REGULARITY RESULTS IN ELLIPTIC AND PARABOLIC TRANSMISSION PROBLEMS

RISULTATI DI REGOLARITÀ PER PROBLEMI DI TRASMISSIONE ELLITTICI E PARABOLICI

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Abstract. This paper surveys recent regularity results for transmission problems in both elliptic and parabolic settings. We also discuss the motivation behind these problems, illustrating them with concrete applications.

Sunto. Questo articolo esamina recenti risultati di regolarità per problemi di trasmissione, sia in contesti ellittici che parabolici. Discutiamo inoltre la motivazione alla base di questi problemi, illustrandoli con applicazioni concrete.

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1. Introduction

In this survey, we discuss recent advances in regularity theory for elliptic and parabolic transmission problems, motivating their study through their connection to free boundary problems.

Transmission problems model physical phenomena in which the behavior changes across some fixed interface and have attracted considerable attention throughout the years, starting with the pioneering work of Picone [37] in elasticity.

In his seminal work, Picone introduced a new mathematical framework to model the behavior of an elastic body C_1, for instance, a dam, embedded or built into a foundation body C_2, such as bedrock, which at rest occupy the domains Ω_1 and Ω_2, respectively. The

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elastic forces acting on C_1 under external loads induce a deformation in C_2, resulting in a displacement along the common interface Γ, shared by $\partial\Omega_1$ and $\partial\Omega_2$. Let V_1 and V_2 denote the displacement vectors of points in Ω_1 and Ω_2, respectively. Under the action of external forces F_i applied to Ω_i, the displacement fields satisfy the following vectorial elliptic transmission problem.

$$\begin{aligned} h_1\Delta V_1 + (h_1+k_1)\nabla\operatorname{div}(V_1) + F_1 = 0 \quad \text{in } \Omega_1, \\ h_2\Delta V_2 + (h_2+k_2)\nabla\operatorname{div}(V_2) + F_2 = 0 \quad \text{in } \Omega_2, \\ V_1 = V_2 \quad \text{in } \Gamma, \qquad \Phi_1(V_1) + \Phi_2(V_2) = 0 \quad \text{on } \Gamma, \\ \Phi_i(V_i) = \varphi_i \quad \text{on } \partial\Omega_i \setminus \Gamma, \end{aligned} \tag{1}$$

where h_i, k_i are the Lamé constants of the body C_i. Moreover, Φ_i represents the pressure vector on the boundary of Ω_i, which is prescribed to be φ_i on $\Omega_i \setminus \Gamma$. Moreover, on the interface Γ, the pressure vectors exerted by C_1 and C_2 must sum to zero, resulting in an equilibrium condition. The pressure vector can be written explicitly in terms of the Lamé constants and the outer normal vector to $\partial\Omega_i$ as

$$\Phi_i(V) = k_i \operatorname{div}(V)\nu + 2h_i\partial_\nu V + k_i(\nu \wedge \operatorname{Rot} V).$$

Other examples of transmission problems can be found for instance in electromagnetic conductivity of materials and mechanics of composite materials.

For the problem (1), Picone established the uniqueness of solutions. In the subsequent years, the existence of weak solutions to the problem posed by Picone, as well as to certain generalizations, was investigated in the works of Lions [35], Stampacchia [42], and Campanato [10]. Beginning in the 1960s, regularity results for transmission problems with an interface and regular coefficients started to be explored. In the non-divergence case, such results were obtained by Schechter [40], while in the divergence form case, related to the so-called diffraction problem, these transmission problems were studied by Oleinik [36], Ladyzhenskaya–Ural'tseva [32], and Borsuk, to whom we also refer to the monograph [6] for further insights.

Building on these classical results, the regularity of solutions to transmission problems with non-smooth interfaces or with coefficients discontinuous along Γ has been investigated. We refer, for instance, to the works of Li and Vogelius [34], Li and Nirenberg [33] and for other recent developments, see [9, 11, 30, 1, 39, 38, 26].

More recently, the connection between the regularity of transmission problems and that of certain elliptic and parabolic free boundary problems has further increased interest in this topic. In [12], De Silva introduced an improvement of flatness technique to address the one-phase case, where the regularity of the limiting problem is transferred to the solution of the free boundary problem. This method has since been extended to various two-phase problems, in which the limiting problem is of transmission type.

In the first section, we recall this connection between transmission problems and free boundary problems. We then focus on more recent results concerning the regularity of transmission problems for fully nonlinear operators. In this context, we discuss the techniques developed to obtain regularity up to the interface for problems in which the operator degenerates with the distance from the interface, results recently obtained by the authors in [25], as well as some interesting open questions.

Finally, in the last section, we will provide an overview of the parabolic counterpart.

2. Preliminaries

Let $d \geqslant 2$ be the space dimension. We denote by $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ a point in $\mathbb{R}^d$. Given a function u we denote with ∇u its gradient, $\nabla' u$ is the gradient only with respect to the variables x' and $u_d = \partial_{x_d} u$. We write $S(d)$ for the set of real symmetric $d \times d$ matrices, endowed with the operator norm $|\cdot|$.

Definition 2.1 (Uniform ellipticity)**.** *We say that $F : S(d) \to \mathbb{R}$ is uniformly (λ, Λ)-elliptic if there exist $0 < \lambda \leqslant \Lambda$ such that*

$$\lambda |N| \leqslant F(M + N) - F(M) \leqslant \Lambda |N|,$$

for every $M, N \in S(d)$, with $N \geqslant 0$.

