander classes $S_{\rho,\delta}^{m+\frac{q-1}{q}k_-}$, where $k_- = \max\{0, -k\}$, $\rho = \delta = 1/q \leq 1/2$. Thus we immediately deduce that $P_0 \in S_q^{2,2}$, $P_{-q} \in S_q^{1,2} \subset S_q^{2,2+\frac{q}{q-1}}$ and finally $P_{-2k} \in S_q^{2,2+\frac{2k}{q-1}}$.

We shall need also the following

Definition 4.1.2 ([2])**.** *Let* Ω *and* Σ *be as above. We define the class* $\mathscr{H}_q^m(\Omega, \Sigma)$ *by*

$$\mathscr{H}_q^m(\Omega, \Sigma) = \cap_{j=1}^\infty S_q^{m-j,-\frac{q}{q-1}j}(\Omega, \Sigma).$$

We write $\mathscr{H}_q^m$ *for* $\mathscr{H}_q^m(\mathbb{R}^2 \times \mathbb{R}^2, \Sigma)$.

Now it is easy to see that P_0, as a differential operator w.r.t. the variable x, depending on the parameters t, $\tau \geq 1$ has a non negative discrete spectrum. Morever the dependence on τ of the eigenvalue is particularly simple, because of (4.1.2). Call $\Lambda_0(t, \tau)$ the lowest eigenvalue of P_0. Then

$$\Lambda_0(t, \tau) = \tau^{\frac{2}{q}} \tilde{\Lambda}_0(t).$$

Moreover Λ_0 has multiplicity one and $\tilde{\Lambda}_0(0) = 0$, since BB^* has a null eigenvalue with multiplicity one. Denote by $\varphi_0(x, t, \tau)$ the corresponding eigenfunction. Because of (4.1.2), we have the following properties of φ_0:

a- For fixed $(t, \tau), \varphi_0$ is exponentially decreasing w.r.t. x as $x \to \pm\infty$. In fact, because of (4.1.2), setting $y = x\tau^{1/q}$, we have that $\varphi_0(y, t, \tau) \sim e^{-y^q/q}$.

b- It is convenient to normalize φ_0 in such a way that $\|\varphi_0(\cdot, t, \tau)\|_{L^2(\mathbb{R}_x)} = 1$. This implies that a factor $\sim \tau^{1/2q}$ appears. Thus we are led to the definition of a Hermite operator (see [9] for more details).

Let $\Sigma_1 = \pi_x \Sigma$ be the space projection of Σ. Then we write

Definition 4.1.3. *We write H_q^m for $\mathscr{H}_q^m(\mathbb{R}^2_{x,t} \times \mathbb{R}_\tau, \Sigma_1)$, i.e. the class of all smooth functions in $\cap_{j=1}^{\infty} S_q^{m-j,-\frac{q}{q-1}j}(\mathbb{R}^2_{x,t} \times \mathbb{R}_\tau, \Sigma_1)$. Here $S_q^{m,k}(\mathbb{R}^2_{x,t} \times \mathbb{R}_\tau, \Sigma_1)$ denotes the set of all smooth functions such that*

$$\left|\partial_t^\alpha \partial_\tau^\beta \partial_x^\gamma a(x,t,\tau)\right| \lesssim (1+|\tau|)^{m-\beta} \left(|x|^{q-1} + \frac{1}{|\tau|^{\frac{q-1}{q}}}\right)^{k-\frac{\gamma}{q-1}}. \tag{4.1.6}$$

Define the action of a symbol $a(x,t,\tau)$ in H_q^m as the map $a(x,t,D_t)\colon C_0^\infty(\mathbb{R}_t) \longrightarrow C^\infty(\mathbb{R}^2_{x,t})$ defined by

$$a(x,t,D_t)u(x,t) = (2\pi)^{-1} \int e^{it\tau} a(x,t,\tau)\hat{u}(\tau) d\tau.$$

Such an operator, modulo a regularizing operator (w.r.t. the t variable) is called a Hermite operator and we denote by OPH_q^m the corresponding class.

We need also the adjoint of the Hermite operators defined in Definition 4.1.3.

Definition 4.1.4. *Let $a \in H_q^m$. We define the map $a^*(x,t,D_t)\colon C_0^\infty(\mathbb{R}^2_{x,t}) \longrightarrow C^\infty(\mathbb{R}_t)$ as*

$$a^*(x,t,D_t)u(t) = (2\pi)^{-1} \int\int e^{it\tau} \overline{a(x,t,\tau)}\hat{u}(x,\tau) dx d\tau.$$

We denote by OPH_q^{*m} the related set of operators.

Lemma 4.1.1. *Let $a \in H_q^m$, $b \in S_q^{m,k}$; then*

(i) *the formal adjoint $a(x,t,D_t)^*$ belongs to OPH_q^{*m} and its symbol has the asymptotic expansion*

$$\sigma(a(x,t,D_t)^*) - \sum_{\alpha=0}^{N-1} \frac{1}{\alpha!} \partial_\tau^\alpha D_t^\alpha \overline{a(x,t,\tau)} \in H_q^{m-N}. \tag{4.1.7}$$

(ii) *The formal adjoint $(a^*(x,t,D_t))^*$ belongs to OPH_q^m and its symbol has the asymptotic expansion*

$$\sigma(a^*(x,t,D_t)^*) - \sum_{\alpha=0}^{N-1} \frac{1}{\alpha!} \partial_\tau^\alpha D_t^\alpha a(x,t,\tau) \in H_q^{m-N}. \tag{4.1.8}$$

(iii) *The formal adjoint $b(x,t,D_x,D_t)^*$ belongs to $OPS_q^{m,k}$ and its symbol has the asymptotic expansion*

$$\sigma(a(x,t,D_x,D_t)^*) - \sum_{\alpha=0}^{N-1} \frac{1}{\alpha!} \partial^\alpha_{(\xi,\tau)} D^\alpha_{(x,t)} \overline{a(x,t,\xi,\tau)} \in S_q^{m-N, k-N\frac{q}{q-1}}. \tag{4.1.9}$$

The following is a lemma on compositions involving the two different types of Hermite operators defined above. First we give a definition of "global" homogeneity:

Definition 4.1.5. *We say that a symbol $a(x,t,\xi,\tau)$ is globally homogeneous (abbreviated g.h.) of degree m, if, for $\lambda \geq 1$, $a(\lambda^{-1/q}x, t, \lambda^{1/q}\xi, \lambda\tau) = \lambda^m a(x,t,\xi,\tau)$. Analogously a symbol, independent of ξ, of the form $a(x,t,\tau)$ is said to be globally homogeneous of degree m if $a(\lambda^{-1/q}x, t, \lambda\tau) = \lambda^m a(x,t,\tau)$.*

Let $f_{-j}(x,t,\xi,\tau) \in S_q^{m,k+\frac{j}{q-1}}$, $j \in \mathbb{N}$, then there exists $f(x,t,\xi,\tau) \in S_q^{m,k}$ such that $f \sim \sum_{j\geq 0} f_{-j}$, i.e. $f - \sum_{j=0}^{N-1} f_{-j} \in S_q^{m,k+\frac{N}{q-1}}$, thus f is defined modulo a symbol in $S_q^{m,\infty} = \cap_{h\geq 0} S_q^{m,h}$.

