# SEMIGROUPS GENERATED BY FIRST ORDER DIFFERENTIAL OPERATORS

# SEMIGRUPPI GENERATI DA OPERATORI DIFFERENZIALI DEL PRIMO ORDINE

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Abstract. The differentiation operator in a space of continuous or $L^p$ functions defined on an interval generates a translation semigroup. Here we illustrate the generalization of this result to general first-order operators, possibly singular.

Sunto. L'operatore di derivazione in uno spazio di funzioni continue o $L^p$ definite su un intervallo genera un semigruppo di traslazioni. Qui illustriamo la generalizzazione di questo risultato a operatori generali del primo ordine, eventualmente singolari.

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## 1. Introduction

We consider a Banach space $X$ of complex valued functions defined on an open interval $I$ of $\mathbb{R}$. The space $X$ can be $L^p$ or a space of continuous functions. We study the semigroup (or the group) of operators in $X$ generated by the (possibly singular) differential operator $G$ such that

$$[Gf](x) = a(x)f'(x) + b(x)f(x),$$

where $a \in C(I, \mathbb{R}^+)$ $b \in C(I, \mathbb{C})$.

In particular we determine necessary and sufficient conditions on the coefficients $a$ and $b$ for the operator $G$ (possibly with suitable conditions at the boundary) to be the infinitesimal generator of a strongly continuous semigroup (or group).

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The results we expose are obtained in collaboration with Simone Creo, Davide Guidetti, Maria Rosaria Lancia and are contained in [2], where complete proofs and bibliographical references can be found.

## 2. Translation semigroups

We recall what happens in the simplest case, i.e. when $Gf = f'$. In this case the operator $G$ (with a suitable domain) generates a left translation semigroup, that is the semigroup $\big(T(t)\big)_{t\in[0,\infty)}$ such that

$$[T(t)f](x) = f(x+t).$$

For a detailed treatment of translation semigroups see [4] Paragraphs I.4.c, II.2.10, II.2.11.

Let $I$ be an open interval of $\mathbb{R}$. We consider the Banach space $X$ with $X$ equal to:

- $BUC(I,\mathbb{C})$, the space of bounded uniformly continuous functions from $I$ to $\mathbb{C}$, endowed with the sup-norm, or a subspace of it;
- $L^p(I,\mathbb{C})$, the space of $p$-summable functions, with $1 \le p < \infty$.

We must distinguish according to whether $I$ is bounded or unbounded (above or below).

(1) Case $I = \mathbb{R}$.

For $t \in \mathbb{R}$ and $f \in X$ let $T(t)f \in X$ be such that

$$[T(t)f](x) = f(x+t).$$

Then $\big(T(t)\big)_{t\in\mathbb{R}}$ is a strongly continuous group of isometries whose infinitesimal generator is the operator $G$ such that

$$\mathcal{D}(G) = \begin{cases} \big\{f \in X \cap C^1(I,\mathbb{C}) \bigm| f' \in X\big\}, & \text{if } X = BUC,\\ \big\{f \in X \cap W^{1,1}_{\mathrm{loc}}(I,\mathbb{C}) \bigm| f' \in X\big\}, & \text{if } X = L^p,\end{cases}$$

$$[Gf](x) = f'(x).$$

(2) Case $I = (\alpha,\infty)$, $\alpha \in \mathbb{R}$.

For $t \in [0,\infty)$ and $f \in X$ let $T(t)f \in X$ be such that

$$[T(t)f](x) = f(x+t).$$

Then $\big(T(t)\big)_{t\in[0,\infty)}$ is a strongly continuous semigroup of contractions (that is $\|T(t)\| \leq 1$) whose infinitesimal generator is the operator $G$ such that

$$\mathcal{D}(G) = \begin{cases} \big\{f \in X \cap C^1(I,\mathbb{C}) \bigm| f' \in X\big\}, & \text{if } X = BUC, \\ \big\{f \in X \cap W^{1,1}_{\text{loc}}(I,\mathbb{C}) \bigm| f' \in X\big\}, & \text{if } X = L^p, \end{cases}$$

$$[Gf](x) = f'(x).$$

(3) Case $I = (\alpha,\beta)$, $\alpha \in \mathbb{R}$ or $\alpha = -\infty$, $\beta \in \mathbb{R}$.

For $t \in [0,\infty)$ and $f \in X$ let $T(t)f \in X$ be such that

$$[T(t)f](x) = \begin{cases} f(x+t), & \text{if } x+t < \beta, \\ 0, & \text{if } x+t \geq \beta, \end{cases}$$

where $X = L^p$ or $X = BUC_0 = \big\{f \in BUC \bigm| \lim_{x\to\beta} f(x) = 0\big\}$. Then $\big(T(t)\big)_{t\in[0,\infty)}$ is a strongly continuous semigroup of contractions whose infinitesimal generator is the operator $G$ such that

$$\mathcal{D}(G) = \begin{cases} \big\{f \in X \cap C^1(I,\mathbb{C}) \bigm| f' \in X\big\}, & \text{if } X = BUC_0, \\ \big\{f \in X \cap W^{1,1}_{\text{loc}}(I,\mathbb{C}) \bigm| f' \in X, \lim\limits_{x\to\beta} f(x) = 0\big\}, & \text{if } X = L^p, \end{cases}$$

$$[Gf](x) = f'(x).$$

In case $\alpha \in \mathbb{R}$ the semigroup is nilpotent, that is $T(t) = 0$ for sufficiently large $t$ $(t \geq \beta - \alpha)$.

