ON THE PROPAGATION OF ISOTROPIC SOBOLEV SINGULARITIES

SULLA PROPAGAZIONE DI SINGOLARITÀ SOBOLEV ISOTROPICHE

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ABSTRACT. The aim of these notes is to examine the propagation of isotropic Sobolev (micro)singularities for an evolution problem whose generator is given by the Weyl quantization of a complex-valued quadratic form on phase space.

SUNTO. L'obiettivo di queste note è quello di esaminare la propagazione di (micro)singolarità Sobolev isotropiche per un problema evolutivo, il cui generatore è dato dalla quantizzazione Weyl di una forma quadratica complessa nello spazio delle fasi.

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1. INTRODUCTION

The goal of these notes is to discuss the results on propagation of isotropic singularities obtained in [12] for an evolution equation of the form

$$\begin{cases} \partial_t u + Au = 0 \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^n, \\ u(0, \cdot) = u_0 \in \mathcal{S}'(\mathbb{R}^n), \end{cases} \tag{1}$$

with $A = \mathrm{Op}^{\mathrm{w}}(a) \in \Psi^2_{\mathrm{iso}}(\mathbb{R}^n)$ the Weyl-quantization of a complex quadratic form on the phase space

$$a : \mathbb{R}^{2n} \longrightarrow \mathbb{C}, \quad (x, \xi) = X \longmapsto a(X) = \langle X, QX \rangle,$$

defined by the symmetric complex matrix $Q \in \mathsf{M}_{2n}(\mathbb{C})$ with $\operatorname{Re} Q \geq 0$.

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The term isotropic is used here because the position variables x and the frequency variables ξ are treated symmetrically. The isotropic theory is applied to study problem (1) due to the isotropic nature of the quadratic form $a(X)$, which enables the analysis of a broader class of meaningful problems. Indeed, our theory encompasses every evolution problem on the whole of $\mathbb{R}^n$ whose generator is a Weyl-quantization of a (complex) quadratic form with positive real part. For instance, with $a(X) = |\xi|^2$ we obtain the Cauchy problem for the Heat equation, with $a(X) = i|\xi|^2$, the Cauchy problem for the (free) Schrödinger equation and finally, with $a(X) = i(|x|^2 + |\xi|^2)$, we obtain the Cauchy problem for the Schrödinger equation of the (purely imaginary) quantum Harmonic oscillator.

We also studied the particular case of more general operators tailored to a complex combination of *quantum Harmonic Oscillators*, such as

$$A = -\Delta_{x'} + |x'|^2 + i(-\Delta_{x''} + |x''|^2),$$

where $x' \in \mathbb{R}^{n_1}$ and $x'' \in \mathbb{R}^{n_2}$, with $n_1 + n_2 = n$ (note that if $n_2 = n$ we recover the standard Harmonic Oscillator). This operator has a symbol that falls within the scope of our study and, as we expect, since the Heat equation has a regularizing behavior, we have that the singularities of the solution of the problem at time $t > 0$ live in the second group of variables (in other words, there is a heat diffusion with respect to some variables, and a quantum diffusion in some others.)

In order to investigate the isotropic microsingularities, we studied the "stratification" of WF_{iso} given by the s-wave-front sets WF^s_{iso}, that are a measure of microlocal B^s-regularity, where B^s is the Shubin-Sobolev space on $\mathbb{R}^n$ (see [6], [18]). The sets WF^s_{iso} allow for a more precise information about the regularity of the distribution that we are dealing with. Indeed, one of the reasons to study such stratification arises from the fact that once the results are achieved in the stratified setting, it is possible to reconstruct the corresponding properties in the "Schwartz regularity" setting.

Our purpose is to examine the propagation of singularities for (1), that is, we aim to relate the isotropic microsingularities of the solution to those of the initial datum $u_0 \in \mathcal{S}'(\mathbb{R}^n)$.

2. History of the problem

The propagation of singularities for problem (1) has already been studied from several perspectives. To properly discuss these approaches, we begin by mentioning that they were made possible thanks to two fundamental works by Hörmander: [10], concerning symplectic classification of quadratic forms, and [9], concerning the isotropic microlocal calculus. Indeed, by such works we are able to give the first rough inclusion

$$WF_{\mathrm{iso}}(e^{-tA}u_0) \subseteq e^{2t\mathrm{Im}\,F} WF_{\mathrm{iso}}(u_0), \quad t>0, \tag{2}$$

where F is the complex matrix that satisfies the equation (with σ the standard complex symplectic form)

$$\sigma(X, FX) = a(X), \quad X \in \mathbb{R}^{2n}, \tag{3}$$

that is called the *Hamilton map* associated with a. The Hamilton map is the linearized version near a critical point of the Hamilton vector field (see [11], Section 21.5), and if a is a *real-valued* quadratic form we have that its Hamilton vector field H_a satisfies

$$H_a(x,\xi) = \begin{pmatrix} \partial_{\xi_1} a(x,\xi) \\ \vdots \\ \partial_{\xi_n} a(x,\xi) \\ -\partial_{x_1} a(x,\xi) \\ \vdots \\ -\partial_{x_n} a(x,\xi) \end{pmatrix} = 2F \begin{pmatrix} x \\ \xi \end{pmatrix}, \quad (x,\xi) \in \mathbb{R}^{2n}. \tag{4}$$

The key step in refining the inclusion (2) was the introduction of the Singular space by M. Hitrik and K. Pravda-Starov in [7]. The *Singular space* associated with a quadratic form $a = a(X)$ is defined by (with F the Hamilton map associated with a as in (3))

$$S = \Big(\bigcap_{j=1}^{+\infty} \mathrm{Ker}\,(\mathrm{Re}\, F (\mathrm{Im}\, F)^j) \Big) \cap \mathbb{R}^{2n}.$$

Equivalently, this space can be characterized as the subspace of the phase space on which all Poisson brackets $H^k_{\mathrm{Im}\,a} \mathrm{Re}\, a$, $k \in \mathbb{N}$ are vanishing, that is (see [8])

$$S = \Big\{ X \in \mathbb{R}^{2n};\ \ (H^k_{\mathrm{Im}\,a}\mathrm{Re}\, a)(X) = 0,\ \forall k \in \mathbb{N} \Big\}.$$

Intuitively S is precisely the space in which the singularities are propagated. More precisely, by exploiting the geometry of the singular space S and its properties, such as

$$(5) \qquad S = \bigcap_{j=1}^{2n-1} \operatorname{Ker}(\operatorname{Re} F(\operatorname{Im} F)^j) \cap \mathbb{R}^{2n} = \Big(\bigcap_{0 \le s \le t} \operatorname{Ker}(\operatorname{Im} e^{-2isF}) \Big) \cap \mathbb{R}^{2n}, \quad \forall t > 0,$$

and

$$(6) \qquad (\operatorname{Re} F)S = \{0\}, \quad (\operatorname{Im} F)S \subseteq S,$$

the authors of [16] proved the following theorem.