Analogously, let f_{-j} be globally homogeneous of degree $m - k\frac{q-1}{q} - \frac{j}{q}$ and such that for every $\alpha, \beta \geq 0$ satisfies the estimates

$$\left| \partial^\gamma_{(t,\tau)} \partial^\alpha_x \partial^\beta_\xi f_{-j}(x,t,\xi,\tau) \right| \lesssim \left(|\xi| + |x|^{q-1} + 1 \right)^{k - \frac{\alpha}{q-1} - \beta}, \qquad (x,\xi) \in \mathbb{R}^2, \tag{4.1.10}$$

for (t,τ) in a compact subset of $\mathbb{R} \times \mathbb{R} \setminus 0$ and every multiindex γ. Then $f_{-j} \in S_q^{m,k+\frac{j}{q-1}}$.

Accordingly, let $\varphi_{-j}(x,t,\tau) \in H_q^{m-\frac{j}{q}}$, then there exists $\varphi(x,t,\tau) \in H_q^m$ such that $\varphi \sim \sum_{j\geq 0} \varphi_{-j}$, i.e. $\varphi - \sum_{j=0}^{N-1} \varphi_{-j} \in H_q^{m-\frac{N}{q}}$, so that φ is defined modulo a symbol regularizing (w.r.t. the t variable.)

Similarly, let φ_{-j} be globally homogeneous of degree $m - \frac{j}{q}$ and such that for every $\alpha, \ell \geq 0$ satisfies the estimates

$$\left| \partial^\beta_{(t,\tau)} \partial^\alpha_x \varphi_{-j}(x,t,\tau) \right| \lesssim \left(|x|^{q-1} + 1 \right)^{-\ell - \frac{\alpha}{q-1}}, \qquad x \in \mathbb{R}, \tag{4.1.11}$$

for (t,τ) in a compact subset of $\mathbb{R} \times \mathbb{R} \setminus 0$ and every multiindex β. Then $\varphi_{-j} \in H_q^{m-\frac{j}{q}}$.

As a matter of fact in the construction below we deal with asymptotic series of homogeneous symbols.

Next we give a brief description of the composition of the various types of operator introduced so far.

Lemma 4.1.2 ([9], Formula 2.4.9). *Let $a \in S_q^{m,k}$, $b \in S_q^{m',k'}$, with asymptotic globally homogeneous expansions*

$$
\begin{aligned}
a &\sim \sum_{j\geq 0} a_{-j}, \qquad a_{-j} \in S_q^{m,k+\frac{j}{q-1}}, \text{ g. h. of degree } m - \frac{q-1}{q}k - \frac{j}{q}\\
b &\sim \sum_{i\geq 0} b_{-i}, \qquad b_{-i} \in S_q^{m',k'+\frac{i}{q-1}}, \text{ g. h. of degree } m' - \frac{q-1}{q}k' - \frac{i}{q}.
\end{aligned}
$$

Then $a \circ b$ is an operator in $OPS_q^{m+m',k+k'}$ with

$$
\sigma(a\circ b) - \sum_{s=0}^{N-1} \sum_{q\alpha+i+j=s} \frac{1}{\alpha!}\sigma\left(\partial_\tau^\alpha a_{-j}(x,t,D_x,\tau)\circ_x D_t^\alpha b_{-i}(x,t,D_x,\tau)\right) \in S_q^{m+m'-N,k+k'}. \tag{4.1.12}
$$

Here $\circ_x$ denotes the composition w.r.t. the x-variable.

Lemma 4.1.3 ([2], Section 5 and [9], Sections 2.2, 2.3). *Let $a \in H_q^m$, $b \in H_q^{m'}$ and $\lambda \in S_{1,0}^{m''}(\mathbb{R}_t \times \mathbb{R}_\tau)$ with homogeneous asymptotic expansions*

$$
\begin{aligned}
a &\sim \sum_{j\geq 0} a_{-j}, \qquad a_{-j} \in H_q^{m-\frac{j}{q}}, \text{ g. h. of degree } m - \frac{j}{q}\\
b &\sim \sum_{i\geq 0} b_{-i}, \qquad b_{-i} \in H_q^{m'-\frac{i}{q}}, \text{ g. h. of degree } m' - \frac{i}{q}\\
\lambda &\sim \sum_{\ell\geq 0} \lambda_{-\ell}, \qquad \lambda_{-\ell} \in S_{1,0}^{m''-\frac{\ell}{q}}, \text{ homogeneous of degree } m'' - \frac{\ell}{q}
\end{aligned}
$$

Then

(i) *$a \circ b^*$ is an operator in $OP\mathscr{H}_q^{m+m'-\frac{1}{q}}(\mathbb{R}^2,\Sigma)$ with*

$$
\sigma(a\circ b^*)(x,t,\xi,\tau) - e^{-ix\xi}\sum_{s=0}^{N-1} \sum_{q\alpha+i+j=s} \frac{1}{\alpha!}\partial_\tau^\alpha a_{-j}(x,t,\tau) D_t^\alpha \hat{\bar{b}}_{-i}(\xi,t,\tau) \in \mathscr{H}_q^{m+m'-\frac{1}{q}-\frac{N}{q}}, \tag{4.1.13}
$$

where the Fourier transform in $D_t^\alpha \hat{\bar{b}}_{-i}(\xi,t,\tau)$ is taken w.r.t. the x-variable.