The operator of left translation is bounded also in $L^\infty$, but in this case the semigroup is not strongly continuous, not even strongly measurable.

## 3. Construction of the semigroup

Let $-\infty \leq r_0 < r_1 \leq \infty$, $a \in C\big((r_0,r_1),\mathbb{R}^+\big)$, $b \in C\big((r_0,r_1),\mathbb{C}\big)$.

To construct the semigroup $\big(T(t)\big)_{t\in[0,\infty)}$ generated by the operator $Gf = au' + b$, we study the Cauchy problem associated to such operator.

Given $f\colon (r_0, r_1) \to \mathbb{C}$, the function $u(t,x) = [T(t)f](x)$ satisfies, in some sense, the Cauchy problem

$$\begin{cases} u_t(t,x) = a(x)u_x(t,x) + b(x)u(t,x), & (t,x) \in [0,\infty) \times (r_0, r_1), \\ u(0,x) = f(x), & x \in (r_0, r_1). \end{cases}$$

Let $v(t,x) = c(x)u(t,x)$, with $c\colon (r_0, r_1) \to \mathbb{C} \setminus \{0\}$ to be determined later. We have

$$v_t - av_x = cu_t - a(cu_x + c'u) = c\Big(u_t - au_x - \frac{c'a}{c}u\Big).$$

If $c'a/c = b$, that is $c'/c = b/a$, then the equation $u_t = au_x + bu$ is equivalent to $v_t = av_x$. If $B\colon (r_0, r_1) \to \mathbb{C}$ is a primitive of $b/a$, then the function $c$ defined by $c(x) = \exp\big(B(x)\big)$ satisfies the equation $c'/c = b/a$. With this choice $v$ is solution of the problem

$$\begin{cases} v_t(t,x) = a(x)v_x(t,x), & (t,x) \in [0,\infty) \times (r_0, r_1), \\ v(0,x) = \exp\big(B(x)\big)f(x), & x \in (r_0, r_1). \end{cases}$$

We solve this problem with the method of characteristics, that is we search for a curve in the $(t,x)$ plane, of equation $x = r(t)$, on which the solution $v$ is constant. It must be

$$\frac{d}{dt} v\big(t, r(t)\big) = 0,$$

that is

$$v_t\big(t, r(t)\big) + r'(t)v_x\big(t, r(t)\big) = 0,$$

or equivalently

$$a\big(r(t)\big)v_x\big(t, r(t)\big) + r'(t)v_x\big(t, r(t)\big) = 0.$$

Hence if $r$ satisfies the differential equation $r'(t) = -a\big(r(t)\big)$, then $v\big(t, r(t)\big)$ does not depend on $t$. If $A\colon (r_0, r_1) \to \mathbb{R}$ is a primitive of $1/a$, then this equation is equivalent to

$$\frac{d}{dt} A\big(r(t)\big) = -1,$$

hence there exists $d \in \mathbb{R}$ such that the solution $r$ satisfies $A\big(r(t)\big) = d - t$. Since $a$ is positive, $A$ is strictly increasing, hence it has an inverse $A^{-1}\colon (s_0, s_1) \to (r_0, r_1)$, where

$$s_0 = \inf A = \lim_{x \to r_0} A(x), \qquad s_1 = \sup A = \lim_{x \to r_1} A(x).$$

Therefore we get $r(t) = A^{-1}(d - t)$.

Since $v\big(t, r(t)\big)$ is constant, taking into account the initial condition, we have

$$v\big(t, A^{-1}(d-t)\big) = v\big(0, A^{-1}(d)\big) = \exp\big(B(A^{-1}(d))\big) f\big(A^{-1}(d)\big).$$

If $x = A^{-1}(d-t)$, then $d = A(x) + t$, therefore

$$v(t,x) = \exp\big(B(A^{-1}(A(x)+t))\big) f\big(A^{-1}(A(x)+t)\big)$$

and

$$u(t,x) = \frac{v(t,x)}{c(x)} = \exp\big(B\big(A^{-1}(A(x)+t)\big) - B(x)\big) f\big(A^{-1}(A(x)+t)\big).$$

This is well defined if $(t,x)$ is such that $A(x)+t$ belongs to $(s_0, s_1)$, the domain of $A^{-1}$. Hence, formally, the operator $Gf = af' + bf$ generates a semigroup defined by

$$[T(t)f](x) = \exp\big(B\big(A^{-1}(A(x)+t)\big) - B(x)\big) f\big(A^{-1}(A(x)+t)\big).$$

As already noted, this is meaningful if $A(x)+t < s_1$. In case $s_1 = \infty$ this condition is always satisfied, otherwise it must be $x < A^{-1}(s_1 - t)$.