Definition 2.2 (Pucci operators)**.** *We define the extremal Pucci operators with ellipticity constants λ and Λ by*

$$\mathcal{M}^+_{\lambda,\Lambda}(M) = \sup_{\lambda I \leqslant A \leqslant \Lambda I} \operatorname{tr}(AM), \qquad \mathcal{M}^-_{\lambda,\Lambda}(M) = \inf_{\lambda I \leqslant A \leqslant \Lambda I} \operatorname{tr}(AM), \quad M \in S(d).$$

For simplicity, we will just call $\mathcal{M}^+$ and $\mathcal{M}^-$ when the ellipticity constants are understood.

3. Elliptic transmission problems

We start by considering viscosity solutions to the following set of equations

$$\begin{cases} F^\pm(D^2u, x) = f^\pm, & \text{in } \Omega^\pm \\ G(u^+_\nu, u^-_\nu, x, \nu) = 0 & \text{on } \Gamma, \end{cases} \tag{2}$$

where ν is the normal of Γ pointing to Ω^+ and $F^\pm$ denotes two distinct (λ, Λ)-elliptic operators. That is, we consider a function which satisfies a PDE in each component of the domain $\Omega^\pm$ and moreover satisfies a transmission condition across the interface separating these components. In the case where $\Omega^\pm$ depend on u, this is called a *free* transmission problem, and $\Gamma = \Gamma(u)$ is a free boundary. The simpler case is when $\Gamma = \{x_d = \Psi(x')\}$ is the graph of a given function Ψ. In this case, it is called a *fixed* transmission problem. In the following we recall the notion of viscosity solution of (2).

Definition 3.1. *Let $u \in \mathrm{USC}(\Omega)$ (resp. $u \in \mathrm{LSC}(\Omega)$). We say that u is a viscosity subsolution (resp. supersolution) of* (2) *if for any φ touching u from above (resp. below) at $x_0 \in \Omega$, the following conditions hold:*

- *if $x_0 \in \Omega^\pm$ and $\varphi \in C^2(B_\delta(x_0) \cap \Omega^\pm)$, then*

$$F(D^2\varphi(x_0), x_0) \geqslant f^\pm(x_0) \qquad (\textit{resp. } F(D^2\varphi(x_0), x_0) \leqslant f^\pm(x_0));$$

- *if $x_0 \in \Gamma$ and $\varphi \in C^1\left(B_\delta(x_0) \cap \overline{\Omega^+}\right) \cup C^1\left(B_\delta(x_0) \cap \overline{\Omega^-}\right)$ then*

$$G(\varphi^+_\nu(x_0), \varphi^-_\nu(x_0), x_0, \nu) \geqslant 0 \qquad (\textit{resp. } G(\varphi^+_\nu(x_0), \varphi^-_\nu(x_0), x_0, \nu) \leqslant 0).$$

We say that $u \in C(\Omega)$ is a viscosity solution of (2) *if it is both a subsolution and a supersolution.*

Now we define useful classes of solutions to the extremal equations, incorporating also the transmission condition.

Definition 3.2 ($\mathcal{S}$ classes)**.** *The class* $\underline{\mathcal{S}}(f^{\pm})$ *denotes the functions* $u \in \mathrm{USC}(\Omega)$ *such that* $\mathcal{M}^{+}(D^2u) \geqslant f^{\pm}$ *in* $\Omega^{\pm}$ *and* $G(u_\nu^+, u_\nu^-, x, \nu) \geqslant 0$ *on* Γ *in the viscosity sense.*

Similarly, the class $\overline{\mathcal{S}}(f^{\pm})$ *denotes the functions* $u \in \mathrm{LSC}(\Omega)$ *such that* $\mathcal{M}^{-}(D^2u) \leqslant f^{\pm}$ *in* $\Omega^{\pm}$ *and* $G(u_\nu^+, u_\nu^-, x, \nu) \leqslant 0$ *on* Γ *in the viscosity sense.*

Furthermore, we call $\mathcal{S}(f^{\pm}) = \underline{\mathcal{S}}(f^{\pm}) \cap \overline{\mathcal{S}}(f^{\pm})$ *and* $\mathcal{S}^*(f^{\pm}) = \underline{\mathcal{S}}(-|f^{\pm}|) \cap \overline{\mathcal{S}}(|f^{\pm}|)$*.*

These definitions concern the general elliptic equation (2), however they easily adapt to the different settings considered in this work.

3.1. **Fixed transmission problem as the limit to a free transmission problem.** In the influential papers by De Silva, Ferrari and Salsa [13, 15], the authors use compactness arguments to study the regularity of solutions to free transmission problems. The limiting case consists of studying the solution to a fixed transmission problem. These ideas are originally from [12].

More precisely, they consider the two-phase elliptic problem given by

$$\begin{cases} F(D^2u) = f, & \text{in } \Omega^+(u) \cup \Omega^-(u) \\ (u_\nu^+)^2 - (u_\nu^-)^2 = 1 & \text{on } \mathcal{F}(u) := \partial\Omega^+(u) \cap \Omega, \end{cases} \tag{3}$$

where

$$\Omega^+(u) := \{x \in \Omega \,:\, u > 0\} \qquad \Omega^-(u) := \{x \in \Omega \,:\, u \leqslant 0\}^o.$$

Here F is a (λ, Λ)-elliptic operator, u_ν^+ and u_ν^- denote the normal derivatives in the inward direction to $\Omega^+(u)$ and $\Omega^-(u)$. The source term f is assumed to be bounded and continuous on $\Omega^{\pm}(u)$.