(ii) $b^* \circ a$ *is an operator in* $OPS_{1,0}^{m+m'-\frac{1}{q}}(\mathbb{R}_t)$ *with*

$$\sigma(b^* \circ a)(t,\tau) - \sum_{s=0}^{N-1} \sum_{q\alpha+j+i=s} \frac{1}{\alpha!} \int \partial_\tau^\alpha \bar{b}_{-i}(x,t,\tau) D_t^\alpha a_{-j}(x,t,\tau) dx \in S_{1,0}^{m+m'-\frac{1}{q}-\frac{N}{q}}(\mathbb{R}_t). \tag{4.1.14}$$

(iii) $a \circ \lambda$ *is an operator in* $OPH_q^{m+m''}$. *Furthermore its asymptotic expansion is given by*

$$\sigma(a \circ \lambda) - \sum_{s=0}^{N-1} \sum_{q\alpha+j+\ell=s} \frac{1}{\alpha!} \partial_\tau^\alpha a_{-j}(x,t,\tau) D_t^\alpha \lambda_{-\ell}(t,\tau) \in H_q^{m+m''-\frac{N}{q}}. \tag{4.1.15}$$

Lemma 4.1.4. *Let* $a(x,t,D_x,D_t)$ *be in the class* $OPS_q^{m,k}(\mathbb{R}^2,\Sigma)$ *and* $b(x,t,D_t)$ *in the class* $OPH_q^{m'}$ *with g.h. asymptotic expansions*

$$\begin{aligned} a &\sim \sum_{j\geq 0} a_{-j}, \qquad a_{-j} \in S_q^{m,k+\frac{j}{q-1}}, \text{ g. h. of degree } m - \frac{q-1}{q}k - \frac{j}{q} \\ b &\sim \sum_{i\geq 0} b_{-i}, \qquad b_{-i} \in H_q^{m'-\frac{i}{q-1}}, \text{ g. h. of degree } m' - \frac{i}{q}. \end{aligned}$$

Then $a \circ b \in OPH_q^{m+m'-k\frac{q-1}{q}}$ *and has a g.h. asymptotic expansion of the form*

$$\sigma(a \circ b) - \sum_{s=0}^{N-1} \sum_{q\ell+i+j=s} \frac{1}{\ell!} \partial_\tau^\ell a_{-j}(x,t,D_x,\tau)(D_t^\ell b_{-i}(\cdot,t,\tau)) \in H_q^{m+m'-k\frac{q-1}{q}-\frac{N}{q}}. \tag{4.1.16}$$

Lemma 4.1.5. *Let* $a(x,t,D_x,D_t)$ *be an operator in the class* $OPS_q^{m,k}(\mathbb{R}^2,\Sigma)$, $b^*(x,t,D_t) \in OPH_q^{*m'}$ *and* $\lambda(t,D_t) \in OPS_{1,0}^{m''}(\mathbb{R}_t)$ *with homogeneous asymptotic expansions*

$$\begin{aligned} a &\sim \sum_{j\geq 0} a_{-j}, \qquad a_{-j} \in S_q^{m,k+\frac{j}{q-1}}, \text{ g. h. of degree } m - \frac{q-1}{q}k - \frac{j}{q} \\ b &\sim \sum_{i\geq 0} b_{-i}, \qquad b_{-i} \in H_q^{m'-\frac{i}{q-1}}, \text{ g. h. of degree } m' - \frac{i}{q} \\ \lambda &\sim \sum_{\ell\geq 0} \lambda_{-\ell}, \qquad \lambda_{-\ell} \in S_{1,0}^{m''-\frac{\ell}{q}}, \text{ homogeneous of degree } m'' - \frac{\ell}{q} \end{aligned}$$

Then

(i) $b^*(x,t,D_t)\circ a(x,t,D_x,D_t)\in OPH_q^{*m+m'-\frac{q-1}{q}k}$ *with g.h. asymptotic expansion*

$$\sigma(b^*\circ a)-\sum_{s=0}^{N-1}\sum_{q\ell+i+j=s}\frac{1}{\ell!}D_t^{\ell}\left(\overline{a_{-j}}(x,t,D_x,\tau)\right)^*\left(\partial_\tau^{\ell}\overline{b_{-i}}(\cdot,t,\tau)\right) \in H_q^{m+m'-k\frac{q-1}{q}-\frac{N}{q}}. \tag{4.1.17}$$

(ii) $\lambda(t,D_t)\circ b^*(x,t,D_t)\in OPH_q^{*m'+m''}$ *with asymptotic expansion*

$$\sigma(\lambda\circ b^*)-\sum_{s=0}^{N-1}\sum_{q\alpha+i+\ell=s}\frac{1}{\alpha!}\partial_\tau^{\alpha}\lambda_{-\ell}(t,\tau)D_t^{\alpha}\overline{b_{-i}}(x,t,\tau)\in H_q^{m'+m''-\frac{N}{q}}. \tag{4.1.18}$$

The proofs of Lemmas 4.1.2 –4.1.4 are obtained with a q-variation of the calculus developed by Boutet de Monvel and Helffer, [2], [9]. The proof of Lemma 4.1.5 is performed taking the adjoint and involves a combinatoric argument; we give here a sketchy proof.

Proof. We prove item (i). The proof of (ii) is similar and simpler.

Since $b^*(x,t,D_t)\circ a(x,t,D_x,D_t)=(a(x,t,D_x,D_t)^*\circ b^*(x,t,D_t)^*)^*$, using Lemma 4.1.1 and 4.1.2, we first compute

$$\begin{aligned}
&\sigma(a(x,t,D_x,D_t)^*\circ b^*(x,t,D_t)^*)\\
&\qquad=\sum_{\alpha,\ell,p,i,j\geq 0}\frac{1}{\ell!\alpha!p!}\partial_\tau^{\alpha+p}D_t^{\alpha}\left(a_{-j}(x,t,D_x,\tau)\right)^*\left(\partial_\tau^{\ell}D_t^{\ell+p}b_{-i}(\cdot,t,\tau)\right)\\
&\qquad\quad=\sum_{\gamma\geq 0}\frac{1}{\gamma!}\partial_\tau^{\gamma}D_t^{\gamma}\left(\sum_{\beta,i,j\geq 0}\frac{1}{\beta!}(-D_t)^{\beta}\left(a_{-j}(x,t,D_x,\tau)\right)^*\left(\partial_\tau^{\beta}b_{-i}(\cdot,t,\tau)\right)\right),
\end{aligned}$$

where $(-D_t)^{\beta}\left(a_{-j}(x,t,D_x,\tau)\right)^*$ denotes the formal adjoint of the operator with symbol $D_t^{\beta}a_{-j}(x,t,\xi,\tau)$ as an operator in the x-variable, depending on (t,τ) as parameters. Here we used Formula (A.2) in the Appendix. Hence

$$\sigma(b^*(x,t,D_t)\circ a(x,t,D_x,D_t)) = \sum_{\ell\geq 0}\frac{1}{\ell!}\partial_\tau^\ell D_t^\ell$$

$$\times\left(\sum_{\gamma\geq 0}\frac{1}{\gamma!}\partial_\tau^\gamma D_t^\gamma\left(\sum_{\beta,i,j\geq 0}\frac{1}{\beta!}(-D_t)^\beta\left(a_{-j}(x,t,D_x,\tau)\right)^*\left(\partial_\tau^\beta b_{-i}(\cdot,t,\tau)\right)\right)\right)^-$$

$$= \sum_{\beta,i,j\geq 0}\frac{1}{\beta!}D_t^\beta\left(\overline{a_{-j}}(x,t,D_x,\tau)\right)^*\left(\partial_\tau^\beta\overline{b_{-i}}(\cdot,t,\tau)\right)$$

$$= \sum_{s\geq 0}\sum_{q\beta+i+j=s}\frac{1}{\beta!}D_t^\beta\left(\overline{a_{-j}}(x,t,D_x,\tau)\right)^*\left(\partial_\tau^\beta\overline{b_{-i}}(\cdot,t,\tau)\right),$$

because of Formula (A.3) of the Appendix. □

4.2. **The actual computation of the eigenvalue.** We are now in a position to start computing the symbol of Λ.