We are in the same situation as the translation semigroup on the interval $(s_0, s_1)$, that is:

(1) Case $s_0 = -\infty, s_1 = \infty$.

We get a group of operators if, for $t \in \mathbb{R}$ and $f$ in some function space from $(r_0, r_1)$ to $\mathbb{C}$, we define

$$[T(t)f](x) = \exp\big(B\big(A^{-1}(A(x)+t)\big) - B(x)\big) f\big(A^{-1}\big(A(x)+t\big)\big).$$

(2) Case $s_0 > -\infty, s_1 = \infty$.

We get a semigroup of operators if, for $t \in [0,\infty)$ and $f$ in some function space from $(r_0, r_1)$ to $\mathbb{C}$, we define

$$[T(t)f](x) = \exp\big(B\big(A^{-1}(A(x)+t)\big) - B(x)\big) f\big(A^{-1}(A(x)+t)\big).$$

(3) Case $s_1 < \infty$.

We get a semigroup of operators if, for $t \in [0,\infty)$ and $f$ in some function space from $(r_0, r_1)$ to $\mathbb{C}$, we define

$$[T(t)f](x) = \begin{cases} \exp\big(B\big(A^{-1}(A(x)+t)\big) - B(x)\big) f\big(A^{-1}(A(x)+t)\big), & \text{if } A(x)+t < s_1, \\ 0, & \text{if } A(x)+t \geq s_1. \end{cases}$$

We recall that $A$ is a primitive of $1/a$, hence we have $s_1 = \sup A = \infty$ if and only if $1/a$ is not integrable near $r_1$, $s_0 = \inf A = -\infty$ if and only if $1/a$ is not integrable near $r_0$.

It is clear that if $f$ is continuous, then $T(t)f$ is continuous (with some caution if $s_1 < \infty$).

Since $A$ is a diffeomorphism it preserves measurability, hence it is easy to show that if $f$ is measurable then $T(t)f$ is measurable.

In case $b = 0$ the formula defining the semigroup goes back to [5], Example 2.

## 4. Estimates of the semigroup

To estimate the semigroup we defined in the previous section it is fundamental the following condition, concerning real valued functions $\varphi$ defined on an interval $I \subseteq \mathbb{R}$; here $L \in [0, \infty)$, $N \in \mathbb{R}$:

$$(C_{L,N}) \qquad \forall x_0, x_1 \in I, \quad x_1 > x_0 \implies \varphi(x_1) - \varphi(x_0) \leq L + N(x_1 - x_0).$$

The condition is satisfied if $\varphi$ is

- Lipschitz continuous ($L = 0$, $N =$ Lipschitz constant);
- decreasing ($L = N = 0$);
- bounded ($L = \sup \varphi - \inf \varphi$, $N = 0$);
- $x \mapsto x^\alpha$, with $0 < \alpha < 1$ ($L = N = 1$).

We say that a semigroup of operators $\big(T(t)\big)_{t \in [0,\infty)}$ is of type $(M, \omega)$, with $M \in [1, \infty)$ and $\omega \in \mathbb{R}$, if $\forall t \in [0, \infty)$ we have

$$\|T(t)\| \leq M e^{\omega t},$$

Analogously we say that a group of operators $\big(T(t)\big)_{t \in \mathbb{R}}$ is of type $(M, \omega)$, with $M \in [1, \infty)$ and $\omega \in [0, \infty)$, if $\forall t \in \mathbb{R}$ we have

$$\|T(t)\| \leq M e^{\omega |t|},$$

If the semigroup or the group is of type $(1, 0)$ then we say that it is a contraction semigroup or group.

We recall that every strongly continuous semigroup or group is of type $(M, \omega)$ for some $M$ and $\omega$ (see [4], Proposition I.5.5).

It is not difficult to estimate $\|T(t)\|$ in $L^p$ spaces in case $b = 0$ and $s_1 = \infty$.

For $p = \infty$ we have

$$\|T(t)f\|_\infty = \sup_x \left\| f\big(A^{-1}(A(x)+t)\big)\right\| \le \|f\|_\infty.$$

Obviously if $f$ is identically 1 then $T(t)f = f$, hence $\|T(t)\| = 1$.