Theorem 2.1 (Theorem 6.2 in [16]). *Let $u_0 \in \mathcal{S}'(\mathbb{R}^n)$ and let $a(X) = \langle X, QX \rangle$ be a complex quadratic form with $Q \in \mathsf{M}_{2n}(\mathbb{C})$ symmetric matrix such that $\operatorname{Re} Q \ge 0$. Then for all given $t > 0$ we have*

$$(7) \qquad WF_{\mathrm{iso}}(e^{-t\mathrm{Op}^{\mathrm{w}}(a)} u_0) \subseteq (e^{tH_{\mathrm{Im} a}}(WF_{\mathrm{iso}}(u_0) \cap S)) \cap S,$$

where S is the singular space associated with a.

This theorem essentially says that the Schwartz singularities of the solution at time t are localized inside the singular space and propagate along the bicharacteristic curves of the imaginary part $\operatorname{Im} a$ of the symbol. In the case of WF^s_{iso}, i.e. for studying the *Shubin-Sobolev* singularities, the inclusion (7) was already investigated using time-frequency tools in [20], resulting in a different loss of derivatives. Motivated by this, we began investigating the relationship between the calculus we introduced and the time-frequency calculus originally developed in [5] and further expanded by numerous authors (see, for example, [13], [17], [19]; see also [3] for the SG-framework). In [12] we showed that the tools we introduced correspond to a complementary viewpoint which ultimately provides essentially equivalent information. However, this alternative approach is more pseudodifferential and flexible, making it better suited in many cases to the geometry of phase space. In recent years, the inclusion has also been studied for an initial condition in the Gelfand-Shilov class and in the anisotropic context (see [2], [1], [21], [22]).

In [12] we studied the inclusion (7) from our complementary point of view, by using the isotropic microlocal theory. We developed the basic properties of the isotropic calculus introduced in [9], aiming to study operations between wave front sets in the stratified setting, which lead to losses of regularity. In particular, in order to analyze the singularities of a tensor product of two tempered distributions we had to keep into account the global regularity of both of them and this produces a first loss of derivatives (the s_* in Theorem 2.2) due to the interaction of the single tempered distributions u and v in the tensor product $u\otimes v \in \mathcal{S}'(\mathbb{R}^n_x \times \mathbb{R}^m_y)$. Moreover, when we considered the "pullback" of a temperate distribution - that satisfies suitable conditions - through an injective map from $\mathbb{R}^m$ to $\mathbb{R}^n$ we encountered an additional loss of derivatives, that depends on the codimension $n-m$, since it basically corresponds to taking the trace of the distribution. In fact, we reduced matters to the case in which the pullback is done via an immersion $L : \mathbb{R}^m \to \mathbb{R}^m \times \{0_{n-m}\}$, and after making sense of the notion of the pullback via an injective map of distributions in a certain class, we show that on sufficiently regular distributions the pullback is the trace on $\{x_{m+1} = \cdots = x_n = 0\}$ and the Trace Theorem leads to a loss of regularity of $(n-m)/2$.

We next describe our main result on the Schwartz kernels that plays a crucial role in our work. Let $\mathcal{K} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^m)$, with $n \geq m$, be a linear operator with Schwartz kernel $K \in \mathcal{S}'(\mathbb{R}^{n+m})$. Then, if $K \in \mathcal{S}(\mathbb{R}^{n+m})$ and $u \in \mathcal{S}(\mathbb{R}^n)$, we may write

$$\mathcal{K}u(x) = \int L^*(K \otimes u)(x,y)dy,$$

where $K \otimes u \in \mathcal{S}(\mathbb{R}^{m+n+n})$ is the function

$$(x,y,z) \mapsto K(x,y)u(z),$$

and L is the map $(x,y) \mapsto (x,y,y)$. Therefore, after extending this expression to those $K \in \mathcal{S}'(\mathbb{R}^{m+n})$ and $u \in \mathcal{S}'(\mathbb{R}^n)$ that satisfy condition (9) below we put together the previous results into the following theorem.

Theorem 2.2. *Let $\mathcal{K} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^m)$, with $n \geq m$, be a linear operator with Schwartz kernel $K \in B^{-r_1}(\mathbb{R}^{m+n})$ for $r_1 \geq 0$. Then, if $u \in \mathcal{S}(\mathbb{R}^n)$ and $\mu > n$ is fixed, we have that for all $s_1 \in \mathbb{R}$*

$$WF^{s_1-\mu}_{\mathrm{iso}}(\mathcal{K}u) \subseteq WF^{s_1}_{\mathrm{iso},X}(K). \tag{8}$$

In addition, the definition of the map $u \mapsto \mathcal{K}u$ *can be extended by continuity to those* $u \in \mathcal{S}'(\mathbb{R}^n)$ *for which*

$$WF_{\rm iso}(u) \cap WF_{{\rm iso},Y}(K) = \emptyset. \tag{9}$$

Finally, if $u \in B^{-r_2}(\mathbb{R}^n)$*, for some* $r_2 \geq 0$*, satisfies* (9) *and* $s_1, s_2 \in \mathbb{R}$ *are such that* $s_* := \min\{s_1 - r_2, s_2 - r_1\} \leq s_1 + s_2$*, then for any fixed* $\mu > n$ *we have*

$$WF_{\rm iso}^{s_*-\mu}(\mathcal{K}u) \subseteq WF_{{\rm iso},X}^{s_1}(K) \cup WF_{\rm iso}^{s_1}(K)' \circ WF_{\rm iso}^{s_2}(u). \tag{10}$$

By using the geometric structure of the kernel $K_{e^{-2itF}}$ of the propagator $\mathcal{K}_{e^{-2itF}} = e^{-tA}$ (see [10]) - and of the associated singular space S - we will specialize Theorem 2.2 to improve the inclusion (7) in the stratified setting giving in addition a lower bound on the loss of derivatives of $4n + \varepsilon$, with ε as small as we wish. This will be our main result concerning the propagation of singularities.