Let us first examine the minimum eigenvalue and the corresponding eigenfunction of $P_0(x,t,D_x,\tau)$ in (4.1.1), as an operator in the x-variable. It is well known that $P_0(x,t,D_x,\tau)$ has a discrete set of non negative, simple eigenvalues depending in a real analytic way on the parameters (t,τ).

P_0 can be written in the form $LL^* + t^{2\ell}L^*L$, where $L = D_x + ix^{q-1}\tau$. The kernel of L^* is a one dimensional vector space generated by $\varphi_{0,0}(x,\tau) = c_0\tau^{\frac{1}{2q}}\exp(-\frac{x^q}{q}\tau)$, c_0 being a normalization constant such that $\|\varphi_{0,0}(\cdot,\tau)\|_{L^2(\mathbb{R}_x)} = 1$. We remark that in this case τ is positive. For negative values of τ the operator LL^* is injective. Denoting by $\varphi_0(x,t,\tau)$ the eigenfunction of P_0 corresponding to its lowest eigenvalue $\Lambda_0(t,\tau)$, we obtain that $\varphi_0(x,0,\tau) = \varphi_{0,0}(x,\tau)$ and that $\Lambda_0(0,\tau) = 0$. As a consequence the operator

$$P = BB^* + B^*(t^{2\ell}+x^{2k})B, \qquad B = D_x + ix^{q-1}D_t, \tag{4.2.1}$$

is not maximally hypoelliptic i.e. hypoelliptic with a loss of $2-\frac{2}{q}$ derivatives.

Next we give a more precise description of the t-dependence of both the eigenvalue Λ_0 and its corresponding eigenfunction φ_0 of $P_0(x,t,D_x,\tau)$.

It is well known that there exists an $\varepsilon > 0$, small enough, such that the operator

$$\Pi_0 = \frac{1}{2\pi i}\oint_{|\mu|=\varepsilon}(\mu I - P_0(x,t,D_x,\tau))^{-1}d\mu$$

is the orthogonal projection onto the eigenspace generated by φ_0. Note that Π_0 depends on the parameters (t,τ). The operator LL^* is thought of as an unbounded operator in $L^2(\mathbb{R}_x)$ with domain $B^2_q(\mathbb{R}_x) = \{u \in L^2(\mathbb{R}_x) \mid x^\alpha D^\beta_x u \in L^2, 0 \leq \beta + \frac{\alpha}{q-1} \leq 2\}$. We have

$$(\mu I - P_0)^{-1} = \left(I + t^{2\ell}\left[-A(I+t^{2\ell}A)^{-1}\right]\right)(\mu I - LL^*)^{-1},$$

where $A = (LL^* - \mu I)^{-1}L^*L$. Plugging this into the formula defining Π_0, we get

$$\Pi_0 = \frac{1}{2\pi i}\oint_{|\mu|=\varepsilon}(\mu I - LL^*)^{-1}d\mu - \frac{1}{2\pi i}t^{2\ell}\oint_{|\mu|=\varepsilon}A(I+t^{2\ell}A)^{-1}(\mu I - LL^*)^{-1}d\mu.$$

Hence

$$\begin{aligned}\varphi_0 = \Pi_0\varphi_{0,0} &= \varphi_{0,0} - t^{2\ell}\frac{1}{2\pi i}\oint_{|\mu|=\varepsilon}A(I+t^{2\ell}A)^{-1}(\mu I - LL^*)^{-1}\varphi_{0,0}d\mu \\ &= \varphi_{0,0}(x,\tau) + t^{2\ell}\tilde{\varphi}_0(x,t,\tau). \qquad (4.2.2)\end{aligned}$$

Since Π_0 is an orthogonal projection then $\|\varphi_0(\cdot,t,\tau)\|_{L^2(\mathbb{R}_x)} = 1$.

As a consequence we obtain that

$$\Lambda_0(t,\tau) = \langle P_0\varphi_0, \varphi_0\rangle = t^{2\ell}\|L\varphi_{0,0}\|^2 + \mathscr{O}(t^{4\ell}). \qquad (4.2.3)$$

We point out that $L\varphi_{0,0} \neq 0$. Observe that, in view of (4.1.2),

$$\begin{aligned}\Lambda_0(t,\mu\tau) &= \min_{\substack{u\in B^2_q\\ \|u\|_{L^2}=1}}\langle P_0(x,t,D_x,\mu\tau)u(x), u(x)\rangle \\ &= \min_{\substack{u\in B^2_q\\ \|u\|_{L^2}=1}}\langle P_0(\mu^{-1/q}x,t,\mu^{1/q}D_x,\mu\tau)\frac{u(\mu^{-1/q}x)}{\mu^{-1/(2q)}}, \frac{u(\mu^{-1/q}x)}{\mu^{-1/(2q)}}\rangle \\ &= \mu^{\frac{2}{q}}\min_{\substack{v\in B^2_q\\ \|v\|_{L^2}=1}}\langle P_0(x,t,D_x,\tau)v(x), v(x)\rangle \\ &= \mu^{\frac{2}{q}}\Lambda_0(t,\tau). \qquad (4.2.4)\end{aligned}$$

This shows that Λ_0 is homogeneous of degree $2/q$ w.r.t. the variable τ.

Since φ_0 is the unique normalized solution of the equation

$$(P_0(x,t,D_x,\tau) - \Lambda_0(t,\tau))u(\cdot,t,\tau) = 0,$$

from (4.1.2) and (4.2.4) it follows that φ_0 is globally homogeneous of degree $1/(2q)$. Moreover φ_0 is rapidly decreasing w.r.t. the x-variable smoothly dependent on (t,τ) in a compact subset of $\mathbb{R}^2 \setminus 0$. Using estimates of the form (4.1.11) we can conclude that $\varphi_0 \in H_q^{1/(2q)}$.