If $p < \infty$ we have

$$\|T(t)f\|_p^p = \int_{r_0}^{r_1} \left| f\big(A^{-1}(A(x)+t)\big)\right|^p \, dx.$$

With the substitution $\xi = A^{-1}(A(x)+t)$, that is $x = A^{-1}(A(\xi)-t)$ and

$$dx = \frac{A'(\xi)}{A'\big(A^{-1}(A(\xi)-t)\big)}\, d\xi = \frac{a\big(A^{-1}(A(\xi)-t)\big)}{a(\xi)}\, d\xi,$$

we get

$$\begin{aligned}\|T(t)f\|_p^p &= \int_{A^{-1}(s_0+t)}^{r_1} |f(\xi)|^p \frac{a\big(A^{-1}(A(\xi)-t)\big)}{a(\xi)}\, d\xi \le \\ &\le \sup_{\xi\in(A^{-1}(s_0+t),r_1)} \frac{a\big(A^{-1}(A(\xi)-t)\big)}{a(\xi)} \|f\|_p^p.\end{aligned}$$

Therefore $\|T(t)\|_p^p$ can be estimated by

$$\begin{aligned}\sup_{\xi\in(A^{-1}(s_0+t),r_1)} \frac{a\big(A^{-1}(A(\xi)-t)\big)}{a(\xi)} &= \sup_{\eta\in(s_0+t,s_1)} \frac{a\big(A^{-1}(\eta-t)\big)}{a\big(A^{-1}(\eta)\big)} = \\ = \sup_{\xi\in(A^{-1}(s_0+t),r_1)} &\exp\Big(-\log\big((a\circ A^{-1})(\eta)\big) + \log\big((a\circ A^{-1})(\eta-t)\big)\Big).\end{aligned}$$

Hence, if $-\log a\circ A^{-1}$ satisfies condition $(C_{L,N})$ for some $L$ and $N$, then

$$\|T(t)\|_p^p \le \exp(L+Nt) = e^L e^{Nt}.$$

That is, $T(t)$ transforms $L^p$ in $L^p$ and the semigroup is of type $(e^L, N)$.

Vice versa suppose $T(t)$ transforms $L^p$ in $L^p$ and $\|T(t)\|_p^p \le Me^{\omega t}$. Let $\eta_0, \eta_1 \in (s_0, s_1)$, with $\eta_0 < \eta_1$, and $h \in \mathbb{R}^+$. If $f$ is the characteristic function of $\big[A^{-1}(\eta_1), A^{-1}(\eta_1+h)\big]$, then

$$\|f\|_p^p = A^{-1}(\eta_1+h) - A^{-1}(\eta_1)$$

and

$$\|T(\eta_1-\eta_0)f\|_p^p = \int_{r_0}^{r_1} \bigl|f\bigl(A^{-1}(A(\xi)+\eta_1-\eta_0)\bigr)\bigr|^p\,d\xi =$$
$$= \int_{s_0}^{s_1} \bigl|f\bigl(A^{-1}(\eta+\eta_1-\eta_0)\bigr)\bigr|^p a\bigl(A^{-1}(\eta)\bigr)\,d\eta = \int_{\eta_0}^{\eta_0+h} a\bigl(A^{-1}(\eta)\bigr)\,d\eta.$$

Hence

$$M^p e^{p\omega(\eta_1-\eta_0)} \geq \frac{\|T(t)f\|_p^p}{\|f\|_p^p} =$$
$$= \frac{h}{A^{-1}(\eta_1+h)-A^{-1}(\eta_1)}\,\frac{1}{h}\int_{\eta_0}^{\eta_0+h} a\bigl(A^{-1}(\eta)\bigr)\,d\eta \xrightarrow[h\to 0]{} \frac{a\bigl(A^{-1}(\eta_0)\bigr)}{(A^{-1})'(\eta_1)} =$$
$$= \frac{a\bigl(A^{-1}(\eta_0)\bigr)}{a\bigl(A^{-1}(\eta_1)\bigr)} = \exp\Bigl(-\log\bigl((a\circ A^{-1})(\eta_1)\bigr)+\log\bigl((a\circ A^{-1})(\eta_0)\bigr)\Bigr)$$

Therefore $-\log a\circ A^{-1}$ satisfies condition $(C_{\log M,\omega})$.

Similar arguments hold also when $s_1<\infty$ and/or $b\neq 0$, allowing to prove the following theorems.

**Theorem 4.1.** *The semigroup $\bigl(T(t)\bigr)_{t\in[0,\infty)}$ is of type $(M,\omega)$ in $L^p\bigl((r_0,r_1),\mathbb{C}\bigr)$ if and only if the function $(\operatorname{Re}B-p^{-1}\log a)\circ A^{-1}$ satisfies condition $(C_{\log M,\omega})$.*

In case $s_1=\infty$ a necessary and sufficient condition in a different form was obtained in [1] Theorem 1.

When $(s_0,s_1)=\mathbb{R}$ we know that $T(t)$ is defined also for negative values of $t$ and we have a group of operators.

**Theorem 4.2.** *If $(s_0,s_1)=\mathbb{R}$, then the group $\bigl(T(t)\bigr)_{t\in\mathbb{R}}$ is of type $(M,\omega)$ in $L^p\bigl((r_0,r_1),\mathbb{C}\bigr)$ if and only if the function $\bigl|(\operatorname{Re}B-p^{-1}\log a)\circ A^{-1}\bigr|$ satisfies condition $(C_{\log M,\omega})$.*

When $b=0$ this characterization was obtained in [3] Proposition 3.1.