3. Basics of isotropic calculus and isotropic microlocal analysis

Notation. We work on $T^*\mathbb{R}^n \cong \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$. Hence, we write $X = (x, \xi) \in \mathbb{R}^{2n}$ for the points in the phase-space and we write $\dot{\mathbb{R}}^{2n} = \mathbb{R}^{2n} \setminus \{0\}$. Recall also that $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$, $D_{x_j} = -i\partial_{x_j}$ and $(\cdot, \cdot)_0$ is the L^2-product on $\mathbb{R}^n$. Furthermore we denote by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, by $\{0_n\}$ the origin on $\mathbb{R}^n$ and, finally, by $\langle \cdot, \cdot \rangle$ the *non-Hermitian* product on $\mathbb{C}^{2n}$ defined by $\langle X, Y \rangle := \sum_i X_i Y_i$ for $X, Y \in \mathbb{C}^{2n}$. We say that a set $\Gamma \subseteq \dot{\mathbb{R}}^{2n}$ is *conic* if it is invariant under multiplication by positive real numbers. Hence a cone (i.e. a conic set) $\Gamma \subseteq \dot{\mathbb{R}}^{2n}$ is uniquely determined by its intersection with the unit sphere $\mathbb{S}^{2n-1}$. Therefore Γ is open, resp. closed, in $\dot{\mathbb{R}}^{2n}$ if Γ is relatively open, resp. closed, in $\dot{\mathbb{R}}^{2n}$ (equivalently, $\Gamma \cap \mathbb{S}^{2n-1}$ is open, resp. closed, in $\mathbb{S}^{2n-1}$). We write $A \approx B$, for $A, B > 0$, if there exists $C > 0$ such that $C^{-1}A \leq B \leq CA$. If $U \subseteq \mathbb{R}^{4n}$, $V \subseteq \mathbb{R}^{2n}$, we write $U \circ V = \{x \in \mathbb{R}^{2n};\ \exists\, y \in V \text{ s.t. } (x, y) \in U\} \subseteq \mathbb{R}^{2n}$.

In this section we recall the basics of the isotropic pseudodifferential calculus introduced by Shubin (see [18]), that is the calculus with respect to the *isotropic* Hörmander metric (see [11], Chapter 18.5)

$$g_{0,X} = \frac{|dX|^2}{\langle X\rangle^2}, \quad X = (x,\xi) \in \mathbb{R}^{2n}.$$

To this end, we begin by recalling the notion of isotropic symbol.

Definition 3.1. *Let $a \in C^\infty(\mathbb{R}^{2n})$ and $m \in \mathbb{R}$. We say that a is an isotropic (or Shubin) symbol of order m, and we write $a \in S^m_{\mathrm{iso}}(\mathbb{R}^n)$, if for all $\alpha \in \mathbb{N}_0^{2n}$ there exists $C_\alpha > 0$ such that*

$$|\partial_X^\alpha a(X)| \le C_\alpha \langle X\rangle^{m-|\alpha|}, \quad X \in \mathbb{R}^{2n}. \tag{11}$$

Shubin symbols of order m form a Fréchet space where, if $a \in S^m_{\mathrm{iso}}(\mathbb{R}^n)$, the semi-norms are given by

$$|a|_k^{(m)} := \max_{|\alpha|\le k} \sup_{X\in\mathbb{R}^{2n}} |\partial_X^\alpha a(X)| \langle X\rangle^{-(m-|\alpha|)}, \quad k \in \mathbb{N}_0. \tag{12}$$

We denote

$$S^\infty_{\mathrm{iso}}(\mathbb{R}^n) = \bigcup_{m\in\mathbb{R}} S^m_{\mathrm{iso}}(\mathbb{R}^n).$$

Note also that

$$\bigcap_{m\in\mathbb{R}} S^m_{\mathrm{iso}}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^{2n}).$$

If $a \in S^m_{\mathrm{iso}}(\mathbb{R}^n)$, we can associate to it a (Weyl-quantized) *pseudodifferential operator* as follows.

Definition 3.2. *Let $a \in S^m_{\mathrm{iso}}(\mathbb{R}^n)$. The Weyl-quantized pseudodifferential operator associated with a is defined as*

$$\mathrm{Op}^{\mathrm{w}}(a)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} a((x+y)/2,\xi) u(y)\, dy d\xi, \quad u \in \mathcal{S}(\mathbb{R}^n).$$

The set of pseudodifferential operators of order m is denoted by

$$\Psi^m_{\mathrm{iso}}(\mathbb{R}^n) := \Big\{ A : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n);\ \exists\, a \in S^m_{\mathrm{iso}}(\mathbb{R}^n),\ A = \mathrm{Op}^{\mathrm{w}}(a) \Big\}.$$

It is easy to verify that $\mathrm{Op}^{\mathrm{w}}(a)$ is a linear operator that acts continuously on $\mathcal{S}(\mathbb{R}^n)$ and extends by duality to a linear continuous operator on $\mathcal{S}'(\mathbb{R}^n)$. The reason for working with Weyl-quantized pseudodifferential operators, rather than the Kohn-Nirenberg quantization (see [11], Chapter 18.1), is that the former satisfy useful properties when dealing with PDEs. To begin with, let $A = \mathrm{Op}^{\mathrm{w}}(a) \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$. Then, its adjoint A^* satisfies

$$A^* = \mathrm{Op}^{\mathrm{w}}(\overline{a}) \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n),$$

which implies the self-adjoint property when a is a real-valued symbol.

Moreover, if χ is a linear symplectic map in $\mathbb{R}^{2n}$, there exists a unitary operator U_χ from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ determined up to a constant factor (cf. [11], Theorem 18.5.9) that satisfies

$$U_\chi^{-1}\mathrm{Op}^{\mathrm{w}}(a)U_\chi = \mathrm{Op}^{\mathrm{w}}(a \circ \chi). \tag{13}$$

Finally, we have the following composition law for Weyl-quantized pseudodifferential operator.