Let us start now the construction of a right parametrix of the operator

$$\begin{bmatrix} P(x,t,D_x,D_t) & \varphi_0(x,t,D_t) \\ \varphi_0^*(x,t,D_t) & 0 \end{bmatrix}$$

as a map from $C_0^\infty(\mathbb{R}^2_{(x,t)}) \times C_0^\infty(\mathbb{R}_t)$ into $C^\infty(\mathbb{R}^2_{(x,t)}) \times C^\infty(\mathbb{R}_t)$. In particular we are looking for an operator such that

$$\begin{bmatrix} P(x,t,D_x,D_t) & \varphi_0(x,t,D_t) \\ \varphi_0^*(x,t,D_t) & 0 \end{bmatrix} \circ \begin{bmatrix} F(x,t,D_x,D_t) & \psi(x,t,D_t) \\ \psi^*(x,t,D_t) & -\Lambda(t,D_t) \end{bmatrix} \equiv \begin{bmatrix} Id_{C_0^\infty(\mathbb{R}^2)} & 0 \\ 0 & Id_{C_0^\infty(\mathbb{R})} \end{bmatrix}. \tag{4.2.5}$$

Here ψ and ψ^* denote operators in $OPH_q^{1/2q}$ and $OPH_q^{*1/2q}$, $F \in OPS_q^{-2,-2}$ and $\Lambda \in OPS_{1,0}^{2/q}$. Here $\equiv$ means equality modulo a regularizing operator.

From (4.2.5) we obtain four relations:

$$P(x,t,D_x,D_t) \circ F(x,t,D_x,D_t) + \varphi_0(x,t,D_t) \circ \psi^*(x,t,D_t) \equiv Id, \tag{4.2.6}$$

$$P(x,t,D_x,D_t) \circ \psi(x,t,D_t) - \varphi_0(x,t,D_t) \circ \Lambda(t,D_t) \equiv 0, \tag{4.2.7}$$

$$\varphi_0^*(x,t,D_t) \circ F(x,t,D_x,D_t) \equiv 0, \tag{4.2.8}$$

$$\varphi_0^*(x,t,D_t) \circ \psi(x,t,D_t) \equiv Id. \tag{4.2.9}$$

We are going to find the symbols F, ψ and Λ as asymptotic series of globally homogeneous symbols:

$$F \sim \sum_{j\geq 0} F_{-j}, \qquad \psi \sim \sum_{j\geq 0} \psi_{-j}, \qquad \Lambda \sim \sum_{j\geq 0} \Lambda_{-j}, \tag{4.2.10}$$

From Lemma 4.1.2 we obtain that

$$\sigma(P \circ F) \sim \sum_{s\geq 0} \sum_{q\alpha+i+j=s} \frac{1}{\alpha!} \sigma(\partial_\tau^\alpha P_{-j}(x,t,D_x,\tau) \circ_x D_t^\alpha F_{-i}(x,t,D_x,\tau)),$$

where we denoted by P_{-j} the globally homogeneous parts of degree $\frac{2}{q}-\frac{j}{q}$ of the symbol of P, so that $P=P_0+P_{-q}+P_{-2k}$. Furthermore from Lemma 4.1.3(i) we may write that

$$\sigma(\varphi_0\circ\psi^*)\sim e^{-ix\xi}\sum_{s\geq 0}\sum_{q\alpha+i=s}\frac{1}{\alpha!}\partial_\tau^\alpha\varphi_0(x,t,\tau)D_t^\alpha\hat{\bar{\psi}}_{-i}(\xi,t,\tau).$$

Analogously Lemmas 4.1.4, (4.1.3)(iii) give

$$\sigma(P\circ\psi)\sim\sum_{s\geq 0}\sum_{q\ell+i+j=s}\frac{1}{\ell!}\partial_\tau^\ell P_{-j}(x,t,D_x,\tau)(D_t^\ell\psi_{-i}(\cdot,t,\tau)),$$

$$\sigma(\varphi_0\circ\Lambda)\sim\sum_{s\geq 0}\sum_{q\alpha+\ell=s}\frac{1}{\alpha!}\partial_\tau^\alpha\varphi_0(x,t,\tau)D_t^\alpha\Lambda_{-\ell}(t,\tau).$$

Finally Lemmas 4.1.5(i) and 4.1.3(ii) yield

$$\sigma(\varphi_0^*\circ F)\sim\sum_{s\geq 0}\sum_{q\ell+j=s}\frac{1}{\ell!}D_t^\ell\left(\overline{F_{-j}}(x,t,D_x,\tau)\right)^*(\partial_\tau^\ell\overline{\varphi_0}(\cdot,t,\tau)),$$

and

$$\sigma(\varphi_0^*\circ\psi)\sim\sum_{s\geq 0}\sum_{q\alpha+j=s}\frac{1}{\alpha!}\int\partial_\tau^\alpha\bar{\varphi}_0(x,t,\tau)D_t^\alpha\psi_{-j}(x,t,\tau)dx.$$

Let us consider the terms globally homogeneous of degree 0. We obtain the relations

$$P_0(x,t,D_x,\tau)\circ_x F_0(x,t,D_x,\tau)+\varphi_0(x,t,\tau)\otimes\psi_0(\cdot,t,\tau) = Id \tag{4.2.11}$$

$$P_0(x,t,D_x,\tau)(\psi_0(\cdot,t,\tau))-\Lambda_0(t,\tau)\varphi_0(x,t,\tau) = 0 \tag{4.2.12}$$

$$(F_0(x,t,D_x,\tau))^*(\varphi_0(\cdot,t,\tau)) = 0 \tag{4.2.13}$$

$$\int\bar{\varphi}_0(x,t,\tau)\psi_0(x,t,\tau)dx = 1. \tag{4.2.14}$$

Here we denoted by $\varphi_0\otimes\psi_0$ the operator $u=u(x)\mapsto\varphi_0\int\bar{\psi}_0 u dx$; $\varphi_0\otimes\psi_0$ must be a globally homogeneous symbol of degree zero.

Conditions (4.2.12) and (4.2.14) imply that $\psi_0=\varphi_0$. Moreover (4.2.12) yields that

$$\Lambda_0(t,\tau)=\langle P_0(x,t,D_x,\tau)\varphi_0(x,t,\tau),\varphi_0(x,t,\tau)\rangle_{L^2(\mathbb{R}_x)},$$

coherently with the notation chosen above. Conditions (4.2.11) and (4.2.13) are rewritten as

$$\begin{aligned}P_0(x,t,D_x,\tau)\circ_x F_0(x,t,D_x,\tau) &= Id-\Pi_0\\ F_0(x,t,D_x,\tau)(\varphi_0(\cdot,t,\tau)) &\in [\varphi_0]^\perp,\end{aligned}$$

whence

$$F_0(x,t,D_x,\tau) = \begin{cases} (P_0(x,t,D_x,\tau)_{|_{[\varphi_0]^\perp \cap B^2_q}})^{-1} & \text{on } [\varphi_0]^\perp \\ 0 & \text{on } [\varphi_0]. \end{cases} \tag{4.2.15}$$

Since P_0 is q-globally elliptic w.r.t. (x,ξ) smoothly depending on the parameters (t,τ), one can show that $F_0(x,t,D_x,\tau)$ is actually a pseudodifferential operator whose symbol verifies (4.1.10) with $m = k = -2$, $j = 0$, and is globally homogeneous of degree $-2/q$.