A significant example of (3) stems from the two-dimensional Prantl-Batchelor flow. The model proposed by Batchelor in [5] to describe the limit of large Reynolds number in the steady Navier-Stokes equation leads to considering a free boundary problem of the form (3). Given a bounded set Ω in the plane and the constants $\mu < 0, \omega > 0$, one seeks to find a curve Γ that splits Ω into two parts Ω_1 and Ω_2 and two functions $\psi_1 \leqslant 0$ and $\psi_2 \geqslant 0$

satisfying

$$\begin{aligned} \Delta\psi_1 &= 0 \quad \text{in } \Omega_1, \\ \Delta\psi_2 &= -\omega \quad \text{in } \Omega_2, \\ |\nabla\psi_2|^2 - |\nabla\psi_1|^2 &= \sigma \geqslant 0 \quad \text{on } \Gamma \\ \psi_1 = \psi_2 \;\text{ on } \Gamma, &\quad \psi_1 = \mu \;\text{ on } \partial\Omega. \end{aligned} \tag{4}$$

The functions ψ_1 and ψ_2 represents the stream functions of an irrotational flow in Ω_1 and of a constant vorticity flow ω in Ω_2. Here Γ plays the role of a free boundary where a discontinuity of the tangential velocities may appear.

The strategy to prove that $\mathcal{F}(u)$ is $C^{1,\alpha}$ for solutions to (3) relies on an improvement of flatness inspired by [12]. Regarding higher regularity, this result is only known for the case of the Laplacian, see [17]. The same problem is still open for general (concave/convex) elliptic operators, although a similar approach as in [17] using the $C^{2,\alpha}$ regularity obtained for the fixed transmission problem in [41] might be possible. We refer to [14, 18] and the references therein for further discussions and open problems on the topic.

We now briefly restate the main result in [15] and explain how the regularity of the limiting transmission problem appears to be crucial in these problems.

Theorem 3.1. *Let u be a viscosity solution of (3) for $\Omega = B_1$. Let $0 \in \mathcal{F}(u)$ and assume $f \in L^\infty(B_1) \cap C(B_1^\pm(u))$. There exists $\eta_0 > 0$ universal such that if*

$$\{x_d \leqslant -\eta\} \subset B_1 \cap \{u_+(x) = 0\} \subset \{x_d \leqslant \eta\}, \tag{5}$$

with $0 \leqslant \eta \leqslant \eta_0$ then $\mathcal{F}(u)$ is $C^{1,\alpha}$ in $B_{1/2}$.

The proof of this result is based on an iterative procedure that improves the control on dyadic balls of our solution around an optimal two-plane configuration. This procedure works nicely as long as the two phases u_+ and u_- are comparable (*nondegenerate case*), while in the situation in which one of the two phases is very small but not negligible (*degenerate case*), one needs to argue similarly to the one-phase case. Let $U_\beta(t)$ be the two plane solution (when $f = 0$) defined as

$$U_\beta(t) = \alpha t_+ - \beta t_-, \quad \beta \geqslant 0,\, \alpha = \sqrt{1+\beta^2},$$

where $t_\pm$ are the positive and negative part of t.

The flatness condition on the free boundary of (5) together with the Lipschitz continuity of u, see [20], implies that u is close in the L^∞-sense to $U_\beta(x_d)$. In the nondegenerate case, calling $\varepsilon = \eta^{1/3}$, the flatness condition on the free boundary can be expressed in terms of the whole solution as follows

$$U_\beta(x_d - \varepsilon) \leqslant u(x) \leqslant U_\beta(x_d + \varepsilon) \quad \text{in } B_1. \tag{6}$$

The proof of Theorem 3.1 relies crucially on the following improvement of flatness.

Lemma 3.1. *Let u be a viscosity solution of (3) for $\Omega = B_1$ with $[u]_{Lip(B_1)} \leqslant L$ and $0 \in \mathcal{F}(u)$. If u satisfies (6) for $0 \leqslant \beta \leqslant L$ and $\|f\|_{L^\infty(B_1)} \leqslant \varepsilon^2\beta$. Then there exists $r_0 > 0$ universal such that for $r \leqslant r_0$ and $0 < \varepsilon \leqslant \varepsilon_0$ for some ε_0 depending on r, then*

$$U_{\beta'}\left(x \cdot \nu_1 - \frac{r\varepsilon}{2}\right) \leqslant u(x) \leqslant U_{\beta'}\left(x \cdot \nu_1 + \frac{r\varepsilon}{2}\right) \quad \text{in } B_r, \tag{7}$$

with $|\nu_1| = 1, |\nu_1 - e_d| \leqslant C\varepsilon$, and $|\beta - \beta'| \leqslant C\beta\varepsilon$ for a universal constant C.

In order to prove Lemma 3.1 we argue by contradiction assuming that there is a sequence of $\varepsilon_k \to 0, \beta_k \geqslant 0$, u_k solutions of (3) in B_1 with a sequence of (λ, Λ)-elliptic operators F_k and right-hand side f_k such that $\|f_k\|_{L^\infty(B_1)} \leqslant \varepsilon_k^2\beta_k$, $0 \in \mathcal{F}(u_k)$ and

$$U_{\beta_k}(x_d - \varepsilon_k) \leqslant u_k(x) \leqslant U_{\beta_k}(x_d + \varepsilon_k) \quad \text{in } B_1,$$

with $\beta_k \leqslant L$ and $\alpha_k^2 = 1 + \beta_k^2$, but the conclusion of the lemma does not hold. Then one proves via a Harnack-type inequality, see [15, Corollary 4.2], compactness and stability, that the sequence of normalized functions

$$\tilde{u}_k(x) := \begin{cases} \dfrac{u_k(x) - \alpha_k x_d}{\alpha_k \varepsilon_k} & x \in B_1^+(u_k) \cup \mathcal{F}(u_k), \\[2ex] \dfrac{u_k(x) - \beta_k x_d}{\beta_k \varepsilon_k} & x \in B_1^-(u_k), \end{cases}$$

converges uniformly (up to a subsequence) to a limit function $\tilde{u}$ and $\mathcal{F}(u_k)$ converges to $B_1 \cap \{x_d = 0\}$ in the Hausdorff distance. As a byproduct, one also proves that $\tilde{u}$ is