Theorem 3.1. *Let $a \in S^{m_1}_{\mathrm{iso}}(\mathbb{R}^n)$ and $b \in S^{m_2}_{\mathrm{iso}}(\mathbb{R}^n)$. Then the composition $\mathrm{Op}^{\mathrm{w}}(a)\mathrm{Op}^{\mathrm{w}}(b)$ is well defined and*

$$\mathrm{Op}^{\mathrm{w}}(a)\mathrm{Op}^{\mathrm{w}}(b) = \mathrm{Op}^{\mathrm{w}}(a\sharp b),$$

where the symbol

$$(a\sharp b)(X) = \Big(e^{i\sigma(D_X;D_Y)/2}a(X)b(Y)\Big)\Big|_{Y=X} \in S^{m_1+m_2}_{\mathrm{iso}}(\mathbb{R}^n)$$

is called the Weyl-product and the map

$$\sharp : S^{m_1}_{\mathrm{iso}}(\mathbb{R}^n) \times S^{m_2}_{\mathrm{iso}}(\mathbb{R}^n) \longrightarrow S^{m_1+m_2}_{\mathrm{iso}}(\mathbb{R}^n),$$

is continuous. Moreover,

$$(a\sharp b)(X) \sim \sum_{j=0}^{+\infty} \frac{(i/2)^j}{j!}\Big(\sigma(D_X;D_Y)\Big)^j a(X)b(Y)\Big|_{Y=X},$$

where for $X = (x,\xi), Y = (y,\eta) \in \mathbb{R}^{2n}$,

$$\sigma(D_X;D_Y) := \sigma(D_x, D_\xi; D_y, D_\eta) = \langle D_\xi, D_y\rangle - \langle D_x, D_\eta\rangle.$$

The next goal of this section is to recall the basics of isotropic microlocal analysis as studied in [12]. For this purpose we first recall the notion of characteristic and elliptic part of an isotropic pseudodifferential operator.

Definition 3.3. *The operator $A = \mathrm{Op}^{\mathrm{w}}(a) \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$ is said to be isotropic elliptic at $X_0 \in \dot{\mathbb{R}}^{2n}$ if there exist an open cone $\Gamma_{X_0} \subseteq \dot{\mathbb{R}}^{2n}$ that contains X_0 and some constants $c, C > 0$, such that*

$$|a(X)| \geq c|X|^m, \quad \forall X \in \Gamma_{X_0},\ |X| \geq C.$$

If A is isotropic elliptic at X_0, for every $X_0 \in \dot{\mathbb{R}}^{2n}$ it is said to be isotropic elliptic. If A is not isotropic elliptic at X_0, it is said to be isotropic characteristic at X_0.

Definition 3.4. *We define the set of the isotropic elliptic points for A as the set*

$$\mathrm{Ell}_{\mathrm{iso}}(A) = \{X_0 \in \dot{\mathbb{R}}^{2n};\ A \text{ is elliptic at } X_0\},$$

and the set of isotropic characteristic points as the set

$$\mathrm{Char}_{\mathrm{iso}}(A) = \dot{\mathbb{R}}^{2n} \setminus \mathrm{Ell}_{\mathrm{iso}}(A).$$

With $A = \mathrm{Op}^{\mathrm{w}}(a)$, we say that a is non-characteristic at $X_0 \in \mathbb{R}^{2n}$ if $X_0 \notin \mathrm{Char}_{\mathrm{iso}}(A)$.

To study the propagation of singularities, another fundamental concept is the operator wave front set, which effectively encodes the microlocal singularities of the operator.

Definition 3.5. *Let $a \in S^m_{\mathrm{iso}}(\mathbb{R}^n)$. We say that $X_0 \notin \mathrm{essconesupp}_{\mathrm{iso}}\, a \subseteq \dot{\mathbb{R}}^{2n}$, if there exists a conic neighborhood Γ_{X_0} of X_0 such that, for all $\alpha \in \mathbb{N}_0^{2n}$, $N \in \mathbb{N}$ there exists $C_{\alpha,N} > 0$ such that*

$$|\partial_X^\alpha a(X)| \leq C_{\alpha,N} \langle X \rangle^{-N}, \quad \forall X \in \Gamma_{X_0}, \quad |X| \geq 1. \tag{14}$$

The set $\mathrm{essconesupp}_{\mathrm{iso}}\, a$ is called the isotropic essential conic support of a. If $A = \mathrm{Op}^{\mathrm{w}}(a) \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$, the isotropic wave front set of A is defined as

$$WF'(A) := \mathrm{essconesupp}_{\mathrm{iso}}\, a \subseteq \dot{\mathbb{R}}^{2n}.$$

Now, since our goal is to study the microlocal Shubin regularity of a distribution, we must recall here the concept of Shubin-Sobolev space (cf. [6], [14], [15]). First of all, for $s \in \mathbb{R}$, we define $\Lambda^s := \mathrm{Op}^{\mathrm{w}}(\langle X \rangle^s)$. Since Λ^s is globally elliptic, we may find $E_s \in \Psi^{-s}_{\mathrm{iso}}(\mathbb{R}^n)$ and $R \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ such that $E_s\Lambda^s = I + R$. Hence, we define the Shubin-Sobolev spaces as follows.

Definition 3.6. *Let $s \in \mathbb{R}$ and let $p \in \mathbb{N}$, $p \geq s$. We define the Shubin-Sobolev space of order $s \in \mathbb{R}$ as*

$$B^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n);\ \Lambda^s u \in L^2(\mathbb{R}^n)\},$$

endowed with the inner product

$$(u,v)_{B^s} := (\Lambda^s u, \Lambda^s v)_0 + \sum_{|\alpha|+|\beta|\leq p} (x^\alpha D_x^\beta Ru, x^\alpha D_x^\beta Rv)_0, \quad u,v \in B^s(\mathbb{R}^n),$$

and norm

$$\|u\|_{B^s} = (u,u)_{B^s}^{1/2}, \quad u \in B^s(\mathbb{R}^n).$$

Remark 3.1. *It is possible to prove (see* [6]*) that a different choice of p gives rise to an equivalent norm.*

Remark 3.2. *Let us point out that another possible way to define such a norm is to consider, for $s \in \mathbb{R}$, $\tilde{\Lambda}^s := (1+|x|^2+|D_x|^2)^{s/2}$, where $(1+|x|^2+|D_x|^2)^{s/2}$ denotes the $s/2$-th power of the operator $(1+|x|^2+|D_x|^2)$, defined via the functional calculus (see* [6], [14]*). By* [6] *we have that $\tilde{\Lambda}^s \in \Psi^s_{\mathrm{iso}}(\mathbb{R}^n)$, and it satisfies $\tilde{\Lambda}^s = \mathrm{Op}^{\mathrm{w}}(\ell_s)$, where $\ell_s(x,\xi) = (|x|^2+|\xi|^2)^{s/2} + r_{s-1}(x,\xi)$, with $r_{s-1} \in S^{s-1}_{\mathrm{iso}}(\mathbb{R}^n)$. Moreover,*

$$\tilde{\Lambda}^{-s} = (\tilde{\Lambda}^s)^{-1}, \quad \tilde{\Lambda}^s\tilde{\Lambda}^{s'} = \tilde{\Lambda}^{s+s'}, \quad (\tilde{\Lambda}^s)^* = \tilde{\Lambda}^s,$$

and finally,

$$\|u\|'_{B^s} := (\tilde{\Lambda}^s u, \tilde{\Lambda}^s u)_0 \approx \|u\|_{B^s}, \quad u \in B^s(\mathbb{R}^n).$$