From now on we assume that $q < 2k$ and that $2k$ is not a multiple of q; the complementary cases are analogous.

Because of the fact that $P_{-j} = 0$ for $j = 1,\dots,q-1$, relations (4.2.11)–(4.2.14) are satisfied at degree $-j/q$, $j = 1,\dots,q-1$, by choosing $F_{-j} = 0$, $\psi_{-j} = 0$, $\Lambda_{-j} = 0$. Then we must examine homogeneity degree -1 in Equations (4.2.6)–(4.2.9). We get

$$\begin{aligned} P_{-q} \circ_x F_0 + P_0 \circ_x F_{-q} + \partial_\tau P_0 \circ_x D_t F_0 & \\ +\varphi_0 \otimes \psi_{-q} + \partial_\tau \varphi_0 \otimes D_t \varphi_0 &= 0 \end{aligned} \tag{4.2.16}$$

$$\begin{aligned} P_0(\psi_{-q}) + P_{-q}(\varphi_0) + \partial_\tau P_0(D_t\varphi_0) & \\ -\Lambda_{-q}\varphi_0 - D_t\Lambda_0\partial_\tau\varphi_0 &= 0 \end{aligned} \tag{4.2.17}$$

$$(F_{-q})^*(\varphi_0) - (D_t F_0^*)(\partial_\tau\varphi_0) = 0 \tag{4.2.18}$$

$$\langle \psi_{-q}, \varphi_0 \rangle_{L^2(\mathbb{R}_x)} + \langle D_t\varphi_0, \partial_\tau\varphi_0 \rangle_{L^2(\mathbb{R}_x)} = 0. \tag{4.2.19}$$

First we solve w.r.t. $\psi_{-q} = \langle \psi_{-q}, \varphi_0 \rangle_{L^2(\mathbb{R}_x)} \varphi_0 + \psi^\perp_{-q} \in [\varphi_0] \oplus [\varphi_0]^\perp$. From (4.2.19) we get immediately that

$$\langle \psi_{-q}, \varphi_0 \rangle_{L^2(\mathbb{R}_x)} = -\langle D_t\varphi_0, \partial_\tau\varphi_0 \rangle_{L^2(\mathbb{R}_x)}. \tag{4.2.20}$$

(4.2.17) implies that

$$P_0(\langle \psi_{-q}, \varphi_0 \rangle \varphi_0) + P_0(\psi^\perp_{-q}) = -P_{-q}(\varphi_0) - \partial_\tau P_0(D_t\varphi_0) + \Lambda_{-q}\varphi_0 + D_t\Lambda_0\partial_\tau\varphi_0.$$

Thus, using (4.2.20) we obtain that

$$[\varphi_0]^\perp \ni P_0(\psi^\perp_{-q}) = -P_{-q}(\varphi_0) - \partial_\tau P_0(D_t\varphi_0) + \Lambda_{-q}\varphi_0 + D_t\Lambda_0\partial_\tau\varphi_0 + \langle D_t\varphi_0, \partial_\tau\varphi_0 \rangle \Lambda_0\varphi_0,$$

whence

$$\Lambda_{-q} = \langle P_{-q}(\varphi_0) + \partial_\tau P_0(D_t\varphi_0) - D_t\Lambda_0\partial_\tau\varphi_0, \varphi_0\rangle_{L^2(\mathbb{R}_x)} - \langle D_t\varphi_0, \partial_\tau\varphi_0\rangle\Lambda_0. \tag{4.2.21}$$

$$\psi_{-q} = -\langle D_t\varphi_0, \partial_\tau\varphi_0\rangle_{L^2(\mathbb{R}_x)}\varphi_0 + F_0\left(-P_{-q}(\varphi_0) - \partial_\tau P_0(D_t\varphi_0) + D_t\Lambda_0\partial_\tau\varphi_0\right), \tag{4.2.22}$$

since, by (4.2.15), $F_0\varphi_0 = 0$. From (4.2.18) we deduce that, for every $u \in L^2(\mathbb{R}_x)$,

$$\Pi_0 F_{-q}u = \langle u, (D_tF_0^*)(\partial_\tau\varphi_0)\rangle_{L^2(\mathbb{R}_x)}\varphi_0 = \left[\varphi_0 \otimes (D_tF_0^*)(\partial_\tau\varphi_0)\right]u$$

Let $-\omega_{-q} = P_{-q} \circ_x F_0 + \partial_\tau P_0 \circ_x D_tF_0 + \varphi_0 \otimes \psi_{-q} + \partial_\tau\varphi_0 \otimes D_t\varphi_0$. Then from (4.2.15), applying F_0 to both sides of (4.2.16), we obtain that

$$(Id - \Pi_0)F_{-q} = -F_0\omega_{-q}.$$

Therefore we deduce that

$$F_{-q} = \varphi_0 \otimes (D_tF_0^*)(\partial_\tau\varphi_0) - F_0\omega_{-q}. \tag{4.2.23}$$

Inspecting (4.2.22), (4.2.23) we see that $\psi_{-q} \in H_q^{\frac{1}{2q}-1}$, globally homogeneous of degree $1/2q - 1$, $F_{-q} \in S_q^{-2,-2+\frac{q}{q-1}}$, globally homogeneous of degree $-2/q - 1$.

From (4.2.21) we have that $\Lambda_{-q} \in S_{1,0}^{2/q-1}$ homogeneous of degree $2/q - 1$. Moreover P_{-q} is $\mathscr{O}(t^{2\ell-1})$, $D_t\varphi_0$ is estimated by $t^{2\ell-1}$, for $t \to 0$, because of (4.2.2), $D_t\Lambda_0$ is also $\mathscr{O}(t^{2\ell-1})$ and $\Lambda_0 = \mathscr{O}(t^{2\ell})$ because of (4.2.3). We thus obtain that

$$\Lambda_{-q}(t, \tau) = \mathscr{O}(t^{2\ell-1}). \tag{4.2.24}$$

This ends the analysis of the terms of degree -1 in (4.2.5).

The procedure can be iterated arguing in a similar way. We would like to point out that the first homogeneity degree coming up and being not a negative integer is $-2k/q$ (we are availing ourselves of the fact that $2k$ is not a multiple of q. If it is a multiple of q, the above argument applies literally, but we need also the supplementary remark that we are going to make in the sequel.)

At homogeneity degree $-2k/q$ we do not see the derivatives w.r.t. t or τ of the symbols found at the previous levels, since they would only account for a negative integer homogeneity degrees.