Hölder continuous in $B_{1/2}$. Moreover, $\tilde{u}$ solves (in the viscosity sense) the following limiting transmission problem

$$\begin{cases} \tilde{F}^{\pm}(D^2\tilde{u}) = 0, & \text{in } B^{\pm}_{1/2} := B_{1/2} \cap \{\pm x_d > 0\} \\ \tilde{\alpha}^2 \tilde{u}^+_d - \tilde{\beta}^2 \tilde{u}^-_d = 0 & \text{on } B'_{1/2} := B_{1/2} \cap \{x_d = 0\}, \end{cases} \tag{8}$$

denoting with $\tilde{\alpha}$ and $\tilde{\beta}$ the limits of α_k and β_k. The operators $\tilde{F}^{\pm}$ appearing in (8), are obtained in the limit as $k \to \infty$ of the sequences of (λ, Λ)-elliptic operators $F^{\pm}_k$ defined by

$$F^+_k(M) := \frac{1}{\alpha_k \varepsilon_k} F_k(\alpha_k \varepsilon_k M), \quad F^-_k(M) := \frac{1}{\beta_k \varepsilon_k} F_k(\beta_k \varepsilon_k M).$$

The first equation in (8) is obtained with standard arguments, see for instance [8, Proposition 2.9], from

$$F^+_k(D^2\tilde{u}_k) = \frac{f_k}{\alpha_k \varepsilon_k} \quad \text{in } B^+_1(u_k), \quad F^-_k(D^2\tilde{u}_k) = \frac{f_k}{\beta_k \varepsilon_k} \quad \text{in } B^-_1(u_k)$$

which implies

$$|F^{\pm}_k(D^2\tilde{u}_k)| \leqslant \varepsilon_k \quad \text{in } B^{\pm}_1(u_k)$$

since $\|f\|_{L^\infty(B_1)} \leqslant \varepsilon_k^2 \beta_k$. Regarding the transmission condition in (8), we formally obtain

$$\begin{aligned} 1 &= |\nabla u^+_k|^2 - |\nabla u^-_k|^2 \\ &= (\alpha_k^2 - \beta_k^2) + \varepsilon_k^2 \left[\alpha_k^2 |\nabla \tilde{u}^+_k|^2 - \beta_k^2 |\nabla \tilde{u}^-_k|^2\right] + 2\varepsilon_k \left[\alpha_k^2 (\tilde{u}_k)^+_d - \beta_k^2 (\tilde{u}_k)^-_d\right], \end{aligned}$$

since $\alpha_k^2 - \beta_k^2 = 1$, dividing by ε_k, we get

$$0 = \varepsilon_k \left[\alpha_k^2 |\nabla \tilde{u}^+_k|^2 - \beta_k^2 |\nabla \tilde{u}^-_k|^2\right] + 2 \left[\alpha_k^2 (\tilde{u}_k)^+_d - \beta_k^2 (\tilde{u}_k)^-_d\right],$$

letting now $k \to \infty$ gives the desired condition on the interface $\{x_d = 0\} \cap B_{1/2}$.

Once we establish that the limit $\tilde{u}$ satisfies (8), then the regularity of this transmission problem can be used to obtain a contradiction. Indeed, knowing that $\tilde{u} \in C^{1,\alpha}(\overline{B^+_{1/2}}) \cap C^{1,\alpha}(\overline{B^-_{1/2}})$ with a universal bound, we get for $r < 1/2$ that

$$|\tilde{u}(x) - (x' \cdot \nabla' \tilde{u}(0) + p_+(x_d)_+ - p_-(x_d)_-)\,| \leqslant C r^{1+\alpha} \quad \text{in } B_r, \tag{9}$$

where $p_\pm := \tilde{u}_d^\pm(0)$. Consequently, the fact that $\tilde{u}_k$ converges uniformly to $\tilde{u}$ leads to a similar estimate for $\tilde{u}_k$ with a different universal constant. If we choose ν_k, α'_k, β'_k so that

$$\nu_k := \frac{1}{\sqrt{1+\varepsilon_k^2|\nabla'\tilde{u}(0)'|^2}}(e_d + \varepsilon_k(\nabla'\tilde{u}(0),0)), \quad \beta'_k := \beta_k(1+\varepsilon_k p_-),$$

then

$$\nu_k = e_d + \varepsilon_k(\nabla'\tilde{u}(0),0) + \tau_1\varepsilon_k^2, \quad \alpha'_k = \sqrt{1+\beta_k'^2} = \alpha_k(1+\varepsilon_k p_+) + \tau_2\varepsilon_k^2,$$

where $|\tau_1|+|\tau_2| \leqslant C$ and using that $\tilde{\alpha}^2 p_+ - \tilde{\beta}^2 p_- = 0$ which gives $\frac{\beta_k^2}{\alpha_k^2}p_- = p_+ + o(1)$. With these choices of parameters ν_k, α'_k, β'_k, the estimate (9) for $\tilde{u}_k$ implies that for k large and for $r \leqslant r_0$, r_0 universal, we reach

$$\tilde{U}_{\beta'_k}\left(x\cdot\nu_k - \frac{r\varepsilon_k}{2}\right) \leqslant \tilde{u}_k(x) \leqslant \tilde{U}_{\beta'_k}\left(x\cdot\nu_k + \frac{r\varepsilon_k}{2}\right) \quad \text{in } B_r, \tag{10}$$

where

$$\tilde{U}_{\beta'_k}(x\cdot\nu) = \begin{cases} \dfrac{U_{\beta'_k}(x\cdot\nu) - \alpha_k x_n}{\alpha_k\varepsilon_k} & x \in B_1^+(U_{\beta'_k}) \cup \mathcal{F}(U_{\beta'_k}), \\ \\ \dfrac{U_{\beta'_k}(x\cdot\nu) - \beta_k x_n}{\beta_k\varepsilon_k} & x \in B_1^-(U_{\beta'_k}). \end{cases}$$

Writing (10) in terms of u_k gives

$$U_{\beta'_k}\left(x\cdot\nu_k - \frac{r\varepsilon_k}{2}\right) \leqslant u_k(x) \leqslant U_{\beta'_k}\left(x\cdot\nu_k + \frac{r\varepsilon_k}{2}\right) \quad \text{in } B_r,$$

which leads to a contradiction with (7).