By Theorem 4.1, the semigroup $\bigl(T(t)\bigr)_{t\in[0,\infty)}$ is a contraction semigroup if and only if $\operatorname{Re}B-p^{-1}\log a$ is decreasing. When $p=\infty$ this is equivalent to the fact that $\operatorname{Re}b$ is non-positive.

Suppose $(s_0,s_1)=\mathbb{R}$. By Theorem 4.2, the group $\bigl(T(t)\bigr)_{t\in\mathbb{R}}$ is a contraction group if and only if $\operatorname{Re}B-p^{-1}\log a$ is constant. If $p<\infty$ this is equivalent to the fact that $a\in C^1$ and $\operatorname{Re}b=p^{-1}a'$. If $p=\infty$ this is equivalent to the fact that $\operatorname{Re}b=0$.

## 5. Strong continuity of the semigroup

Every strongly continuous semigroup is of type $(M,\omega)$ for some $M$ and $\omega$. Therefore the fact that $(\operatorname{Re} B - p^{-1}\log a)\circ A^{-1}$ satisfies condition $(C_{L,N})$ for some $L\in\mathbb{R}^+$ and $N\in\mathbb{R}$ is necessary for the strong continuity of the semigroup.

As in the case of the translation semigroup, the semigroup is never continuous in $L^\infty$.

**Theorem 5.1.** *If the semigroup $\big(T(t)\big)_{t\in[0,\infty)}$ is of type $(M,\omega)$ in $L^p\big((r_0,r_1),\mathbb{C}\big)$, with $1\le p<\infty$, for some $M$ and $\omega$, then it is strongly continuous.*

The theorem follows from the fact that for every $f$ continuous and with compact support in $(r_0,r_1)$ we have

$$\lim_{t\to 0}\big\|T(t)f-f\big\|_p=0$$

and the space of compactly supported continuous functions is dense in $L^p$.

It is easy to prove the following theorem.

**Theorem 5.2.** *In case $b=0$ the semigroup $\big(T(t)\big)_{t\in[0,\infty)}$ is strongly continuous in the space*

$$\big\{f\in L^\infty\big((r_0,r_1),\mathbb{C}\big)\ \big|\ f\circ A^{-1}\in BUC\big\},\qquad \textit{if } s_1=\infty,$$
$$\big\{f\in L^\infty\big((r_0,r_1),\mathbb{C}\big)\ \big|\ f\circ A^{-1}\in BUC_0\big\},\qquad \textit{if } s_1\in\mathbb{R}.$$

## 6. Infinitesimal generator

To determine the infinitesimal generator of the semigroup we calculate its resolvent operator and we invert it.

For every strongly continuous semigroup of type $(M,\omega)$ the set $\{\lambda\in\mathbb{C}\mid \operatorname{Re}\lambda>\omega\}$ is contained in the resolvent set of the infinitesimal generator $G$ and $(\lambda I-G)^{-1}$ is the Laplace transform of the semigroup, that is

$$(\lambda I-G)^{-1}f=\int_0^\infty e^{-\lambda t}T(t)f\,dt.$$

(see [4], Theorem II.1.10). This fact allows to easily obtain a representation of the resolvent operator.

**Theorem 6.1.** *Suppose that the semigroup* $(T(t))_{t\in[0,\infty)}$ *is strongly continuous of type* $(M,\omega)$ *in* $L^p((r_0,r_1),\mathbb{C})$*, with* $1 \le p < \infty$*, and let* $G$ *be its infinitesimal generator. If* $\lambda \in \mathbb{C}$ *is such that* $\operatorname{Re}\lambda > \omega$*, then* $\lambda \in \rho(G)$*; moreover* $\forall f \in L^p((r_0,r_1),\mathbb{C})$ *we have*

$$[(\lambda I - G)^{-1} f](x) = \int_x^{r_1} \exp\bigl(B(\xi) - B(x) - \lambda(A(\xi) - A(x))\bigr) \frac{1}{a(\xi)} f(\xi)\, d\xi.$$

In case $a = 1$ and $b = 0$ we get again the standard form of the resolvent of a translation semigroup:

$$\bigl((\lambda I - G)^{-1} f\bigr)(x) = \int_x^{r_1} \exp\bigl(-\lambda(\xi - x)\bigr) f(\xi)\, d\xi.$$

**Theorem 6.2.** *Suppose that the semigroup* $\bigl(T(t)\bigr)_{t\in[0,\infty)}$ *is strongly continuous of type* $(M,\omega)$ *in* $L^p\bigl((r_0,r_1),\mathbb{C}\bigr)$*, with* $1 \le p < \infty$*, and let* $G$ *be its infinitesimal generator. Put*

$$Y = \bigl\{f \in L^p\bigl((r_0,r_1),\mathbb{C}\bigr) \cap W^{1,1}_{\mathrm{loc}}\bigl((r_0,r_1),\mathbb{C}\bigr) \bigm| af' + bf \in L^p\bigr\}.$$

*Then*

$$\mathcal{D}(G) = \begin{cases} Y, & \text{if } s_1 = \infty, \\ \bigl\{f \in Y \bigm| \lim\limits_{x\to r_1} a(x)^{1/p} f(x) = 0\bigr\}, & \text{if } s_1 \in \mathbb{R} \end{cases}$$

*and, for* $f \in \mathcal{D}(G)$*,*

$$Gf = af' + bf.$$

We give a sketch of the proof of this theorem.