In particular condition (4.2.7) for homogeneity degree $-2k/q$ reads as

$$P_0\psi_{-2k} + P_{-2k}\varphi_0 - \varphi_0\Lambda_{-2k} = 0.$$

Taking the scalar product of the above equation with the eigenfunction φ_0 and recalling that $\|\varphi_0(\cdot, t, \tau)\|_{L^2(\mathbb{R}^x)} = 1$, we obtain that

$$\Lambda_{-2k}(t, \tau) = \langle P_{-2k}\varphi_0, \varphi_0\rangle_{L^2(\mathbb{R}_x)} + \langle P_0\psi_{-2k}, \varphi_0\rangle_{L^2(\mathbb{R}_x)}. \tag{4.2.25}$$

Now, because of the structure of P_{-2k}, $\langle P_{-2k}\varphi_0, \varphi_0\rangle_{L^2(\mathbb{R}_x)} > 0$, while the second term on the right, which is equal to $\langle \psi_{-2k}, \varphi_0\rangle \bar{\Lambda}_0$, vanishes for $t = 0$. Thus if t is small enough we deduce that

$$\Lambda_{-2k}(t, \tau) > 0. \tag{4.2.26}$$

From this point on the procedure continues exactly as above.

We have thus proved the

Theorem 4.2.1. *The operator Λ defined in (4.2.5) is a pseudodifferential operator with symbol $\Lambda(t, \tau) \in S^{2/q}_{1,0}(\mathbb{R}_t)$. Moreover, if j_0 is a positive integer such that $j_0 q < 2k < (j_0 + 1)q$, the symbol of Λ has an asymptotic expansion of the form*

$$\Lambda(t, \tau) \sim \sum_{j=0}^{j_0} \Lambda_{-jq}(t, \tau) + \sum_{s\geq 0} \left(\Lambda_{-2k-sq}(t, \tau) + \Lambda_{-(j_0+1)q-sq}(t, \tau)\right). \tag{4.2.27}$$

Here Λ_{-p} has homogeneity $2/q - p/q$ and

a-
$$\Lambda_{-jq}(t, \tau) = \mathscr{O}(t^{2\ell-j}) \qquad \text{for } j = 0, \dots, j_0.$$

b- Λ_{-2k} *satisfies (4.2.26).*

4.3. **Hypoellipticity of P.** In this section we give a different proof of the C^∞ hypoellipticity of P. This is accomplished by showing that the hypoellipticity of P follows from the hypoellipticity of Λ and proving that Λ is hypoelliptic if condition (1.2) is satisfied. As a matter of fact the hypoellipticity of P is equivalent to the hypoellipticity of Λ, so that the structure of Λ in Theorem 4.2.1, may be used to prove assertion (iii) in Theorem 1.1 (see [3].)

We state without proof the following

Lemma 4.3.1. (a) *Let* $a \in S_q^{m,k}$*, properly supported, with* $k \leq 0$*. Then* $\operatorname{Op} a$ *is continuous from* $H^s_{loc}(\mathbb{R}^2)$ *to* $H_{loc}^{s-m+k\frac{q-1}{q}}(\mathbb{R}^2)$*.*

(b) *Let* $\varphi \in H_q^{m+\frac{1}{2q}}$*, properly supported. Then* $\operatorname{Op}\varphi$ *is continuous from* $H^s_{loc}(\mathbb{R})$ *to* $H^{s-m}_{loc}(\mathbb{R}^2)$*. Moreover* $\varphi^*(x,t,D_t)$ *is continuous from* $H^s_{loc}(\mathbb{R}^2)$ *to* $H^{s-m}_{loc}(\mathbb{R})$*.*

Mirroring the argument above, we can find symbols $F \in S_q^{-2,-2}$, $\psi \in H_q^{1/2q}$ and $\Lambda \in S^{2/q}_{1,0}$ as in (4.2.10), such that

$$
\begin{bmatrix} F(x,t,D_x,D_t) & \psi(x,t,D_t) \\ \psi^*(x,t,D_t) & -\Lambda(t,D_t) \end{bmatrix} \circ \begin{bmatrix} P(x,t,D_x,D_t) & \varphi_0(x,t,D_t) \\ \varphi_0^*(x,t,D_t) & 0 \end{bmatrix} \equiv \begin{bmatrix} Id_{C_0^\infty(\mathbb{R}^2)} & 0 \\ 0 & Id_{C_0^\infty(\mathbb{R})} \end{bmatrix}. \tag{4.3.1}
$$

From (4.3.1) we get the couple of relations

$$
F(x,t,D_x,D_t) \circ P(x,t,D_x,D_t) = Id - \psi(x,t,D_t) \circ \varphi_0^*(x,t,D_t) \tag{4.3.2}
$$

$$
\psi^*(x,t,D_t) \circ P(x,t,D_x,D_t) = \Lambda(t,D_t) \circ \varphi_0^*(x,t,D_t). \tag{4.3.3}
$$

Proposition 4.3.1. *If* Λ *is hypoelliptic with a loss of* δ *derivatives, then* P *is also hypoelliptic with a loss of derivatives equal to*

$$
2\frac{q-1}{q} + \max\{0, \delta\}.
$$

Proof. Assume that $Pu \in H^s_{loc}(\mathbb{R}^2)$. From Lemma 4.3.1 we have that $FPu \in H^{s+2/q}_{loc}(\mathbb{R}^2)$. By (4.3.2) we have that $u - \psi\varphi_0^* u \in H^{s+2/q}_{loc}(\mathbb{R}^2)$. Again, using Lemma 4.3.1, $\psi^* Pu \in H^s_{loc}(\mathbb{R})$, so that, by (4.3.3), $\Lambda\varphi_0^* u \in H^s_{loc}(\mathbb{R})$. The hypoellipticity of Λ yields then that $\varphi_0^* u \in H_{loc}^{s+\frac{2}{q}-\delta}(\mathbb{R})$. From Lemma 4.3.1 we obtain that $\psi\varphi_0^* u \in H_{loc}^{s+\frac{2}{q}-\delta}(\mathbb{R})$. Thus $u = (Id - \psi\varphi_0^*)u + \psi\varphi_0^* u \in H_{loc}^{s+\frac{2}{q}-\max\{0,\delta\}}$. This proves the proposition. ☐

Next we prove the hypoellipticity of Λ under the assumption that $\ell > k/q$.

First we want to show that there exists a smooth non negative function $M(t,\tau)$, such that

$$
M(t,\tau) \leq C|\Lambda(t,\tau)|, \qquad |\Lambda^{(\alpha)}_{(\beta)}(t,\tau)| \leq C_{\alpha,\beta} M(t,\tau)(1+|\tau|)^{-\rho\alpha+\delta\beta}, \tag{4.3.4}
$$

where α, β are non negative integers, C, $C_{\alpha,\beta}$ suitable positive constants and the inequality holds for t in a compact neighborhood of the origin and $|\tau|$ large. Moreover ρ and δ are such that $0 \leq \delta < \rho \leq 1$.