Therefore, in order to apply this argument, a crucial step is understanding the regularity of solutions to (8).

The regularity of the problem (8) is treated in [15]. We briefly recap the ideas and the techniques used in a more general framework. Indeed, in a subsequent work, see [16], the same authors study the regularity of solutions to a fixed transmission problem up to the interface with a source term that differs on each side. They considered the following problem

$$\begin{cases} F^\pm(D^2u) = f^\pm, & \text{in } B_1^\pm \\ \\ au_d^+ - bu_d^- = 0 & \text{on } B'_1, \end{cases} \tag{11}$$

where $a > 0$, $b \geqslant 0$ and $f^\pm \in C^{0,1}(B_1^\pm) \cap L^\infty(B_1)$.

They obtain local $C^{1,\alpha}$ regularity up to the flat interface $\{x_d = 0\}$, that is, for every $x_0 \in B_{1/2}$, there exists an affine function $\ell_{x_0}(x)$ such that, for $r \leqslant 1/4$,

$$\|u - \ell_{x_0}\|_{L^\infty(B_r(x)\cap B_1^\pm)} \leqslant Cr^{1+\alpha},$$

where C depends on the ellipticity constants, the dimension, $\|u\|_{L^\infty(B_1)}$ and $\|f^\pm\|_{C^{0,1}(B_1)}$.

The main idea behind this result is the following. They prove that for any $h > 0$ small, if u is a solution to (11), e' is any unit vector in the x' direction and $u_h(x) := u(x - he')$, is a translation of u in a direction parallel to the interface $\{x_d = 0\}$, then $u - u_h$ belongs to the class of functions $\mathcal{S}^*(\gamma_{f^\pm}(h))$ (recall Definition 3.2) for the transmission problem (11) where $\gamma_{f^\pm}$ is the modulus of continuity of $f^\pm$ defined as

$$\gamma_{f^\pm}(h) := \sup_{|x-y|\leqslant h} |f^\pm(x) - f^\pm(y)|, \qquad \text{for } h > 0. \tag{12}$$

For functions in this class, it is possible to obtain $C^{0,\alpha}$ estimates, see [8, Proposition 4.10]. This implies $C^{0,\alpha}$ regularity of the derivatives in directions parallel to $\{x_d = 0\}$. In the remaining direction e_d, they note that they can now re-frame the equation as

$$\begin{cases} F^+(D^2u) = f^+, & \text{in } B_1^+ \\ u = g & \text{on } B_1', \end{cases}$$

where $g \in C^{1,\alpha}$. This follows from the first step, since in B_1', $g = u$ is $C^{1,\alpha}$ in the directions perpendicular to e_d. Solutions of this equation are $C^{1,\alpha}$ in the e_d direction at the interface, which finally gives the full $C^{1,\alpha}$ regularity of u in all directions.

3.2. **Elliptic fixed curved interface.**

As previously observed, the transmission condition in (8), which arises in the limiting profile of a two-phase elliptic free boundary problem, requires that the solution meets the interface at a constant prescribed angle. In what follows, we will consider another more general and rather natural transmission condition that prescribes the jump of the normal derivatives across the interface as a given function g. Different types of conditions along the transmission surface were examined in the seminal work of Schechter, see [40], where a comprehensive regularity theory is developed for the case of smooth interfaces.

When dealing with domains of lower regularity, new difficulties arise that require the use of different techniques. In this direction, the work of Caffarelli, Soria-Carro, and Stinga in [9] addresses the case of harmonic functions on each side of an interface Γ with only $C^{1,\alpha}$ regularity. In particular, regularity for this problem is established through a novel geometric approach based on the mean value property and the maximum principle. This approach relies on stability results that allow, at small scales, to approximate solutions to the curved-interface problem with solutions to flat-interface problems.

More recently, in [41], Soria-Carro and Stinga consider the generalization of such a problem to the realm of fully nonlinear operators. In particular, they consider u to be a viscosity solution of

$$\begin{cases} F^{\pm}(D^2u) = f^{\pm} & \text{in } \Omega^{\pm} = \{\pm(x_d - \Psi(x')) > 0\} \\ u_\nu^+ - u_\nu^- = g & \text{on } \Gamma = \{x_d - \Psi(x') = 0\}, \end{cases} \tag{13}$$

where the curved interface Γ is the graph of a given function Ψ. When $g \in C^{0,\alpha}$ and $\Psi \in C^{1,\alpha}$, they obtain again that solutions are $C^{1,\alpha}$ up to the interface assuming a closeness condition between F^+ and F^- in the sense that there exists θ small depending only on d, λ, Λ such that

$$\sup_{M \in S(d)\setminus\{0\}} \frac{|F^+(M) - F^-(M)|}{|M|} \leqslant \theta. \tag{14}$$

Besides this technical hypothesis on the operators (which was removed recently in [28]), this work improves the result by [16] in the following ways. They remove the $C^{0,1}$ assumption on $f^{\pm}$, consider a more general transmission condition, and allow the interface to be curved.