Let $\lambda \in \mathbb{C}$ be such that $\operatorname{Re}\lambda > \omega$ and $f \in Y$. For a. e. $\xi, x \in (r_0,r_1)$ we have

$$(1) \quad \frac{d}{d\xi} \exp\bigl(B(\xi) - B(x) - \lambda(A(\xi) - A(x))\bigr) f(\xi) = \\ = \exp\bigl(B(\xi) - B(x) - \lambda(A(\xi) - A(x))\bigr) \left(f'(\xi) + \frac{b(\xi) - \lambda}{a(\xi)} f(\xi)\right),$$

hence, by Theorem 6.1, we have

$$[(\lambda I - G)^{-1}(\lambda f - af' - bf)](x) = \\ = \int_x^{r_1} \exp\bigl(B(\xi) - B(x) - \lambda(A(\xi) - A(x))\bigr) \left(-f'(\xi) + \frac{\lambda - b(\xi)}{a(\xi)} f(\xi)\right) d\xi = \\ = -\lim_{z\to r_1} \exp\bigl(B(z) - B(x) - \lambda(A(z) - A(x))\bigr) f(z) + f(x).$$

Therefore, if

$$\lim_{z\to r_1} \exp\bigl(B(z)-B(x)-\lambda(A(z)-A(x))\bigr)f(z)=0, \tag{2}$$

then $f\in\mathcal{D}(G)$ and $(\lambda I-G)^{-1}(\lambda f-af'-bf)=f$.

The proof of (2) is different depending on whether it is $s_1=\infty$ or $s_1\in\mathbb{R}$.

Suppose $s_1=\infty$ and let $\delta\in\mathbb{R}^+$ be such that $\operatorname{Re}\lambda-\delta>\omega$. For $x\in(r_0,r_1)$ and $z\in(x,r_1)$ from (1) we get

$$\begin{aligned}&\Bigl|\exp\bigl(B(z)-B(x)-(\lambda-\delta)(A(z)-A(x))\bigr)f(z)\Bigr| =\\ &=\Bigl|f(x)+\\ &\quad+\int_x^z \exp\bigl(B(\xi)-B(x)-(\lambda-\delta)(A(\xi)-A(x))\bigr)\Bigl(f'(\xi)+\frac{b(\xi)-(\lambda-\delta)}{a(\xi)}f(\xi)\Bigr)\,d\xi\Bigr|.\end{aligned}$$

Since $(\operatorname{Re}B-p^{-1}\log a)\circ A^{-1}$ satisfies condition $(C_{\log M,\omega})$, Hölder's inequality allows to estimate this quantity with

$$|f(x)|+\frac{M\|af'+(b-\lambda+\delta)f\|_{L^p}}{p'^{1/p'}(\operatorname{Re}\lambda-\delta-\omega)^{1/p'}a(x)^{1/p}}.$$

Hence

$$\begin{aligned}&\Bigl|\exp\bigl(B(z)-B(x)-\lambda(A(z)-A(x))\bigr)f(z)\Bigr|\le\\ &\le\exp\bigl(-\delta(A(z)-A(x))\bigr)\Bigl(|f(x)|+\frac{M\|af'+(b-\lambda+\delta)f\|_{L^p}}{p'^{1/p'}(\operatorname{Re}\lambda-\delta-\omega)^{1/p'}a(x)^{1/p}}\Bigr)\xrightarrow[z\to r_1]{}0,\end{aligned}$$

since $A(z)\to s_1=\infty$.

Suppose $s_1\in\mathbb{R}$ and let $f$ be such that $\lim_{x\to r_1}a(x)^{1/p}f(x)=0$. We have

$$\begin{aligned}&\Bigl|\exp\bigl(B(z)-B(x)-\lambda(A(z)-A(x))\bigr)f(z)\Bigr|=\\ &=\exp\bigl(\operatorname{Re}B(z)-\operatorname{Re}B(x)-\operatorname{Re}\lambda(A(z)-A(x))-p^{-1}\bigl(\log(a(z))-\log(a(x))\bigr)\bigr)\cdot\\ &\qquad\cdot\frac{a(z)^{1/p}}{a(x)^{1/p}}|f(z)|\le\\ &\le M\exp\bigl((\omega-\operatorname{Re}\lambda)(A(z)-A(x))\bigr)\frac{a(z)^{1/p}}{a(x)^{1/p}}|f(z)|\le M\frac{a(z)^{1/p}}{a(x)^{1/p}}|f(z)|\xrightarrow[z\to r_1]{}0.\end{aligned}$$

This proves that $Y$ or $\bigl\{f\in Y \bigm| \lim_{x\to r_1}a(x)^{1/p}f(x)=0\bigr\}$ is contained in $\mathcal{D}(G)$ and for $f$ in such space $Gf=af'+bf$.