These spaces can be seen as the isotropic version of standard Sobolev spaces and encode both the Sobolev regularity and the decay at infinity. To clarify this concept further, let us point out that, for $k \in \mathbb{N}_0$,

$$B^k(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n); \quad x^\alpha D_x^\beta u \in L^2(\mathbb{R}^n), \quad \forall |\alpha|+|\beta| \leq k\}$$

and

$$\|u\|^2_{B^k} \approx \sum_{|\alpha|+|\beta|\leq k} \|x^\alpha \partial_x^\beta u\|_0^2.$$

Furthermore, we proved the following Sobolev boundedness property in the isotropic context.

Theorem 3.2. *Let $A \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$. Then, for all $s \in \mathbb{R}$, $A : B^s(\mathbb{R}^n) \to B^{s-m}(\mathbb{R}^n)$ is bounded.*

Using those spaces we recall the definition of isotropic wave front set (see [4] and [12]).

Definition 3.7. *Let $u \in \mathcal{S}'(\mathbb{R}^n)$. The Isotropic Wave Front Set of u is defined as*

$$WF_{\mathrm{iso}}(u) = \bigcap_{\substack{A \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n) \\ Au \in \mathcal{S}(\mathbb{R}^n)}} \mathrm{Char}_{\mathrm{iso}}(A)$$

and the Isotropic Wave Front Set of u of order $s \in \mathbb{R}$ is defined as

$$WF^s_{\mathrm{iso}}(u) = \bigcap_{\substack{A \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n) \\ Au \in B^s(\mathbb{R}^n)}} \mathrm{Char}_{\mathrm{iso}}(A).$$

To conclude, let us mention that the so-called microlocality and the microlocal elliptic regularity (see (15) and (16) below, respectively) still hold in the isotropic setting.

Theorem 3.3. *Let $A \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. Then, for all $s \in \mathbb{R}$*

$$WF^{s-m}_{\mathrm{iso}}(Au) \subseteq WF'(A) \cap WF^s_{\mathrm{iso}}(u) \tag{15}$$

and

$$WF^s_{\mathrm{iso}}(u) \subseteq WF^{s-m}_{\mathrm{iso}}(Au) \cup \mathrm{Char}_{\mathrm{iso}}(A). \tag{16}$$

4. Propagation of isotropic singularities

In this section we state the results on propagation of singularities established through the properties of the isotropic microlocal calculus proved in [12]. Recall that we focus on a problem of the form

$$\begin{cases} \partial_t u + \mathrm{Op^w}(a)u = 0 \quad \text{in } \ \mathbb{R}^+ \times \mathbb{R}^n, \\ u(0, \cdot) = u_0 \in \mathcal{S}'(\mathbb{R}^n), \end{cases} \tag{17}$$

where $a(X) = \langle X, QX \rangle$ is a (complex) quadratic form defined by the symmetric matrix $Q \in \mathsf{M}_{2n}(\mathbb{C})$ satisfying $\mathrm{Re}\, Q \geq 0$.

To study such a problem, it is necessary to recall first that by [10] (p. 426) we know that there exists a C^0-contraction semigroup $\{e^{-t\mathrm{Op^w}(a)}\}_{t \geq 0}$ on $L^2(\mathbb{R}^n)$. Moreover for any

$t \geq 0$, by Theorem 5.12 and Proposition 5.8 in [10], we have that $e^{-t\mathrm{Op}^{\mathrm{w}}(a)}$ is a continuous map

$$e^{-t\mathrm{Op}^{\mathrm{w}}(a)} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

and extends to a continuous map

$$e^{-t\mathrm{Op}^{\mathrm{w}}(a)} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n).$$

Then, as anticipated in the Introduction, for $u_0 \in \mathcal{S}'(\mathbb{R}^n)$, the goal is to study the regularity of $u = e^{-t\mathrm{Op}^{\mathrm{w}}(a)}u_0$ that solves problem (17).

Before examining the general case, in order to motivate our study, let us investigate the propagation of singularities for the evolution equation

$$\begin{cases} \partial_t u + \mathrm{Op}^{\mathrm{w}}(a)u = 0, & \text{in } \ \mathbb{R}^+ \times \mathbb{R}^n \\ u(0,\cdot) = u_0 \in \mathcal{S}'(\mathbb{R}^n), \end{cases}$$

where

$$a(X) = a_R(X) + ia_I(X), \quad a_R(X) = |X'|^2, \ \ a_I(X) = |X''|^2, \tag{18}$$

with $X' = (x', \xi')$ for $x' \in \mathbb{R}^{n_1}$ and $\xi' \in \mathbb{R}^{n_1}$, $X'' = (x'', \xi'')$ for $x'' \in \mathbb{R}^{n_2}$ and $\xi'' \in \mathbb{R}^{n_2}$ and $n_1, n_2 \in \mathbb{N}$ satisfy $n_1 + n_2 = n$. Therefore, we have

$$\mathrm{Op}^{\mathrm{w}}(a) = -\Delta_{x'} + |x'|^2 + i(-\Delta_{x''} + |x''|^2),$$

and we may rewrite it as

$$\mathrm{Op}^{\mathrm{w}}(a) = \mathrm{Op}^{\mathrm{w}}(a_R) + i\mathrm{Op}^{\mathrm{w}}(a_I). \tag{19}$$

In [12] we studied how the operator $\mathrm{Op}^{\mathrm{w}}(a)$ propagates singularities. Roughly speaking, we proved that in the first n_1 variables it destroys the singularities (that is, it is regularizing), since in those variables it behaves like a heat evolution operator in n_1 variables. More precisely, we obtained the following theorem.