The approach pursued in [41] to treat curved interfaces deviates considerably from that employed by [16]. The key ingredient from which the regularity theory is developed is a new ABP estimate. To achieve this, the authors use a perturbation of u that does not touch its convex envelope on Γ, constructing a suitable auxiliary function via Hopf's Lemma. The ABP is then employed with the construction of new barriers to obtain Harnack's inequality, and from there, the Hölder regularity follows. In order to get Hölder differentiability the authors approximate the solution at small scales distinguishing two cases. In the case where g is close to 0, then using a stability argument, it is proven that the solution of (13) is close to a differentiable function across Γ. If g is far from 0, then the solution u is approximated with solutions of flat interface problems. Then they used an argument in the spirit of the perturbation method introduced by Caffarelli in [7] to iterate the procedure at all scales.

Moreover, assuming additionally that $F^+ = F^-$ is convex, $g \in C^{1,\alpha}$, $f \in C^{0,\alpha}$ and $\Psi \in C^{2,\alpha}$, they obtain that u is $C^{2,\alpha}$ up to the interface. The regularity assumptions on g, f, Ψ are all natural. The assumption that $F^\pm$ needs to be convex is also natural since this is what allows for the Evans-Krylov theory to apply [8]. The additional assumption $F^+ = F^-$, however, is not natural and should not be necessary to still obtain this result; however, this remains an interesting open question. The reason why Soria-Carro and Stinga needed this assumption is related to the argument described in the previous subsection. An essential initial step in proving the Evans-Krylov theory is obtaining $C^{1,1}$ regularity. In this context, we want to obtain this regularity at the interface. Therefore, we consider the second variation of u in each direction e, that is,

$$\partial^2_{hh} u(x) = u(x + he) - 2u(x) + u(x - he).$$

When e is perpendicular to e_d, we readily check that $u_h(x) = u(x + he)$ and $u_{-h}(x) = u(x - he)$ still satisfy the same equation, and thus by convexity of the operators, $\partial^2_{hh} u$ belongs to a class $\underline{S}$ (see [8, Corollary 5.9]), therefore we can obtain some regularity in this direction. However, for $e = e_d$, this argument fails, because if $x_d \approx 0$, then $x + he_d$ and $x - he_d$ belong to different components of Ω. Therefore, if F^+ and F^- are different operators, then u_h and u_{-h} satisfy different equations and thus $\partial^2_{hh} u$ does not belong to the $\underline{S}$ class.

3.3. **Degenerate elliptic transmission problem.** More recently, in [25], the authors of the current paper were able to extend the results of [41] to a class of transmission problems ruled by an equation that degenerates as a power of the distance to the interface

$$\begin{cases} |x_d - \Psi(x')|^{a(x)} F^\pm(D^2 u) = f^\pm, & \text{in } \Omega^\pm := B_1 \cap \{\pm(x_d - \Psi(x')) > 0\} \\ u^+_\nu - u^-_\nu = g & \text{on } \Gamma := B_1 \cap \{x_d = \Psi(x')\}, \end{cases} \tag{15}$$

where $0 \leqslant a < 1$. The interior regularity theory of the problem (15) without a transmission condition, that is $g = 0$, was studied by the second author and Sire in [29].

The problem (15) is motivated by a class of equations that appear naturally in the analysis of singular manifolds, particularly those with conic or conic-edge singularities (see, for instance, [21, 22]). In these settings, one encounters elliptic and parabolic equations where the coefficients degenerate as they approach a submanifold. A key feature is that the degeneracy exponent is explicitly linked to the sharpness of the edge angle, which itself can be variable. A significant challenge arises when one attempts to study these problems, without a transmission condition, using C viscosity solutions. We observe that the uniqueness of solutions fails drastically. This failure is due to the hypersurface, where the ellipticity degenerates, creating a natural interface that effectively disconnects the domain into two components. The strategy developed in [29] to circumvent this issue is to consider L^p viscosity solutions that do not see sets of zero measure. Our goal is to restore uniqueness within the context of C viscosity solutions by imposing a transmission condition across this interface.

The results in [25] generalize those of [41] in multiple ways. We analyze operators with a variable degeneracy exponent at the interface, establishing sharp pointwise regularity that depends on the exponent's pointwise value. We also present a more direct proof for the $C^{1,\alpha}$ regularity.

In the following the constant $\alpha_0(\lambda, \Lambda, d)$ always refers to the universal exponent corresponding to the interior $C^{1,\alpha_0}(B_{1/2})$ regularity of solutions to the problem $F(D^2u) = 0$ in B_1 where F is any (λ, Λ) uniformly elliptic operator, see [8, Corollary 5.7]. Henceforth $\omega(x)$ denotes the function $\omega(x) = |x_d - \Psi(x')|^{a(x)}$. With $f^\pm \in L^p_\omega(\Omega^\pm)$ we mean that $f^\pm \omega^{-1} \in L^p(\Omega^\pm)$. We recall the notion of Hölder spaces with variable exponent. We say that $u \in C^{1,\alpha(\cdot)}(\Omega)$ if $u \in C^1(\Omega)$ and

$$[u]_{C^{1,\alpha(\cdot)}(\Omega)} := \sup_{x \neq y \in \Omega} \frac{|\nabla u(x) - \nabla u(y)|}{|x-y|^{\alpha(x)}} < \infty. \tag{16}$$

We define the norm $\|u\|_{C^{1,\alpha(\cdot)}(\Omega)} = \|u\|_{L^\infty(\Omega)} + \|\nabla u\|_{L^\infty(\Omega)} + [u]_{C^{1,\alpha(\cdot)}(\Omega)}$.

The main result of [25] is the following.