With similar arguments it is possible to show that $\mathcal{D}(G)$ is contained in $Y$ or in $\big\{f \in Y \mid \lim_{x\to r_1} a(x)^{1/p} f(x) = 0\big\}$.

**Theorem 6.3.** *Suppose $b = 0$ and put*

$$X = \begin{cases} \big\{f \in L^\infty\big((r_0, r_1), \mathbb{C}\big) \mid f \circ A^{-1} \in BUC\big\}, & \text{if } s_1 = \infty, \\ \big\{f \in L^\infty\big((r_0, r_1), \mathbb{C}\big) \mid f \circ A^{-1} \in BUC_0\big\}, & \text{if } s_1 \in \mathbb{R}. \end{cases}$$

*Let $G$ be the infinitesimal generator of the semigroup $\big(T(t)\big)_{t\in[0,\infty)}$. Then*

$$\mathcal{D}(G) = \big\{f \in X \cap C^1\big((r_0, r_1), \mathbb{C}\big) \mid af' \in X\big\},$$

*and, for $f \in \mathcal{D}(G)$,*

$$Gf = af'.$$

## 7. Examples

In this section we study some examples of operators $Gf = af' + bf$ in $L^p$ spaces, with $1 \le p < \infty$.

*Example* 7.1. Let $(r_0, r_1) = (-1, 1)$, $a(x) = 1 - x^2$, and $b(x) = \beta x$, with $\beta \in \mathbb{R}$. A primitive of $1/a$ is the function $A(x) = (1/2)\log\big((1 + x)/(1 - x)\big) = \operatorname{arctanh} x$ hence the image of $A$ is $\mathbb{R}$ and, for $y \in \mathbb{R}$, we have $A^{-1}(y) = \tanh y$. A primitive of $b/a$ is the function $B(x) = -(\beta/2)\log(1 - x^2)$.

For $x \in \mathbb{R}$ we have

$$\big|(\operatorname{Re} B - p^{-1}\log a)\big(A^{-1}(y)\big)\big| = \Big|\Big(-\frac{\beta}{2} - \frac{1}{p}\Big)\log(1 - \tanh^2 y)\Big| = \Big|\beta + \frac{2}{p}\Big|\log(\cosh y).$$

This function is Lipschitz continuous with Lipschitz constant $|\beta + 2/p|$, hence it satisfies condition $(C_{0,N})$, with $N = |\beta + 2/p|$. By Theorems 4.2 and 6.2 the operator $G$ defined by

$$\mathcal{D}(G) = \big\{f \in L^p\big((-1, 1), \mathbb{C}\big) \cap W^{1,1}_{\text{loc}}\big((-1, 1), \mathbb{C}\big) | x \mapsto (1 - x^2)f'(x) + \beta x f(x) \in L^p\big\},$$
$$[Gf](x) = (1 - x^2)f'(x) + \beta x f(x),$$

generates a strongly continuous group in $L^p\big((-1, 1), \mathbb{C}\big)$ for every $p \in [1, \infty)$.

We have

$$A^{-1}(A(x) + t) = \tanh(\operatorname{arctanh} x + t) = \frac{\tanh(\operatorname{arctanh} x) + \tanh t}{1 + \tanh(\operatorname{arctanh} x)\tanh t} = \frac{x\cosh t + \sinh t}{\cosh t + x\sinh t}$$

and

$$B\big(A^{-1}(A(x)+t)\big) - B(x) = -\frac{\beta}{2}\log\left(1 - \frac{(x\cosh t + \sinh t)^2}{(\cosh t + x\sinh t)^2}\right) + \frac{\beta}{2}\log(1-x^2) =$$
$$= \beta\log(\cosh t + x\sinh t).$$

Therefore the group is such that

$$[T(t)f](x) = (\cosh t + x\sinh t)^{\beta} f\left(\frac{x\cosh t + \sinh t}{\cosh t + x\sinh t}\right).$$

*Example* 7.2. Let $(r_0, r_1) = \mathbb{R}$, $a(x) = 1 + x^2$, and $b(x) = \beta x$, with $\beta \in \mathbb{R}$. A primitive of $1/a$ is the function $A(x) = \arctan x$ hence the image of $A$ is $(-\pi/2, \pi/2)$ and, for $y \in (-\pi/2, \pi/2)$, we have $A^{-1}(y) = \tan y$. A primitive of $b/a$ is the function $B(x) = (\beta/2)\log(1 + x^2)$.