We actually need to check the above estimates for Λ only when τ is positive and large.

Let us choose $\rho = 1$, $\delta = \frac{k}{\ell q} < 1$ and

$$M(t,\tau) = \tau^{\frac{2}{q}}\left(t^{2\ell} + \tau^{-\frac{2k}{q}}\right),$$

for $\tau \geq c \geq 1$. It is then evident, from Theorem 4.2.1, that the first of the conditions (4.3.4) is satisfied. The second condition in (4.3.4) is also straightforward for $\Lambda_0 + \Lambda_{-2k}$, because of (4.2.26) and (4.2.3). To verify the second condition in (4.3.4) for Λ_{-jq}, $q \in \{1,\dots,j_0\}$, we have to use property a- in the statement of Theorem 4.2.1. Finally the verification is straightforward for the lower order parts of the symbol in Formula (4.2.27). Using Theorem 22.1.3 of [10], we see that there exists a parametrix for Λ. Moreover from the proof of the above quoted theorem we get that the symbol of any parametrix satisfies the same estimates that Λ^{-1} satisfies, i.e.

$$|D_t^\beta D_\tau^\alpha \Lambda(t,\tau)| \leq C_{\alpha,\beta}\left[\tau^{\frac{2}{q}}\left(t^{2\ell} + \tau^{-\frac{2k}{q}}\right)\right]^{-1}(1+\tau)^{-\alpha+\frac{k}{\ell q}\beta} \leq C_{\alpha,\beta}(1+\tau)^{\frac{2k}{q}-\frac{2}{q}-\alpha+\frac{k}{\ell q}\beta},$$

for t in a compact set and $\tau \geq C$. Thus the parametrix obtained from Theorem 22.1.3 of [10] has a symbol in $S^{\frac{2k}{q}-\frac{2}{q}}_{1,\frac{k}{\ell q}}$.

We may now state the

Theorem 4.3.1. *Λ is hypoelliptic with a loss of $\frac{2k}{q}$ derivatives, i.e. $\Lambda u \in H^s_{loc}$ implies that $u \in H^{s+\frac{2}{q}-\frac{2k}{q}}_{loc}$.*

Theorem 4.3.1 together with Proposition 4.3.1 prove assertion (i) of Theorem 1.1.

A. Appendix

We prove here a well-known formula for the adjoint of a product of two pseudodifferential operators using just symbolic calculus. Let a, b symbols in $S^0_{1,0}(\mathbb{R}_t)$. We want to show that

$$(a\#b)^* = b^*\#a^*, \tag{A.1}$$

where # denotes the usual symbolic composition law (a higher dimensional extension involves just a more cumbersome notation.)

We may write

$$\begin{aligned}(a\#b)^* &= \sum_{\ell,\alpha\geq 0} \frac{(-1)^\alpha}{\alpha!\ell!}\partial_\tau^\ell D_t^\ell \left(\partial_\tau^\alpha \bar{a} D_t^\alpha \bar{b}\right) \\ &= \sum_{\ell,\alpha\geq 0}\sum_{r,s\leq\ell} \frac{(-1)^\alpha}{\alpha!\ell!}\binom{\ell}{r}\binom{\ell}{s}\partial_\tau^{\alpha+r} D_t^{\ell-s}\bar{a}\ \partial_\tau^{\ell-r} D_t^{\alpha+s}\bar{b}.\end{aligned}$$

Let us change the summation indices according to the following prescription; $j = \alpha + r$, $\beta + j = \ell - s$, $i = \alpha + s$, so that $\ell - r = i + \beta$, we may rewrite the last equality in the above formula as

$$(a\#b)^* = \sum_{i,j,\beta\geq 0}\sum_{s\leq i} \frac{(-1)^{i-s}}{(i-s)!(\beta+j+s)!}\binom{\beta+j+s}{j-i+s}\binom{\beta+j+s}{s}\partial_\tau^{i+\beta} D_t^i \bar{b}\ \partial_\tau^j D_t^{\beta+j}\bar{a}.$$

Let us examine the s-summation; we claim that

$$\sum_{s=0}^{i} \frac{(-1)^{i-s}}{(i-s)!}\frac{1}{(\beta+i)!(j-i+s)!}\binom{\beta+j=s}{s} = \frac{1}{\beta!i!j!}.$$

This is actually equivalent to

$$\sum_{s=0}^{i}(-1)^{i-s}\binom{i}{s}\binom{\beta+j+s}{\beta+i} = \binom{\beta+j}{j}.$$

Setting $i - s = \nu \in \{0, 1, \dots, i\}$, the above relation is written as

$$\sum_{\nu=0}^{i}(-1)^{\nu}\binom{i}{\nu}\binom{\beta+i+j-\nu}{\beta+i} = \binom{\beta+j}{j},$$

and this is precisely identity (12.15) in W. Feller [8], vol. 1.

Thus we may conclude that

$$\begin{aligned}(a\#b)^* &= \sum_{i,j,\beta}\frac{1}{\beta!i!j!}\partial_\tau^{i+\beta} D_t^i\bar{b}\partial_\tau^j D_t^{j+\beta}\bar{a} \\ &= \sum_{\beta\geq 0}\frac{1}{\beta!}\partial_\tau^\beta\left(\sum_{i\geq 0}\frac{1}{i!}\partial_\tau^i D_t^i\bar{b}\right) D_t^\beta\left(\sum_{j\geq 0}\frac{1}{j!}\partial_\tau^j D_t^j\bar{a}\right) \\ &= b^*\#a^*.\end{aligned}$$

This proves (A.1).

As a by-product of the above argument we get the following identity

$$\sum_{i,j,\beta} \frac{1}{\beta! i! j!} \partial_\tau^{i+\beta} D_t^i \bar{b} \partial_\tau^j D_t^{j+\beta} \bar{a} = \sum_{\ell,\alpha\geq 0} \frac{(-1)^\alpha}{\alpha! \ell!} \partial_\tau^\ell D_t^\ell \left(\partial_\tau^\alpha \bar{a} D_t^\alpha \bar{b} \right), \tag{A.2}$$

which is the purpose of the present Appendix.

We would like to point out that the relation $(a^*)^* = a$ rests on the identity

$$\sum_{\ell\geq 0} \frac{1}{\ell!} \partial_\tau^\ell D_t^\ell \overline{\left(\sum_{\alpha\geq 0} \frac{1}{\alpha!} \partial_\tau^\alpha D_t^\alpha \bar{a} \right)} \tag{A.3}$$

$$= \sum_{s\geq 0} \frac{1}{s!} \left(\sum_{\ell+\alpha=s} \frac{s!}{\ell! \alpha!} (-1)^\alpha \right) \partial_\tau^s D_t^s a = \sum_{s\geq 0} \frac{1}{s!} (1-1)^s \partial_\tau^s D_t^s a = a.$$

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