Theorem 3.2. *Let $u \in C(B_1)$ be a viscosity solution to the transmission problem* (15) *where $F^\pm$ are (λ, Λ)-elliptic operators which satisfy* (14). *Assume that $f^\pm \in C(\Omega^\pm) \cap L^\infty(\Omega^\pm) \cap L^p_\omega(\Omega^\pm)$ with $p = d(1+1/\bar{a})/2 > d$, $g \in C^{\alpha_0}_c(\Gamma)$, $\Psi \in C^{1,\alpha_0}(\overline{B_1'})$, and the function a has modulus of continuity satisfying*

$$\limsup_{t\to 0} \ln\left(\frac{1}{t}\right)\gamma_a(t) = 0,$$

and $\bar{a} := \max_{B_1} a(x) < 1$. Then, $u \in C^{1,\alpha(x_0)}(x_0)$ with $\alpha(x_0) = \min\{\alpha_0^-, 1 - a(x_0)\}$. Moreover, it is endowed with the local estimate up to the interface

$$\|u\|_{C^{1,\alpha(\cdot)}\left(\overline{\Omega}^\pm_{1/2}\right)} \leqslant C\left(\|u\|_{L^\infty(B_1)} + \|g\|_{C^{\alpha_0}(\Gamma)} + \|f^+\|_{L^p_\omega(\Omega^+)} + \|f^-\|_{L^p_\omega(\Omega^-)}\right) \tag{17}$$

where $\Omega^\pm_{1/2} := \Omega^\pm \cap B_{1/2}$ and C is a universal constant depending only on $d, \lambda, \Lambda, \bar{a}$ α_0, and $\|\Psi\|_{C^{1,\alpha_0}(\overline{B_1'})}$, and α has the same modulus of continuity of a.

In particular, since the estimates are uniform with respect to the point, as a direct consequence of our theorem, we additionally get a uniform local regularity result, which is $u \in C^{1,\alpha}(\overline{\Omega}^\pm_{1/2})$ for $\alpha = \min\{\alpha_0^-, 1-\bar{a}\}$. Note also that a simple consequence of this result is that solutions of (15) are Lipschitz continuous across the interface. This is the best we can expect for this problem with $g \not\equiv 0$, since it prescribes a jump of the gradient.

The expression $\alpha = \min\{\alpha_0^-, 1-a\}$ should be understood in the following sense: if $\alpha_0 > 1-a$, then solutions are $C^{1,\alpha}$ for $\alpha = 1-a$. On the other hand, if $\alpha_0 \leqslant 1-a$, then solutions are $C^{1,\alpha}$ for every $\alpha < \alpha_0$.

Remark 3.1. *The regularity given by Theorem 3.2 is optimal. In fact, even in the simple case when $F^\pm(M) = \mathrm{tr}(M)$, one can construct a solution of* (15) *which is no more than $C^{1,\alpha}$ up to the interface. Indeed, by taking $\Psi \equiv 0$, $g \equiv 1$ and $a(x) \equiv a \in (0,1)$ so that $a \geqslant 1-\alpha$, the function $u(x) = |x_d|^{1+\alpha} + \frac{1}{2}|x_d|$ is a solution of the flat transmission problem*

$$\begin{cases} |x_d|^a \Delta u = f, & \text{in } B_1^\pm := B_1 \cap \{\pm x_d > 0\} \\ u_\nu^+ - u_\nu^- = 1 & \text{on } B_1' := B_1 \cap \{x_d = 0\} \end{cases} \tag{18}$$

with $f(x) = C_{d,\alpha}|x_d|^{a+\alpha-1}$ where $a + \alpha - 1 \geqslant 0$.

3.3.1. *Strategy of the proof of Theorem 3.2.* The strategy developed to prove Theorem 3.2 combines the arguments in [29] about fully nonlinear equations with degeneracy of distance-type, with the treatment of fully nonlinear transmission problems due to [41], together with some ideas from [27] to obtain sharper results. The presence of a degeneracy in the interface poses new difficulties and requires new arguments that will considerably differ from [41], particularly when constructing barriers.

The approach is to first establish the ABP, the Harnack inequality, the Hölder regularity of solutions of (15), and then, in the spirit of [41], obtain an approximation lemma which relates our equation with the limiting profile where both $f^{\pm}$ and g are zero. Once these tools are available, the main idea to prove Theorem 3.2 is to adapt the argument developed by Caffarelli in [7] which consists of importing the improved regularity for the limiting profile (corresponding to the case $g = 0$ and $f^{\pm} = 0$) to our equation. This is done via geometric iterations, where we subsequently approximate our solution by affine functions and rescale, zooming into a point in the interface.

To get a two-sided improvement up to the interface, we need to consider different affine functions from each side. By writing explicitly how these affine functions depend on the transmission condition g, we are able to maintain the smallness assumption on $\|g\|_{C^{\alpha_0}}$ which considerably simplifies the proof, since the case when $g \approx 0$ is much simpler than the case where g is large. Furthermore, the proper rescaling is dependent on the degeneracy, which has variable exponent. Thus to obtain the sharp regularity, a careful application of the argument developed in [27] has to be performed, by considering a different rescaling power in each iteration. Once the pointwise regularity at the interface is obtained, this is patched with the classical interior regularity in the usual way.

4. Parabolic transmission problems

Another interesting problem is the parabolic counterpart

$$
\begin{cases}
\partial_t u - F^{\pm}(D^2 u) = f^{\pm} & \text{in } \Omega^{\pm} := \mathcal{C}_1 \cap \{\pm(x_d - \Psi(x', t)) > 0\} \\
u^+_\nu - u^-_\nu = g & \text{on } \Gamma := \mathcal{C}_1 \cap \{x_d = \Psi(x', t)\},
\end{cases}
\tag{19}
$$

where $\mathcal{C}_1 := B_1 \times (-1, 0]$ and ν denotes the spatial unit normal vector to Γ pointing towards Ω^+. The literature in this topic is still rather limited and consists mostly of papers studying free boundary problems which study a particular fixed transmission problem as useful tool, see [19, 31]. In fact, as in the elliptic case, fixed transmission problems are closely related to free boundary problems.