We have

$$(ReB - p^{-1}\log a)\big(A^{-1}(y)\big) = \frac{\beta}{2}\log(1 + \tan^2 y) - \frac{1}{p}\log(1 + \tan^2 y) = \left(\frac{2}{p} - \beta\right)\log(\cos y).$$

Since the domain of $A^{-1}$ is bounded, condition $(C_{L,N})$ is satisfied for some $L$ and $N$ if and only if the set

$$\left\{\left(\frac{2}{p} - \beta\right)\log\Big(\frac{\cos y_1}{\cos y_0}\Big) : -\frac{\pi}{2} < y_0 \le y_1 < \frac{\pi}{2}\right\}$$

is bounded from above. If $2/p - \beta = 0$, that is $\beta = 2/p$, condition $(C_{0,0})$ is obviously satisfied. If $2/p - \beta < 0$ then

$$\lim_{y_0 \to -\pi/2}\left(\frac{2}{p} - \beta\right)\log\Big(\frac{\cos y_1}{\cos y_0}\Big) = \infty$$

and if $2/p - \beta > 0$ then

$$\lim_{y_1 \to \pi/2}\left(\frac{2}{p} - \beta\right)\log\Big(\frac{\cos y_1}{\cos y_0}\Big) = \infty.$$

Hence if $2/p - \beta \neq 0$, then condition $(C_{L,N})$ is not satisfied.

By TheoremS 4.1 and 6.2 the operator $G$ defined by

$$\mathcal{D}(G) =$$
$$= \big\{f \in L^p(\mathbb{R}, \mathbb{C}) \cap W^{1,1}_{\text{loc}}(\mathbb{R}, \mathbb{C}) : x \mapsto (1+x^2)f'(x) + \beta x f(x) \in L^p, \lim_{x\to\infty} x^{\beta/p} f(x) = 0\big\},$$
$$[Gf](x) = (1+x^2)f'(x) + \beta x f(x),$$

generates a strongly continuous semigroup in $L^p(\mathbb{R},\mathbb{C})$ if and only if $\beta = 2/p$. The semigroup is a contraction semigroup.

If $t < \pi$ and $x < A^{-1}(\pi/2 - t) = \tan(\pi/2 - t) = \cot t$, we have

$$A^{-1}(A(x)+t) = \tan(\arctan x + t) = \frac{\tan(\arctan x) + \tan t}{1 - \tan(\arctan x)\tan t} = \frac{x + \tan t}{1 - x\tan t} = \frac{x\cos t + \sin t}{\cos t - x\sin t}.$$

Hence

$$B\big(A^{-1}(A(x)+t)\big) - B(x) = \frac{\beta}{2}\log\left(1 + \frac{(x\cos t + \sin t)^2}{(\cos t - x\sin t)^2}\right) - \frac{\beta}{2}\log(1+x^2) =$$

$$= -\beta\log(\cos t - x\sin t).$$

Therefore the semigroup is such that

$$[T(t)f](x) = \begin{cases} (\cos t - x\sin t)^{-\beta} f\left(\dfrac{x\cos t + \sin t}{\cos t - x\sin t}\right), & \text{if } t < \pi \text{ and } x < \cot t, \\ 0, & \text{otherwise.} \end{cases}$$

*Example* 7.3. Let $(r_0, r_1) = (1, \infty)$, $a(x) = x\log x$, and $b(x) = \beta\log x$ with $\beta \in \mathbb{R}$. A primitive of $1/a$ is the function $A(x) = \log(\log x)$ hence the image of $A$ is $\mathbb{R}$ and, for $y \in \mathbb{R}$, we have $A^{-1}(y) = e^{e^y}$. A primitive of $b/a$ is the function $B(x) = \beta\log x$.

We have

$$(ReB - p^{-1}\log a)\big(A^{-1}(y)\big) = \beta\log(e^{e^y}) - \frac{1}{p}\log\big(e^{e^y}\log(e^{e^y})\big) = \beta e^y - \frac{1}{p}e^y - \frac{1}{p}y.$$

Obviously this function satisfies condition $(C_{L,N})$ for some $L$ and $N$ if and only if $\beta \le 1/p$. If $\beta = 1/p$ then $\big|(ReB - p^{-1}\log a)\big(A^{-1}(y)\big)\big| = y/p$, and this function satisfies condition $(C_{0,1/p})$. Therefore, by Theorems 4.1, 4.2 and 6.2, the operator $G$ defined by

$$\mathcal{D}(G) = \big\{f \in L^p\big((1,\infty),\mathbb{C}\big) \cap W^{1,1}_{\text{loc}}\big((1,\infty),\mathbb{C}\big) | x \mapsto x\log(x)f'(x) + \beta\log x\, f(x) \in L^p\big\},$$

$$[Gf](x) = x\log x\, f'(x) + \beta\log x\, f(x),$$

generates a strongly continuous semigroup in $L^p\big((1,\infty),\mathbb{C}\big)$ if and only if $\beta \le 1/p$. In case $\beta = 1/p$ it generates a group.

We have

$$A^{-1}(A(x)+t) = \exp\big(\exp(\log(\log x) + t)\big) = \exp(e^t\log x) = x^{e^{e^t}}.$$

Hence

$$B\big(A^{-1}(A(x)+t)\big) - B(x) = \beta \log(x^{e^t}) - \beta \log x = \beta(e^t - 1) \log x.$$

Therefore the semigroup is such that

$$[T(t)f](x) = x^{\beta(e^t-1)} f(x^{e^t}).$$

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