Theorem 4.1. *Let $u_0 \in \mathcal{S}'(\mathbb{R}^n)$, a be as in* (18) *and let $t > 0$ be fixed. Then*

$$WF_{\mathrm{iso}}(e^{-t\mathrm{Op}^{\mathrm{w}}(a)}u_0) \subseteq \Big(\{0_{n_1}\} \times \mathbb{R}^{n_2}\Big) \times \Big(\{0_{n_1}\} \times \mathbb{R}^{n_2}\Big). \tag{20}$$

One of the purposes of our theory is to refine the inclusion (20) and thereby obtain more precise information about the propagation in the second group of variables (see Example 4.1). To this end, we studied the propagation of singularities in the general case, namely for a quadratic form $a(X)$ with the aforementioned properties. In this context, let us stress that the proof of our main theorem relies on two key ingredients. The first is the symplectic classification of such quadratic forms, as developed in [10]. The second is the microlocal theory of isotropic pseudodifferential operators, introduced in [9] and further expanded in [12]. We are now ready to state the main theorem.

Theorem 4.2. *Let $t > 0$ be fixed and let $e^{-t\mathrm{Op}^{\mathrm{w}}(a)} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be the solution operator of the problem (17). Then, for any given $\varepsilon > 0$ (small), the Schwartz kernel $K_{t'}$ of $e^{-t'\mathrm{Op}^{\mathrm{w}}(a)}$ belongs to $B^{-(n+\varepsilon)}(\mathbb{R}^{2n})$, uniformly in $t' \in (0, t]$.*
Moreover, on defining $r_0 = \inf\{r \geq 0;\ K_{t'} \in B^{-r}(\mathbb{R}^{2n}),\ \forall t' \in (0, t]\}$, we have that if $u_0 \in \mathcal{S}'(\mathbb{R}^n)$ and $\mu > 2(n + r_0)$, then for all $s \in \mathbb{R}$

$$WF^{s-\mu}_{\mathrm{iso}}(e^{-tA}u_0) \subseteq (e^{tH_{\mathrm{Im}a}}(WF^{s}_{\mathrm{iso}}(u_0) \cap S)) \cap S, \tag{21}$$

where S is the singular space associated with the quadratic form $a = a(X)$.

Proof. To begin with, let us mention the key observation on the propagator. By Theorem 5.12 in [10] we have that the propagator of our problem (17) for $t \geq 0$ can be written as

$$e^{-t\mathrm{Op}^{\mathrm{w}}(a)} = \mathcal{K}_{e^{-2itF}},$$

where $\mathcal{K}_{e^{-2itF}} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is the linear operator with Schwartz kernel

$$K_{e^{-2itF}} = (2\pi)^{-(n+N)/2} \sqrt{\det \begin{pmatrix} p''_{\theta\theta}/i & p''_{\theta y} \\ p''_{x\theta} & ip''_{xy} \end{pmatrix}} \int e^{ip(x,y,\theta)} d\theta \in \mathcal{S}'(\mathbb{R}^{2n}). \tag{22}$$

In this kernel the quadratic form p is a form that defines the Lagrangian λ associated with its twisted Lagrangian $\lambda' := \mathrm{graph}(e^{-2itF})$ (see [10], [12] and [16] for more details). This expression shows that the kernel of the propagator belongs to a class of distributions, called Gaussian distributions (see [10]), for which fundamental information about singularities is well understood. In the first place, in [12] we proved the following lemma, concerning the global regularity of the Kernel.

Lemma 4.1. *Let $t > 0$ be fixed. Then for $\varepsilon > 0$ we have*

$$K_{e^{-2itF}} \in B^{-n-\varepsilon}(\mathbb{R}^{2n}).$$

At the same time, as for the microlocal properties, we have the following result.

Lemma 4.2. *Let $a(x,\xi)$ be a quadratic form defined by a symmetric matrix Q with $\operatorname{Re} Q \geq 0$, and F its Hamilton map. Then (with the notation above) for all $t > 0$ we have*

$$WF_{\text{iso}}(K_{e^{-2itF}}) \subseteq \{(x, y, \xi, -\eta) \in \dot{\mathbb{R}}^{4n};\ X = e^{-2itF}Y,\ \operatorname{Im}\ e^{-2itF}Y = 0\}.$$

Next, by using the properties of the isotropic microlocal calculus among with Lemma 4.2 we proved the following fundamental lemma.

Lemma 4.3. *Let $t > 0$ be fixed and let $K_{e^{-2itF}}$ be the kernel of the operator $\mathcal{K}_{e^{-2itF}}$ satisfying $K_{e^{-2itF}} \in B^{-r}(\mathbb{R}^{2n})$ and $u_0 \in B^{-r_0}(\mathbb{R}^n)$, for some $r_0, r \geq 0$. Let also $s_K, s \in \mathbb{R}$ be such that $s_* := \min\{s - r, s_K - r_0\} \leq s + s_K$. Then, for all fixed $\mu > n$, we have*

$$\begin{aligned} WF^{s_*-\mu}_{\text{iso}}(e^{-t\operatorname{Op}^{\mathrm{w}}(a)}u_0) &\subseteq WF^{s_K}_{\text{iso}}(K_{e^{-2itF}})' \circ WF^{s}_{\text{iso}}(u_0) \\ &\subseteq e^{-2itF}(WF^{s}_{\text{iso}}(u_0) \cap \operatorname{Ker}_{\mathbb{R}}(\operatorname{Im} e^{-2itF})). \end{aligned} \tag{23}$$

where $\operatorname{Ker}_{\mathbb{R}}(\operatorname{Im} e^{-2itF}) = \operatorname{Ker}(\operatorname{Im} e^{-2itF}) \cap \mathbb{R}^{2n}$.

At this point, by choosing s_K sufficiently large (see [12] for more details), we may rewrite inclusion (23) as (with $s \in \mathbb{R}$, $\mu' > n + r$ and $r \leq n + \varepsilon$)

$$WF^{s-\mu'}_{\text{iso}}(e^{-t\operatorname{Op}^{\mathrm{w}}(a)}u_0) \subseteq e^{-2itF}(WF^{s}_{\text{iso}}(u_0) \cap \operatorname{Ker}_{\mathbb{R}}(\operatorname{Im}\ e^{-2itF})).$$

We next note that, since $WF^{s}_{\text{iso}}(u_0) \subseteq \mathbb{R}^{2n}$ and e^{-2itF} is an invertible linear map (see Lemma 5.2 in [16]), we get

$$\begin{aligned} &e^{-2itF}(WF^{s}_{\text{iso}}(u_0) \cap \operatorname{Ker}_{\mathbb{R}}(\operatorname{Im} e^{-2itF})) \\ &= e^{-2itF}(WF^{s}_{\text{iso}}(u_0) \cap \operatorname{Ker}(\operatorname{Im} e^{-2itF}) \cap \mathbb{R}^{2n}) \\ &= (e^{-2itF}WF^{s}_{\text{iso}}(u_0)) \cap (e^{-2itF}(\operatorname{Ker}(\operatorname{Im} e^{-2itF}) \cap \mathbb{R}^{2n})). \end{aligned}$$