A particularly important parabolic free boundary problem is the well-known two-phase Stefan problem, studied by Athanasopoulos, Caffarelli and Salsa in the celebrated papers [2, 3, 4] is given by

$$(20)\qquad \begin{cases} a_1\partial_t u - \Delta u = 0 & \text{in } \Omega_T^+(u) := (\Omega \times [0,T]) \cap \{u > 0\}, \\ a_2\partial_t u - \Delta u = 0 & \text{in } \Omega_T^-(u) := (\Omega \times [0,T]) \cap \{u \leqslant 0\}^0, \\ \dfrac{\partial_t u^+}{|\nabla u^+|} = \left|\nabla u^+\right| - \left|\nabla u^-\right| = -\dfrac{\partial_t u^-}{|\nabla u^-|} & \text{on } F(u) := (\Omega \times [0,T]) \cap \partial\{u > 0\}, \end{cases}$$

where $a_1, a_2 > 0$ denote the heat capacities of the two states. In these papers, they develop a viscosity approach to study properties of the solutions to (20) and their free boundaries. They use monotonicity formulas, in the style of Alt-Caffarelli-Friedman, blow-up analysis and compactness arguments, interior and boundary Harnack inequalities, and perturbation techniques through continuous families of supersolutions.

More recently in [19], De Silva, Forcillo and Savin develop a new strategy to study the regularity of flat free boundaries to the one-phase Stefan problem. This strategy aligns closely to the ideas developed in the elliptic setting by De Silva and her collaborators. The method introduced in [19] has been recently employed to deal also with the inhomogeneous one-phase Stefan problem, see [23, 24]. Therefore, using a similar (but considerably more complicated) argument as the one described in Section 3.1 applied to the two-phase equation (20), one is led to study a parabolic fixed transmission problem.

An interesting difficulty arises when trying to obtain higher regularity for the free boundary of solutions to (20). The usual strategy is to use the Hodograph transform to bootstrap the regularity. However, this argument is reliant on $C^{2,\alpha}$ estimates up to the interface for a fixed interface problem with different operators. As we mentioned in Section 3.2, this problem is open even in the elliptic case.

4.1. **Parabolic fixed interface.** Recently, in [28], the second author together with Soria-Carro extended to the parabolic case part of the results in [41]. The main result obtained is the following.

Theorem 4.1 ($C^{1,\alpha}$ regularity)**.** *Let $0 < \alpha < \alpha_0$. Assume that $\Psi \in C^{1,\alpha}(\overline{\mathcal{C}_1'})$, $\Psi \not\equiv 0$, $g \in C^{0,\alpha}(\Gamma)$, and $f^\pm$ satisfy that there are constants $C_{f^\pm} > 0$ such that, for any $(x_0, t_0) \in \mathcal{C}_1$,*

$$\left(\fint_{\mathcal{C}_r(x_0,t_0)\cap\Omega^\pm} |f^\pm|^{d+1}\, dxdt \right)^{\frac{1}{d+1}} \leqslant C_{f^\pm} r^{\alpha-1}, \quad \text{for all } r > 0 \text{ small.}$$

If u is a bounded viscosity solution to (19), then $u \in C^{1,\alpha}(\overline{\Omega}^\pm_{1/2})$, with

$$\|u^\pm\|_{C^{1,\alpha}(\overline{\Omega}^\pm_{1/2})} \leqslant C\Big(\|u\|_{L^\infty(\mathcal{C}_1)} + \|g\|_{C^{0,\alpha}(\Gamma)} + C_{f^-} + C_{f^+}\Big),$$

where $C > 0$ depends only on $\|\Psi\|_{C^{1,\alpha}(\overline{\mathcal{C}_1'})}$, d, λ, Λ and α.

In addition to extending the work in [41] to the parabolic setting, they also refine and improve upon several aspects of the original work. First, they remove the assumption of proximity between the operators $F^\pm$, i.e.,

$$\sup_{M\in S(d)\setminus\{0\}} \frac{|F^+(M) - F^-(M)|}{|M|} \leqslant \theta \ll 1,$$

which was required in the case $g \approx 0$ (see [41, Lemma 5.2]). They show that the stability argument presented in [41, Lemma 5.6] for $g \approx 1$ remains valid when $g \approx 0$, thereby eliminating the need for the proximity condition. Second, they provide a simplified proof of the stability argument by introducing a more direct and streamlined approach, avoiding the *closedness* result (see [41, Lemma 5.1]).

One of the main tools developed in [29] is a novel ABP-Krylov-Tso estimate for super-solutions to (19). Its proof differs from the elliptic case mainly because of the different character of the parabolic convex envelope. Furthermore, a new Hopf lemma for domains satisfying an interior $C^{1,\mathrm{Dini}}$ condition is obtained, which might be of independent interest.

Another essential tool is the Harnack inequality, which presents additional challenges in this setting. In the elliptic case, a standard strategy involves applying the interior Harnack inequality in a small ball contained on one side of the interface, and constructing a barrier that satisfies the transmission condition. By comparing this barrier with the solution, one can transfer the information from the initial ball across the interface. In the parabolic case, a similar approach can be employed. However, because of the waiting time in the parabolic Harnack inequality, the construction of a suitable barrier in this geometry becomes more difficult.

While the transmission condition in (19) is purely spatial, applications like the Stefan problem (20) naturally involve conditions depending on the time derivative as well. This more complex case remains an open area, as regularity results are, to our knowledge, not yet established.

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