Now, setting $Z = e^{-2itF}W$, with $Z, W \in \mathbb{C}^{2n}$,

$$Z \in \operatorname{Ker}(\operatorname{Im}\ e^{2itF}) \cap \mathbb{R}^{2n} \Longleftrightarrow W \in \operatorname{Ker}(\operatorname{Im} e^{-2itF}) \cap \mathbb{R}^{2n},$$

Therefore, it follows that

$$e^{-2itF}(WF^s_{\mathrm{iso}}(u_0)\cap \mathrm{Ker}(\mathrm{Im}\, e^{-2itF})) = e^{-2itF}(WF^s_{\mathrm{iso}}(u_0))\cap \mathrm{Ker}(\mathrm{Im}\, e^{2itF})\cap \mathbb{R}^{2n},$$

and then

$$WF^{s-\mu'}_{\mathrm{iso}}(e^{-t\mathrm{Op^w}(a)}u_0)\subseteq (e^{-2itF}WF^s_{\mathrm{iso}}(u_0))\cap \mathrm{Ker}(\mathrm{Im}\, e^{2itF})\cap \mathbb{R}^{2n}. \tag{24}$$

Now by Theorem 5.9 in [10] for all $t\geq 0$ the linear continuous operator

$$\mathcal{K}_{e^{-2itF}}:\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}'(\mathbb{R}^n),$$

extends to a continuous map

$$\mathcal{K}_{e^{-2itF}}:\mathcal{S}'(\mathbb{R}^n)\to\mathcal{S}'(\mathbb{R}^n),$$

satisfying

$$\mathcal{K}_{e^{-2i(t_1+t_2)F}}=\mathcal{K}_{e^{-2it_1F}}\mathcal{K}_{e^{-2it_2F}}$$

and, then, for all $t_1, t_2\geq 0$

$$e^{-(t_1+t_2)\mathrm{Op^w}(a)}=e^{-t_1\mathrm{Op^w}(a)}e^{-t_2\mathrm{Op^w}(a)}, \tag{25}$$

Now choosing $t_1,t_2>0$ such that $t_1+t_2=t$ and using (24) and (25), with $\mu=2\mu'$ and $\mu'>n+r$, we get

$$\begin{aligned}
&WF^{s-2\mu'}_{\mathrm{iso}}(e^{-t\mathrm{Op^w}(a)}u_0)\\
&=WF^{s-2\mu'}_{\mathrm{iso}}(e^{-t_1\mathrm{Op^w}(a)}e^{-t_2\mathrm{Op^w}(a)}u_0)\\
&\subseteq (e^{-2it_1F}WF^{s-\mu'}_{\mathrm{iso}}(e^{-t_2\mathrm{Op^w}(a)}u_0))\cap \mathrm{Ker}(\mathrm{Im}\, e^{2it_1F})\cap\mathbb{R}^{2n}\\
&\subseteq \Big(e^{-2it_1F}((e^{-2it_2F}(WF^s_{\mathrm{iso}}(u_0))\cap \mathrm{Ker}(\mathrm{Im}\, e^{2it_2F})\cap\mathbb{R}^{2n})\Big)\cap \mathrm{Ker}(\mathrm{Im}\, e^{2it_1F})\cap\mathbb{R}^{2n}\\
&\subseteq (e^{-2itF}WF^s_{\mathrm{iso}}(u_0))\cap\Big((e^{-2it_1F}(\mathrm{Ker}(\mathrm{Im}\, e^{2it_2F})\cap\mathbb{R}^{2n})\Big)\cap \mathrm{Ker}(\mathrm{Im}\, e^{2it_1F})\cap\mathbb{R}^{2n}\\
&\subseteq (e^{-2itF}WF^s_{\mathrm{iso}}(u_0))\cap \mathrm{Ker}(\mathrm{Im}\, e^{-2it_1F})\cap\mathbb{R}^{2n}.
\end{aligned}$$

Hence,

$$WF^{s-\mu}_{\mathrm{iso}}(e^{-t\mathrm{Op^w}(a)}u_0)\subseteq (e^{-2itF}WF^s_{\mathrm{iso}}(u_0))\cap\Big(\bigcap_{0\leq t'\leq t}\mathrm{Ker}(\mathrm{Im}\, e^{-2it'F})\Big)\cap\mathbb{R}^{2n}$$

and, then, by (5),

$$WF^{s-\mu}_{\text{iso}}(e^{-t\text{Op}^{\text{w}}(a)}u_0) \subseteq (e^{-2itF}WF^{s}_{\text{iso}}(u_0)) \cap S. \tag{26}$$

If we now take $X \in (e^{-2itF}WF^{s}_{\text{iso}}(u_0)) \cap S$, we find $Y \in WF^{s}_{\text{iso}}(u_0) \subseteq \mathbb{R}^{2n}$ such that $X = e^{-2itF}Y \in S \subseteq \mathbb{R}^{2n}$. Then

$$Y = e^{2itF}X = \sum_{j=0}^{+\infty} \frac{(2itF)^j X}{j!} = \sum_{j=0}^{+\infty} \frac{(-2t\text{Im}F)^j X}{j!} = e^{-2t\text{Im}F}X, \tag{27}$$

since, by (6), $(iF)^j X = (-\text{Im}\, F)^j X$ for all $j \geq 0$. We deduce that

$$X = e^{2t\text{Im}F}Y \in (e^{2t\text{Im}F}WF^{s}_{\text{iso}}(u_0)) \cap S.$$

Conversely, from (27), if $X \in (e^{2t\text{Im}F}WF^{s}_{\text{iso}}(u_0)) \cap S$, then

$$Y = e^{-2t\text{Im}F}X = e^{2itF}X \in WF^{s}_{\text{iso}}(u_0).$$

On the other hand this implies that

$$\begin{aligned}(e^{-2itF}WF^{s}_{\text{iso}}(u_0)) \cap S &= (e^{2t\text{Im}F}WF^{s}_{\text{iso}}(u_0)) \cap S \\ &= e^{2t\text{Im}F}(WF^{s}_{\text{iso}}(u_0) \cap (e^{-2t\text{Im}F}S)).\end{aligned} \tag{28}$$

Now, since by (6) the singular space S is invariant with respect to $\text{Im}\, F$, we get

$$e^{-2t\text{Im}F}S \subseteq S.$$

Moreover, as the linear map $e^{-2t\text{Im}F}$ is invertible, we have that

$$e^{-2t\text{Im}F}\Big|_S : S \to S,$$

is an injective endomorphism and since S is finite dimensional it is also surjective. Therefore

$$e^{-2t\text{Im}F}S = S.$$

Hence, putting together (26) and (28) we obtain that

$$WF^{s-\mu}_{\text{iso}}(e^{-t\text{Op}^{\text{w}}(a)}u_0) \subseteq (e^{2t\text{Im}F}(WF^{s}_{\text{iso}}(u) \cap S)) \cap S,$$

and hence, by (4)

$$WF^{s-\mu}_{\text{iso}}(e^{-t\text{Op}^{\text{w}}(a)}u_0) \subseteq (e^{tH_{a_I}}(WF^{s}_{\text{iso}}(u) \cap S)) \cap S.$$

This ends the proof of the Theorem, $\square$

We conclude this work by observing that, in the context of our toy model, Theorem 4.2 has enabled a refinement of the rough inclusion (23).

Example 4.1. *In this example we consider the Cauchy problem*

$$\begin{cases} \partial_t u + \mathrm{Op}^{\mathrm{w}}(a)u = 0, & \textit{in } \mathbb{R}^+ \times \mathbb{R}^n, \\ u(0,\cdot) = u_0 \in \mathcal{S}'(\mathbb{R}^n), \end{cases}$$

where

$$a(X) = |X'|^2 + i|X''|^2.$$

Let now $0 < t < \pi/2$ be fixed. In this case, for $t' \in]0,t]$ we have

$$\begin{aligned} &K_{e^{-2it'F}}(x,y) \\ &= \exp\Big(-\frac{1}{2}((|x'|^2 + |y'|^2)\cosh(2t') - 2\langle x', y'\rangle)/\sinh(2t')\Big)/\sqrt{2\pi\sinh(2t')} \\ &\quad \times \exp\Big(\frac{i}{2}((|x''|^2 + |y''|^2)\cos(2t') - 2\langle x'', y''\rangle)/\sin(2t')\Big)/\sqrt{2\pi i\sin(2t')}, \end{aligned}$$

where here $\sqrt{2\pi i \sin(2t')}$ is a fixed complex root of $z = 2\pi i \sin(2t')$.
Then, for $\varepsilon > 0$, and for all $t' \in]0,t]$ we have (see [12]*)*

$$K_{e^{-2it'F}} \in H^{-n_2-\varepsilon,0}(\mathbb{R}^{2n}) \subseteq B^{-n_2-\varepsilon}(\mathbb{R}^{2n}).$$

Moreover,

$$Q = \left(\begin{array}{cc|cc} I_{n_1} & 0_{n_1\times n_2} & \multicolumn{2}{c}{\multirow{2}{*}{$0_{n\times n}$}} \\ 0_{n_2\times n_1} & iI_{n_2} & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$0_{n\times n}$}} & I_{n_1} & 0_{n_1\times n_2} \\ & & 0_{n_2\times n_1} & iI_{n_2} \end{array}\right),$$

and so

$$F = JQ = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{$0_{n\times n}$}} & I_{n_1} & 0_{n_1\times n_2} \\ & & 0_{n_2\times n_1} & iI_{n_2} \\ \hline -I_{n_1} & 0_{n_1\times n_2} & \multicolumn{2}{c}{\multirow{2}{*}{$0_{n\times n}$}} \\ 0_{n_2\times n_1} & -iI_{n_2} & & \end{array}\right).$$

Therefore

$$\mathrm{Im} F = \frac{F-\bar{F}}{2i} = \left(\begin{array}{c|c} 0_{n\times n} & \begin{matrix} 0_{n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & I_{n_2} \end{matrix} \\ \hline \begin{matrix} 0_{n_1\times n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & -I_{n_2} \end{matrix} & 0_{n\times n} \end{array}\right)$$

and

$$\mathrm{Re} F = \frac{F+\bar{F}}{2} = \left(\begin{array}{c|c} 0_{n\times n} & \begin{matrix} I_{n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & 0_{n_2} \end{matrix} \\ \hline \begin{matrix} -I_{n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & 0_{n_2} \end{matrix} & 0_{n\times n} \end{array}\right).$$

Hence, since $(\mathrm{Re}\, F \mathrm{Im}\, F) = 0$*, in this case*

$$S = \mathrm{Ker}(\mathrm{Re}\, F) \cap \mathbb{R}^{2n} = (\{0_{n_1}\} \times \mathbb{R}^{n_2}) \times (\{0_{n_1}\} \times \mathbb{R}^{n_2})$$

and as

$$e^{2t\mathrm{Im}\, F} = \left(\begin{array}{cc|cc} I_{n_1} & 0_{n_1\times n_2} & 0_{n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \cos(2t) I_{n_2} & 0_{n_2\times n_1} & \sin(2t) I_{n_2} \\ \hline 0_{n_1} & 0_{n_1\times n_2} & I_{n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & -\sin(2t) I_{n_2} & 0_{n_2\times n_1} & \cos(2t) I_{n_2} \end{array}\right)$$

we may conclude that, denoting $\pi_{n_2} : (x,\xi) \mapsto (x'',\xi'')$ *the projection onto the* n_2*-group of variables, for* $0 < t < \pi/2$*, for* $s \in \mathbb{R}$ *and for* $\varepsilon > 0$*,*

$$\begin{aligned} & WF_{\mathrm{iso}}^{s-2n-2n_2-\varepsilon}(e^{-t\mathrm{Op}^{\mathrm{w}}(a)} u_0) \\ & \subseteq \Big(e^{2t\mathrm{Im} F}(WF^s_{\mathrm{iso}}(u_0) \cap S) \cap S \Big) \\ & = \Big(\{0_{n_1}\} \times \big\{ \cos(2t) x'' + \sin(2t)\xi'';\ (x'',\xi'') \in \pi_{n_2} WF^s_{\mathrm{iso}}(u_0) \big\} \Big) \\ & \quad \times \Big(\{0_{n_1}\} \times \big\{ -\sin(2t) x'' + \cos(2t)\xi'';\ (x'',\xi'') \in \pi_{n_2} WF^s_{\mathrm{iso}}(u_0) \big\} \Big). \end{aligned} \tag{29}$$

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